15-859N — Spectral Graph Theory and Scientific Computing — Fall 2007

Gary Miller

Assignment 1 Due date: October 25

1 Effective Perron-Frobenius Theorem

[10 points]

a) The Perron-Frobenius theorem tell us that if A is the adjacency matrix of a weighted, strongly connected, directed graph, then A has a strictly positive eigenvector v with positive eigenvalue λ_1. Moreover, all other eigenvalues λ_i of A satisfy $|\lambda_i| < \lambda_1$. It turns out that this theorem is easy to prove in the case that A is stochastic, that is $A1 = 1$. In this problem, we will show how to reduce to the case that $A1 = c1$, at least when A has no zero entries. That is, we will prove that every such A is similar to a constant times a stochastic matrix. We will do this algorithmically. Let $A_0 = A$. Then, set $s^{(i)} = A_i 1$, and $D_i = \text{diag}(s^{(i)})$. We then set $A_{i+1} = D_i^{-1} A_i D_i$. We will show that the sequence of matrices A_i is converging to a constant times a stochastic matrix.

(a) Let $s^{(i)}_{\max} = \max_j s^{(i)}$ denote the maximum row-sum in A_i, and $s^{(i)}_{\min}$ denote the minimum row sum. Prove that

$$s^{(i+1)}_{\max} - s^{(i+1)}_{\min} \leq s^{(i)}_{\max} - s^{(i)}_{\min}$$

(b) For a matrix A, let $\min(A)$ denote the minimum entry of A. Set

$$\gamma_i = \frac{\min(A_i)}{s^{(i)}_{\max}}$$

Prove that

$$s^{(i+1)}_{\max} - s^{(i+1)}_{\min} \leq (1 - \gamma_i) s^{(i)}_{\max} - s^{(i)}_{\min}$$

(c) Prove that for every matrix A, there exists a constant ϵ such that $\gamma_i \geq \epsilon$, for all i.

Taken together, these statements show that if A has no zero entries, then the sequence A_i approaches a multiple of a stochastic matrix. With a little analysis, one can extend this to show that A is similar to a multiple of a stochastic matrix.

(extra credit) Extend this analysis to the case in which A can have zero entries, but is the weighted adjacency matrix of a strongly connected directed graph.

2 Experimental Problem

[10 points]

This problem can be summarized as "get a graph and tell me a little about it". To be more specific, you should find a graph that you are going to use in experimental work throughout the class (although you can switch graphs later if you become unhappy with your original choice). The graph should be from some "real-world" source. That is, it should not be constructed at random, or come from an algebraic construction. The graph should have at least 1,000 nodes, and preferably at least 10,000. The best way to get a graph is probably to write a script to read it or grab it from the web. Interesting sources include databases of papers (arxiv, ncstrl), fragments of the web (Wikipedia, links internal to a university, etc.). Wget is a useful tool for downloading a large portion of the web, but it is overkill since you only need the links. This part could take a lot of work.

Here's what you should report:

1. What is your graph, and where did you get it.
2. How are you storing your graph?
3. How many nodes are there in your graph?
4. How many edges does it have?
5. How many nodes does it have of degree 1?
6. How many nodes does it have of degree 2?
7. What tools are you using?

3 Estimating λ_2 for a complete balences binary tree

[10 points]

Determnine upper and lower bounds on λ_2 for a matrix A where A is the Laplacian of the complete balenced binary tree on n nodes. You should get a result of the form:

$$\lambda_2 = \Theta(f(n))$$

where $f(n)$ is some function of n.