15-859NN Spectral Graph Theory, Spring 2021

Homework 3 **Due: Friday Apr 9th in class**

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Instructions. Collaboration is permitted in groups of size at most three. You must write the names of your collaborators on your solutions and must write your solutions alone.

Question	Points	Score
1	25	
2	15	
3	25	
4	25	
Total:	90	

(25) 1. **Weyl's Inequality**

In class we proved the following inequality:

Theorem 0.1. *Let $A \in \mathbb{R}^{nxn}$ matrix with singular values $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ and eignvalues $|\lambda_1| \geq \cdots \geq |\lambda_n|$ then*

$$\prod_{i=1}^{k} |\lambda_i| \leq \prod_{i=1}^{k} \sigma_i \quad 1 \leq k \leq n \tag{1}$$

In class we claimed the following Corollary to Weyl:

Corollary 1. *If A,σ, and λ is on the theorem above then*

$$\sum_{i=1}^{k} |\lambda_i| \leq \sum_{i=1}^{k} \sigma_i \quad 1 \leq k \leq n \tag{2}$$

By applying the logarithm function to both sides of equation 1 we get

$$\sum_{i=1}^{k} \log |\lambda_i| \leq \sum_{i=1}^{k} \log(\sigma_i) \quad 1 \leq k \leq n \tag{3}$$

Prove the following theorem and use it to prove the corollary.

Theorem 0.2. *Given two sequences* $a_1 \geq \cdots \geq a_n \geq 0$ *and* $b_1 \geq \cdots \geq b_n \geq 0$ *such that*

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \quad 1 \leq k \leq n \tag{4}$$

and $f(0) = 0$, *is convex, and non-decreasing then*

$$\sum_{i=1}^{k} f(a_i) \leq \sum_{i=1}^{k} f(b_i) \quad 1 \leq k \leq n \tag{5}$$

Hints:

1. Try induction on n. Suppose for some $1 \leq k < n$ we have $\sum_{i=1}^{k} f(a_i) = \sum_{i=1}^{k} f(b_i)$ what can you say?
2. Assuming $0 < \alpha \leq \beta$ show that

$$f(\alpha) + f(\beta) \leq f(\alpha - \epsilon) + f(\beta + \epsilon)$$

for $\epsilon \leq \alpha$.

(15) 2. **λ_2 of Balanced Binary Tree**

Let T_n be a rooted balanced binary tree with n vertices (i.e. the depth of the left and right subtrees of every node differ by 1 or less). Show that λ_2 of its Laplacian is $\Theta(1/n)$.

Hint: It should be easy, using methods from class, to see that $1/(n \log n) = O(\lambda_2)$ and $\lambda_2 = O(1/n)$. You will need to tune our lower bound method.

(25) 3. **Eigenvalues of Cartesian Products**

Let $G = (V, E, w)$ and $H = (V', E', w')$ be two non-negatively weighted simple graphs. Let $G \otimes H = (\bar{V}, \bar{E}, \bar{w})$ be their Cartesian product, where:

- The vertices are $\bar{V} = V \times V'$
- The edges are $\bar{E} = \{((x, x'), (y, y')) \,|\, [x = y \wedge (x', y') \in E'] \vee [x' = y' \wedge (x, y) \in E]\}$
- $\bar{w}((x, x'), (x, y')) = w'(x', y')$ and $\bar{w}((x, x'), (y, x')) = w(x, y)$.

1. Show that the eigenvalues of $L_{G \otimes H}$ are the direct sum of those of L_G and L_H. That is if the eigenvalues of L_G are $\{\lambda_1, \ldots, \lambda_n\}$ and those of L_H are $\{\mu_1, \ldots, \mu_m\}$ the those of $L_{G \otimes H}$ are $\{\lambda_i + \mu_j \,|\, 1 \leq i \leq n,\, 1 \leq j \leq m\}$
2. Show that the eigenvectors of $L_{G \otimes H}$ are the direct product of those of L_G and L_H. The direct product of column vectors $\bar{a}$ and $\bar{b}$ is the matrix ab^T, reformatted (flattened) as a vector.

(25) 4. **A Bad Example for Spectral Partitioning**

Define **threshold spectral partitioning** of a possibly weighted graph $G = (V, E)$ to be the vertex partition one gets by; 1) Finding the eigenvector x for λ_2, 2) Sorting the vertices by their value in x, and 3) Returning the best threshold cut.

Let P_n^ϵ, for n even, be the weighted path graph on n vertices where all the edges have unit weight except the middle one which has weight ϵ.

Consider the Cartesian product $M_{cn}^\epsilon = P_{c\sqrt{n}} \otimes P_{\sqrt{n}}^\epsilon$

1. Show that threshold spectral partitioning on the graph M_{cn}^ϵ will generate a quotient cut of size $\Omega(1/\sqrt{n})$ for c sufficiently large and $\epsilon = 1/\sqrt{n}$ while the best cut is of size $O(1/n)$.
2. How big does c need to be for this to happen?