Multi-coloured Hamilton cycles in random edge-coloured graphs

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June 11, 2001

Abstract

We define a space of random edge-coloured graphs $\mathcal{G}_{n,m,\kappa}$ which correspond naturally to edge κ-colourings of $G_{n,m}$. We show that there exist constants $K_0,\ K_1 \leq 21$ such that provided $m \geq K_0 n \log n$ and $\kappa \geq K_1 n$ then a random edge coloured graph contains a multi-coloured Hamilton cycle with probability tending to 1, as the number of vertices n tends to infinity.

1 Introduction

Let $\mathcal{G}_{n,m,\kappa}$ denote the space of random edge-coloured graphs, defined as follows: Each G in $\mathcal{G}_{n,m,\kappa}$ has vertex set $[n]$, edge set $E_m(G)$ of size m, and each edge is coloured with a label from $[\kappa]$. Thus $|\mathcal{G}_{n,m,\kappa}| = \binom{N}{m}\kappa^m$ where $N = \binom{n}{2}$. The elements of $\mathcal{G}_{n,m,\kappa}$ are given the uniform measure. A random edge-coloured graph $G_{n,m,\kappa}$ is a graph G sampled uniformly at random from $\mathcal{G}_{n,m,\kappa}$.

Given $G \in \mathcal{G}_{n,m,\kappa}$, a subset S of the edges of G is *multi-coloured*, if no two edges of S have the same colour. We are interested in conditions on n, m, κ which imply that **whp**[1] $G_{n,m,\kappa}$ contains a multi-coloured Hamilton cycle. We also consider the corresponding randomly arc-coloured random digraph $D_{n,m,\kappa}$ which is defined in the analogous way. We prove

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[1]A sequence of events $\mathcal{E}_n$ is said to occur *with high probability* (**whp**) if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

Theorem 1 *There exist constants $K_0, K_1 > 0$ such that if $m \geq K_0 n \log n$ and $\kappa \geq K_1 n$ then* **whp** *$G_{n,m,\kappa}$ (resp. $D_{n,m,\kappa}$) contains a multi-coloured Hamilton cycle.*

We note that Theorem 1 is best possible up to a constant factor, for we need at least n distinct colours and the underlying graph (resp. digraph) does not contain a Hamilton cycle **whp** until $m > n \log n/2$. The values of the constants, $(K_0, K_1 = 21)$, we use in the proof of Theorem 1 are, however, not the best possible values.

There are two general types of results on multi-coloured structures: **whp** existence under random colouring and guaranteed existence under adversarial colouring. When considering adversarial (worst-case) colouring, the guaranteed existence of multi-coloured structures, is called an Anti-Ramsey property.

Erdös, Nesetril and Rödl [5], Hahn and Thomassen [8] and Albert, Frieze and Reed [1] (correction in Rue [10]) considered colourings of the edges of the complete graph K_n where no colour is used more than k times. It was shown in [1] that if $k \leq n/64$, then there must be a multi-coloured Hamilton cycle. Cooper and Frieze [3] proved a random graph threshold for this property to hold in almost every graph in the space studied.

With respect to random colouring, Janson and Wormald [9] gave conditions for the existence of a multi-coloured Hamilton cycle in a random regular graph. We also mention that Frieze and Mckay [7] found a tight threshold for the existence of a multi-coloured spanning tree.

2 A sequence of random graphs

Because we are concerned with monotone properties, we can work entirely with the independent model $G_{n,p,\kappa}$ where $p = m/N$ and the underlying uncoloured graph is $G_{n,p}$. Let p_1 satisfy $1 - p = (1 - p_1)^2$. Let D_{n,p_1} be the random digraph where each arc occurs independently with probability p_1. Suppose now that we randomly colour the arcs of D_{n,p_1} with κ colours to obtain the random coloured graph $D_{n,p_1,\kappa}$. Ignoring orientation gives us the random graph $G_{n,p,\kappa}$, provided we make a random choice from the two possible colours when coalescing the edges of directed 2-cycles.

Next let $\mathcal{D}_{d-out,\kappa}$ denote the following set of arc-coloured *digraphs*: Each $D \in \mathcal{D}_{d-out,\kappa}$ has vertex set $[n]$, each vertex has out-degree d and the arcs of D are multi-coloured by $[\kappa]$ i.e. no colour is used more than once. Thus $|\mathcal{D}_{d-out,\kappa}| = \binom{n-1}{d}^n \binom{\kappa}{dn}(dn)!$. The arc-coloured digraph $D_{d-out,\kappa}$ is chosen uniformly at random from $\mathcal{D}_{d-out,\kappa}$. In this paper we will be concerned with $d = O(1)$, in particular we assume that $d = 5$ from now on.

The central idea of this paper is to use a network flow algorithm to take $D_{n,p_1,\kappa}$ and, conditional on an event of probability 1-o(1), return as output, a multi-coloured sub-digraph D. The distribution of D will be that of $D_{d-out,\kappa}$. If we ignore orientation in $D_{d-out,\kappa}$ and *delete* parallel edges then we obtain the random multi-coloured graph $G_{d-out,\kappa}$. Ignoring colours now gives us the random graph G_{d-out}. If it is known **whp** that G_{d-out} is Hamiltonian, then we will have proved that **whp** $G_{n,p,\kappa}$ contains the required multicoloured H. To prove Theorem 1 for $G_{n,m,\kappa}$ we only have to do this for $d \geq 5$ and then apply the result of Frieze and Łuczak [6] which states that such a graph is Hamiltonian **whp**. There is a technical point here. In the usual construction of G_{d-out} we coalesce rather than delete parallel edges. It is not difficult to see that the proof of [6] is easily modifiable to handle this. On the other hand the result of Cooper and Frieze [4] that G_{4-out} is Hamiltonian **whp** seems to run into difficulty.

2.1 Network Flow Construction

We define a flow network $\mathcal{N}$ as follows. $\mathcal{N}$ has source s and sink t. The vertex set W consists of s, t, the set of colours $C = [\kappa]$ and the set $V = [n]$ of vertices of the $D_{n,p_1,\kappa}$ under consideration. For each colour $x \in C$ there is an arc (s, x) in $\mathcal{N}$ of capacity 1. There is an arc (x, v) in $\mathcal{N}$ of infinite capacity for every $v \in V$ for which there is an arc (v, w) in $D_{n,p_1,\kappa}$ with tail v and colour x. Finally, for each vertex $v \in V$ there is an arc (v, t) of capacity d.

For $S \subseteq C$, let $N(S) = \{v : \ x \in S, \ v \in V, \ (x, v) \in \mathcal{N}\}$ be the out-neighbour set of S in $\mathcal{N}$. A cut of finite capacity can be obtained from a set $S \subseteq C$ and $N(S) \subseteq V$. Let $T = N(S)$, $W = \{s\} \cup S \cup T$, and let $\overline{W} = (C \setminus S) \cup (V \setminus T) \cup \{t\}$. The capacity of the cut $(W : \overline{W})$ is $\kappa - |S| + d|T|$. Applying the max-flow min-cut theorem we see that $\mathcal{N}$ admits a flow of value dn if and only if, for all $S \subseteq C$,

$$\kappa - |S| + d|N(S)| \geq dn. \tag{1}$$

We estimate the probability that (1) is not true because, for some set S, $|N(S)| < n - (\kappa - |S|)/d$. I.e. there exists a set of colours S of size s and a set of vertices $\overline{T}$ of size $|\overline{T}| > (\kappa - s)/d$ such that every arc of D whose tail is in $\overline{T}$ has a colour in $C \setminus S$.

p_1 satisfies $1 - p = (1 - p_1)^2$ and so $p_1 \geq p/2$, for $1 - \sqrt{1-p} \geq p/2$ for $p \geq 0$. We see therefore that $np_1 \geq K_0 \log n$.

Let $\mathcal{E}$ denote the subset of D_{n,p_1} for which $\delta^+(D_{n,p_1}) > np_1/2$.

We first estimate $\mathbf{Pr}(\overline{\mathcal{E}})$. By the Chernoff inequality,

$$\mathbf{Pr}(\delta^+(D_{n,p_1}) \leq np_1/2) \leq ne^{-np_1/8} = O(n^{1-K_0/8}), \tag{2}$$

which is $O(n^{-13/8})$ for $K_0 \geq 21$. Thus $\mathbf{Pr}(\overline{\mathcal{E}}) = o(1)$

Let

$$L(s) = 2\binom{\kappa}{s}\binom{n}{\lceil(\kappa-s)/d\rceil}\left(\frac{\kappa-s}{\kappa}\right)^{(\kappa-s)np_1/(2d)}$$

be an upper bound on the probability that some set of size s does not satisfy (1) conditional on $\mathcal{E}$. The range of s we need to consider is between $\kappa - dn + 1$ and $\kappa - 1$. For, if $|S| < \kappa - dn$ then (1) is true with $N(S) = \emptyset$, and if $s = \kappa$ then as $\delta^+(D) \geq np_1/2$, $\overline{T} = \emptyset$.

The probability that (1) is not satisfied is bounded by Θ where

$$\Theta = \mathbf{Pr}(\overline{\mathcal{E}}) + \sum_{s=\kappa-dn+1}^{\kappa-1} L(s). \tag{3}$$

As $\mathbf{Pr}(\overline{\mathcal{E}}) = o(1)$, we can concentrate on the summation term in (3).

Now, choosing $\kappa \geq 21n$, and putting $\lceil(\kappa-s)/d\rceil = (\kappa-s)/d + f_s$, $0 \leq f_s < 1$,

$$\begin{aligned}
\sum_{s=\kappa-dn+1}^{\kappa-1} L(s) &\leq 2ned \sum_{s=\kappa-dn+1}^{\kappa-1} \left(\frac{\kappa e}{\kappa-s}\left(\frac{ned}{\kappa-s}\right)^{1/d}\left(\frac{\kappa-s}{\kappa}\right)^{(K_0\log n)/(2d)}\right)^{\kappa-s} \\
&\leq 2ned \sum_{s=\kappa-dn+1}^{\kappa-1} \left(3\left(\frac{\kappa-s}{\kappa}\right)^{K_0\log n/(2d)-1/d-1}\right)^{\kappa-s} \\
&\leq 2ned \sum_{s=\kappa-dn+1}^{\kappa-1} \left(3\left(\frac{dn-1}{edn}\right)^{(K_0\log n)/(2d)-1/d-1}\right)^{\kappa-s} &(4)\\
&\leq 2ed^2n^2 \exp\left\{-\frac{K_0\log n}{2d} + \frac{1}{d} + 3\right\}, &(5)
\end{aligned}$$

which is $O(n^{-1/10})$ for $K_0 \geq 21$ when $d = 5$.

Thus **whp** $\mathcal{N}$ contains a flow of value nd. The capacities of $\mathcal{N}$ are integral and so we can assume this flow is integral. It decomposes into nd (s,t)-paths, each of which assigns a colour x to a vertex v. By construction a colour can be assigned at most once to an edge and each vertex is assigned d colours. For each assignment of a colour x to a vertex v we choose (randomly from D) an arc of colour x which has tail v. We thus obtain a multi-coloured member of $\mathcal{D}_{d-out,\kappa}$. It is easy to argue that the underlying uncoloured digraph is distributed as $D_{d-out,\kappa}$. Indeed we could start with $D_{n,p_1,\kappa}$ and then replace each arc (v,w) by $(v,\pi_v(w))$ where the π_v, $v \in V$ are independent permutations of $V \setminus \{v\}$. After this transformation the digraph is still distributed as $D_{n,p_1,\kappa}$. We run the network flow algorithm and **whp** we obtain a multi-coloured member of $\mathcal{D}_{d-out,\kappa}$.

By replacing each arc (v,w) by $(v,\pi_v^{-1}(w))$ we obtain a subgraph of the original $D_{n,p_1,\kappa}$ which is distributed as $D_{d-out,\kappa}$. For those cases where both (v,w) and (w,v) are selected by the algorithm to be edges of $D_{n,p_1,\kappa}$ we simply delete this edge. We have to do this because of the possibility that the network algorithm chooses a different colour for $\{v,w\}$ to the one chosen in going from $D_{n,p_1,\kappa}$ to $G_{n,p,\kappa}$.

In summary, **whp** $D_{n,p_1,\kappa}$ contains a multi-coloured subgraph which is distributed as $D_{d-out,\kappa}$. Ignoring orientation we obtain a graph which **whp** contains a Hamilton cycle. This verifies Theorem 1 for the case of undirected graphs.

Consider now the directed case i.e. we start with $D_{n,p,\kappa}$. We first split this into two independent copies D_1, D_2 of $D_{n,p_1,\kappa}$. We then use a slightly modified network. Now we have vertices s, t, C and two copies V_1, V_2 of V. The s, C edges are as before and there are V_1, t and V_2, t edges of capacity d (now we can take $d = 3$). We join $x \in C$ to $v \in V_1$ by an infinite capacity arc if V_1 contains an arc of colour x and tail v. We join $x \in C$ to $v \in V_2$ by an infinite capacity arc if V_1 contains an arc of colour x and head v. The network flow algorithm constructs a random multi-coloured 3-in,3-out digraph, which **whp** has a Hamilton cycle, by the result of [4], even after removing parallel arcs. This is why we take $d = 3$ and appeal to the proof of the result in [2], that a random 3-in,3-out digraph is Hamiltonian **whp**. The proof there will survive the deletion of parallel arcs.

As a final remark, we did not really make arguments about Hamiltonicity only about constructing a random subgraph which is distributed as $D_{d-out,\kappa}$. Clearly, other monotone graph properties can be treated in this manner.

Finally we mention two natural related problems: Suppose we fix K_0 at the threshold value $\frac{1}{2} + o(1)$. What is the least value of $K_1 = K_1(n)$ for which $G_{n,p,\kappa}$ contains a multi-coloured Hamilton cycle **whp**? Similarly, if we fix $K_1 = 1$, what is the least value of $K_0 = K_0(n)$ for which $G_{n,p,\kappa}$ contains a multi-coloured Hamilton cycle **whp**? It is prudent to observe that we must take $K_0 \geq 1 + o(1)$ so that **whp** each colour occurs at least once.

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This research was sponsored in part by National Science Foundation (NSF) grant no. CCR-0122581.