# 15-859NN Spectral Graph Theory, Spring 2021

**Homework 2** **Due: Monday Mar 22 in class**

Gary Miller

**Instructions.** Collaboration is permitted in groups of size at most three. You must write the names of your collaborators on your solutions and must write your solutions alone.

| Question | Points | Score |
|---|---|---|
| 1 | 25 | |
| 2 | 25 | |
| 3 | 25 | |
| Total: | 75 | |

(25) 1. **Resistance Theorem**

In this problem we show that if we have a graph $G$ with two attachment vertices $a$ and $b$ and we only have attachment to these two vertices then we can replace the entire graph with a single edge from $a$ to $b$ with resistance $R_{ab}$ in $G$.

Prove the following theorem.

**Theorem 0.1.** *Let $a$ and $b$ be two vertices of a graph $G$ with Laplacian $L$. Let $R_{ab}$ the effective resistance in $G$ from $a$ to $b$ and $H_{ab}$ the Laplacian of the unit weight single edge graph from $a$ to $b$, i.e., $H_{ab} = \chi_{ab}\chi_{ab}^T$. Then*

$$\text{For all } x \neq 0 \; x^T H_{ab} x \leq \mu \cdot x^T L x \text{ if and only if } R_{ab} \leq \mu \tag{1}$$

*and there exist an $x$ such that for all $\alpha$, the inequality holds with equality for $x + \alpha\bar{1}$.*

*In the language of Loewner Order, defined below, show that if $a$ and $b$ are two distinct vertices of $G$ then*

$$H_{ab} \preceq \mu \cdot L \text{ if and only if } R_{ab} \leq \mu \tag{2}$$

Hint: First prove the theorem for the case $x = L^+ w$. At some point in your proof you will need to use Cauchy-Schwartz.

(25) 2. **Loewner Inequalities**

1.

**Definition 1.** *Let $T$ be a spanning tree of a unit weight graph $G = (V, E)$. The stretch, $str(e)$ of an edge $e = (a, b) \in E$ is the number of edges on the path in $T$ from $a$ to $b$. While $str_T(G) = \sum_{e \in E} str(e)$ is the stretch of $G$ in $T$.*

Using the path embedding argument prove that for any spanning tree $T$ of a connected unweighted graph $G$,

$$L_T \preceq L_G \preceq str_T(G) L_T$$

2. Let $A$ and $B$ be symmetric positive definite $n$ by $n$ matrices.
   Show that $A \preceq \alpha B$ if and only if $\lambda_{\max}(B^{-1}A) \leq \alpha$

3. Let $G = (V, E, w)$ and $H = (V, E', w')$ be two weighted graphs. We generalize the stretch for trees to general graphs as follows. Lets define the stretch between of $G$ in $H$ as

$$str_H(G) = \sum_{e \in E} w_G(e) \cdot ER_H(e)$$

   Using Part 2 prove that for any subgraph $H$ of a connected weighted graph $G$,

$$L_H \preceq L_G \preceq str_H(G) L_H$$

   where $str_H(G)$ is the stretch of $G$ in $H$.

4. If $A \preceq B$ and $A$ and $B$ are both PSD, then is $A^2 \preceq B^2$ always true? Prove this statement, or else give a counterexample.
   Hint: Write $x^T A^2 x$ as $(Ax)^T (Ax)$ and choose $A$ and $B$ to be graph Laplacians.

5. In this problem we show how to generalize the Loewner order to matrices which need not have the same vertex set.
   Let that $G = (V, E, w)$ and $H = (V', E', w')$ be two weighted graphs such that $W \subset V \cap V'$.

   **Definition 2.** *Let $x$ vary over column vectors corresponding to the vertices in $W$, $y$ vary over column vectors corresponding to the vertices $V \setminus W$, and $y'$ vary over column vectors corresponding to the vertices $V' \setminus W$. We define the Loewner inequality $L_G \preceq_W L_H$ if for all $x$:*

$$\min_y \begin{pmatrix} x \\ y \end{pmatrix}^T L_G \begin{pmatrix} x \\ y \end{pmatrix} \leq \min_{y'} \begin{pmatrix} x \\ y' \end{pmatrix}^T L_H \begin{pmatrix} x \\ y' \end{pmatrix}$$

   (a) Let $G = (V, E, w)$ be a weighted graph over the vertices $V$. Suppose that $A$ is the Schur complement of $L_G$ after pivoting out some single vertex/variable, say, $v_n$ from $L_G$. Show that $A$ is the Laplacian of a graph, say $H$ on the vertices $V_1, \ldots, V_{n-1}$.

(b) Furthermore, suppose we pivoted out from $G = (V, E, w)$ all the vertices $v \in W \subset V$. Denote the residual graph with $H = (V', E', w')$ where $V' = V \setminus W$. Show that:

$$L_H \preceq_{V'} L_G \preceq_{V'} L_H$$

(25) 3. **Constant Time Sampling**

Suppose we are given a set of real numbers $r_1, r_2, \ldots, r_n > 0$ such that $\sum_{i=1}^{n} r_i = 1$. and a constant time algorithm which generates number uniformly in the unit interval.

The goal is to computes the random variable $R \in \{1, \ldots, n\}$ such that the probability that $R = i$ is $r_i$.

It is easy to see how to compute a sample of $R$ in $O(\log n)$ time per sample. How?

Give an algorithm that preprocesses $r_1, \ldots, r_n$ such that samples can be generated in constant time per sample, worst case. Your algorithm should do the preprocessing in $O(n)$ time.

You may assume that you have access to unit time uniform $[0 - 1]$ random number generator and arithmetic operation are also unit time.