15-859NN Spectral Graph Theory, Spring 2021

Homework 1 **Due: Friday February 19 in class**

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Instructions. Collaboration is permitted in groups of size at most three. You must write the names of your collaborators on your solutions and must write your solutions alone.

In class on February 19th we will only go over the first two problems. The third we go over the next week.

(25) 1. **Pseudoinverses**

Wikipedia defines the Moore-Penrose inverse or Pseudoinverse to be the operation that takes an $n \times m$ matrix A and returns an $m \times n$ A^+ satisfying the following constraints. We will assume that A is real this will simpify the constraints to:

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)^T = AA^+$
4. $(A^+A)^T = A^+A$

Recall that in class we defined the pseudoinverse of a symmetric real matrix M to be $U\Lambda^+U^T$ where $M = U\Lambda U^T$ is the spectral decomposition of M.

(a) Show that our definition agrees with the Moore-Penrose inverse and it is the only solution.

(b) In the case where A is real symmetric do we need all four axioms or constraints?

(25) 2. **Effective Resistance and Selfloops**

1. Given a connected graph of conductors $G = (V, E, c)$ (for simplicity assume $V = [n]$), let L be its Laplacian. Fix a destination $n \in V$, and let L_n be obatined from L by deleting the last row and last column.
   (a) Prove that L_n has full rank.
   (b) Prove that $(L_n^{-1})_{i,i}$ is equal to the effective resistance between i and n.
2. Given a connected graph $G = (V, E, c)$ (for simplicity assume $V = [n]$), let L be its Laplacian. Fix a destination $n \in V$. We defined the hitting time h_i to be the expected number of steps we need to take from i to n. For any $i \neq n$, we have $h_i = 1 + \sum_j \frac{c_{i,j}}{c_i} h_j$ with $h_n = 0$, which can be modeled written as $L_n h = [c_1, ..., c_n]^T$ where L_n is obtained from L by deleting the last row and column.
   (a) Suppose that we add a self-loop of positive weight at some vertices of G to make G'. Show that the hitting time is still a solution of $L_n h = [c'_1, ..., c'_{n-1}]^T$ where $c'_i \geq c_i$ for every i. Note that L_n is still obtained from the Laplacian of G.

(b) Intuition tells that adding a self-loop should increase the hitting time of every vertex. Prove it formally by showing that if h, h' satisfy $L_n h = c$ and $L_n h' = c'$ where $c \leq c'$ (which means that $c_i \leq c'_i$ for all i), $h \leq h'$.

(c) Does the above fact also holds when L_n is replaced by any other positive-definite matrix? In other words, if A is a symmetric, positive definite matrix (so it is invertible), $b \leq b'$, and $Ax = b, Ax' = b'$, is it true that $x \leq x'$?

(25) 3. **Resistance in the Mesh Graph**

(a) Suppose that $M = (V, E)$ is the $\sqrt{n}$ by $\sqrt{n}$ mesh graph where each edge is a unit size resistor and the lower left most vertex is a and the right most upper vertex is b.

It is known that the effective resistance $R_{ab} = \Omega(\log n)$.

Prove that the effective resistance is $R_{ab} \geq \frac{1}{2} \log n$.

Hint: Use Rayleigh's Monotonicity law by adding extra conductors to M so that one can use the first two parts of the problem.

Hint: If graph G has an edge between a, b with 0 (or extremely low) resistance, what would be an equivalent G' that preserves effective resistance between pairs of vertices. Try to work this out with a triangle.

(b) Problem: Show how to construct an efficient unit electrical flow from a to b using Polya's Urn as a way to set the current flow in graph M and thus get a tight upper bound on R_{ab}.

Recall that Polya's urn is a famous sampling model used in probability. Let's say you had an urn with red and green balls. You choose one ball at random, note the color, and replace the ball in the urn along with another ball of the same color. The resulting model is called Polya's urn process.

Hint explain how to think of Polya's process as a random walk on the mesh graph.