A geometric preferential attachment model of networks

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Abstract

We study a random graph G_n that combines certain aspects of geometric random graphs and preferential attachment graphs. The vertices of G_n are n sequentially generated points $x_1, x_2, \ldots, x_n$ chosen uniformly at random from the unit sphere in $\boldsymbol{R}^3$. After generating x_t, we randomly connect it to m points from those points in $x_1, x_2, \ldots, x_{t-1}$ which are within distance r. Neighbours are chosen with probability proportional to their current degree. We show that if m is sufficiently large and if $r \geq \log n / n^{1/2-\beta}$ for some constant β then **whp** at time n the number of vertices of degree k follows a power law with exponent 3. Unlike the preferential attachment graph, this geometric preferential attachment graph has small separators, similar to experimental observations of [7]. We further show that if $m \geq K \log n$, K sufficiently large, then G_n is connected and has diameter $O(m/r)$ **whp**.

1 Introduction

Recently there has been much interest in understanding the properties of real-world large-scale networks such as the structure of the Internet and the World Wide Web. For a general introduction to this topic, see Bollobás and Riordan [8], Hayes [21], Watts [32], or Aiello, Chung and Lu [2]. One approach is to model these networks by random graphs. Experimental studies by Albert, Barabási, and Jeong [3], Broder et al [12], and Faloutsos, Faloutsos, and Faloutsos [20] have demonstrated that in the World Wide Web/Internet the proportion of vertices of a given degree follows an approximate inverse power law i.e. the proportion of vertices of degree k is approximately $Ck^{-\alpha}$ for some constants C, α. The classical models of random graphs introduced by Erdős and Renyi [18] do not have power law degree sequences, so they are not suitable for modeling these networks. This has driven the development of various alternative models for random graphs.

One approach is to generate graphs with a prescribed degree sequence (or prescribed expected degree sequence). This is proposed as a model for the web graph by Aiello, Chung,

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and Lu in [1]. Mihail and Papadimitriou also use this model [27] in their study of large eigenvalues, as do Chung, Lu, and Vu in [14].

An alternative approach, which we will follow in this paper, is to sample graphs via some generative procedure which yields a power law distribution. There is a long history of such models, outlined in the survey by Mitzenmacher [29]. We will use an extension of the preferential attachment model to generate our random graph. The preferential attachment model has been the subject of recently revived interest. It dates back to Yule [33] and Simon [31]. It was proposed as a random graph model for the web by Barabási and Albert [4], and their description was elaborated by Bollobás and Riordan [9] who showed that at time n, **whp** the diameter of a graph constructed in this way is asymptotic to $\frac{\log \log n}{\log n}$. Subsequently, Bollobás, Riordan, Spencer and Tusnády [11] proved that the degree sequence of such graphs does follow a power law distribution.

The random graph defined in the previous paragraph has good expansion properties. For example, Mihail, Papadimitriou and Saberi [28] showed that **whp** the preferential attachment model has conductance bounded below by a constant. This is in contrast to what has sometimes been found experimentally, for example by Blandford, Blelloch and Kash [7]. Their results seem to suggest the existence of smaller separators than implied by random graphs with the same average degree. The aim of this paper is to describe a random graph model which has *both* a power-law degree distribution and which has small separators.

We study here the following process which generates a sequence of graphs $G_t, t = 1, 2, \ldots, n$. The graph $G_t = (V_t, E_t)$ has t vertices and e_t edges. Here V_t is a subset of S, the surface of the sphere in $\boldsymbol{R}^3$ of radius $\frac{1}{2\sqrt{\pi}}$ (so that $area(S) = 1$).

For $u \in S$ and $r > 0$ we let $B_r(u)$ denote the spherical cap of radius r around u in S. More precisely, $B_r(u) = \{x \in S : ||x - u|| \leq r\}$.

1.1 The random process

- **Time step 1:** To initialize the process, we start with G_1 containing a single vertex x_1 chosen at random in S. The edge (multi)set consists of m loops at x_1.

- **Time step $t + 1$:** We choose vertex x_{t+1} uniformly at random in S and add it to G_t. If $V_t \cap B_r(x_{t+1})$ is nonempty, we add m random edges (x_{t+1}, y_i), $i = 1, 2, \ldots, m$ incident with x_{t+1}. Here, each y_i is chosen from $V_t \cap B_r(x_{t+1})$ and for $x \in V_t \cap B_r(x_{t+1})$,

$$\mathbf{Pr}(y_i = x) = \frac{\deg_t(x)}{D_t(B_r(x_{t+1}))},$$

  where $\deg_t(x)$ denotes the degree of vertex x in G_t and $V_t(U) = V_t \cap U$ and $D_t(U) = \sum_{x \in V_t(U)} \deg_t(x)$.

  If $V_t \cap B_r(x_{t+1})$ is empty then we add m loops at x_{t+1}.

Let $d_k(t)$ denote the number of vertices of degree k at time t.

We will prove the following:

Theorem 1

(a) *If $0 < \beta < 1/2$ is constant and $r \geq n^{\beta-1/2}\log n$ and m is a sufficiently large constant then there exists a constant $c > 0$ such that* **whp**

$$d_k(n) = \frac{cn}{k(k+1)(k+2)} + O(n^{1-\gamma}),$$ [1]

for some $0 < \gamma < 1$.

(b) *If $r = o(1)$ then* **whp** *V_n can be partitioned into $T, \bar{T}$ such that $|T|, |\bar{T}| \sim n/2$, and there are at most $4\sqrt{\pi} rnm$ edges between T and $\bar{T}$.*

(c) *If $r \geq n^{-1/2}\log n$ and $m \geq K\log n$ and K is sufficiently large then* **whp** *G_n is connected.*

(d) *If $r \geq n^{-1/2}\log n$ and $m \geq K\log n$ and K is sufficiently large then* **whp** *G_n has diameter $O(\log n/r)$.*

We note that geometric models of trees with power laws have been considered in [19], [5] and [6].

1.2 Some definitions

There exists some constant c_0 such that for any $u \in S$, we have

$$A_r = Area(B_r(u)) = c_0 n^{2\beta-1}(\log n)^2.$$

Given $u \in S$, we define

$$V_t(u) = V_t(B_r(u))$$

and

$$D_t(u) = D_t(B_r(u)).$$

Given $v \in V_t$, we have

$$\deg_t(v) = m + \deg_t^-(v), \tag{1}$$

where $\deg_t^-(v)$ is the number of edges of G_t that are incident to v and were added by vertices that chose v as a neighbor.

Given $U \subseteq S$, let $D_t^-(U) = \sum_{v \in V_t(U)} \deg_t^-(v)$. We also define $D_t^-(u) = D_t^-(B_r(u))$. Notice that $D_t(U) = m|V_t(U)| + D_t^-(U)$.

We write $\overline{d}_k(t)$ to denote the expectation of $d_k(t)$. We also localize these notions: given $U \subseteq S$ and $u \in S$ we define $d_k(t, U)$ to be the number of vertices of degree k at time t in U and $d_k(t, u) = d_k(t, B_r(u))$.

[1] Asymptotics are taken as $n \to \infty$

2 Small separators

Theorem 1 part (b) is the easiest part to prove. We use the geometry of the instance to obtain a sparse cut. Consider partitioning the vertices using a great circle of S. This will divide V into sets T and $\bar{T}$ which each contain about $n/2$ vertices. More precisely, we have

$$\mathbf{Pr}\left[|T| < (1-\epsilon)n/2\right] = \mathbf{Pr}\left[|\bar{T}| < (1-\epsilon)n/2\right] \le e^{-\epsilon^2 n/6}.$$

Since edges only appear between vertices within distance r, only vertices appearing in the strip within distance r of the great circle can appear in the cut. Since $r = o(1)$, this strip has area less than $3r\sqrt{\pi}$, so, letting U denote the vertices appearing in this strip, we have

$$\mathbf{Pr}\left[|U| \ge 4\sqrt{\pi}rn\right] \le e^{-\sqrt{\pi}rn/9}.$$

vertices. Even if every one of the vertices chooses its m neighbors on the opposite side of the cut, this will yield at most $4\sqrt{\pi}rnm$ edges **whp**. So the graph has a cut with $\frac{e(T,\bar{T})}{|T||\bar{T}|} \le \frac{17\sqrt{\pi}rm}{n}$ with probability at least $1 - e^{-\Omega(rn)}$.

3 Proving a power law

3.1 Establishing a recurrence for $\overline{d}_k(t)$

Our approach to proving Theorem 1 part (a) is to find a recurrence for $\overline{d}_k(t)$.

We define $\overline{d}_{m-1}(t) = 0$ for all integers t with $t > 0$. Let $\eta_1(t)$ denote the probability that $V_t \cap B_r(x_{t+1}) = \emptyset$ so $\eta_1(t) = (1 - A_r)^t$. Let $\eta_2(t)$ denote the probability that a parallel edge is created. Thus

$$\eta_2(t) = O\left(\sum_{i=m}^{k} d_i(t, x_{t+1}) i^2 / D_t(x_{t+1})^2\right) = O(k/D_t(x_{t+1})).$$

Then for $k \ge m$,

$$\begin{aligned}\mathbf{E}\left[d_k(t+1) \mid G_t, x_{t+1}\right] = d_k(t) &+ m d_{k-1}(t, x_{t+1}) \frac{k-1}{D_t(x_{t+1})} \\ &- m d_k(t, x_{t+1}) \frac{k}{D_t(x_{t+1})} + 1_{k=m} + O(\eta_1(t) + \eta_2(t)). \quad (2)\end{aligned}$$

Let

$$\alpha = \frac{1}{400} \qquad \text{and} \qquad \gamma = \frac{\alpha^2}{2(1-\alpha^2)},$$

and let $\mathcal{A}_t$ be the event

$$\{|D_t(x_{t+1}) - 2mA_r t| < A_r m t^{1-\gamma}\}.$$

Then, because $\mathbf{E}[d_k(t,x_{t+1})] \leq k^{-1}\mathbf{E}[m|V_t(B_{2r}(x_{t+1}))|] \leq k^{-1}m(4A_rt)$ and $d_k(t,x_{t+1}) \leq k^{-1}D_t(x_{t+1})$, we have

$$\begin{aligned}
&\mathbf{E}\left[\frac{d_k(t,x_{t+1})}{D_t(x_{t+1})}\right] \\
&= \mathbf{E}\left[\frac{d_k(t,x_{t+1})}{D_t(x_{t+1})} \;\middle|\; \mathcal{A}_t\right]\mathbf{Pr}\left[\mathcal{A}_t\right] + \mathbf{E}\left[\frac{d_k(t,x_{t+1})}{D_t(x_{t+1})} \;\middle|\; \neg\mathcal{A}_t\right]\mathbf{Pr}\left[\neg\mathcal{A}_t\right] \\
&= \frac{\mathbf{E}\left[d_k(t,x_{t+1}) \mid \mathcal{A}_t\right]}{2mA_rt}\mathbf{Pr}\left[\mathcal{A}_t\right] + O\left(\frac{t^{-\gamma}}{k}\right) + O\left(\frac{\mathbf{Pr}\left[\neg\mathcal{A}_t\right]}{k}\right) \\
&= \frac{\mathbf{E}\left[d_k(t,x_{t+1})\right]}{2mA_rt} + O\left(\frac{t^{-\gamma}}{k}\right) \\
&\qquad + \left(O\left(\frac{1}{k}\right) - \frac{\mathbf{E}\left[d_k(t,x_{t+1}) \mid \neg\mathcal{A}_t\right]}{2mA_rt}\right)\mathbf{Pr}\left[\neg\mathcal{A}_t\right] \\
&= \frac{\mathbf{E}\left[d_k(t,x_{t+1})\right]}{2mA_rt} + O\left(\frac{t^{-\gamma}}{k}\right) + O\left(\frac{1}{k} + \frac{1}{A_r}\right)\mathbf{Pr}\left[\neg\mathcal{A}_t\right].
\end{aligned}$$

In Lemmas 1 and 3 below we prove that $\mathbf{E}\left[d_k(t,x_{t+1})\right] = A_r\overline{d}_k(t)$ and that if $t \geq n^{1-\alpha}$ then $\mathbf{Pr}\left[\neg\mathcal{A}_t\right] = O\left(n^{-2}\right)$. Therefore if $t \geq n^{1-\alpha}$ then

$$\mathbf{E}\left[\frac{d_k(t,x_{t+1})}{D_t(x_{t+1})}\right] = \frac{\overline{d}_k(t)}{2mt} + O\left(\frac{t^{-\gamma}}{k}\right). \tag{3}$$

In a similar way

$$\mathbf{E}\left[\frac{d_{k-1}(t,x_{t+1})}{D_t(x_{t+1})}\right] = \frac{\overline{d}_{k-1}(t)}{2mt} + O\left(\frac{t^{-\gamma}}{k-1}\right). \tag{4}$$

Now note that

$$\eta_2(t) \leq \mathbf{Pr}(\neg\mathcal{A}_t) + O(k/tA_r).$$

Taking expectations on both sides of Eq. (2) and using Eq. (3) and Eq. (4), we see that if $k \leq n^{1-\alpha} \leq t$ then

$$\overline{d}_k(t+1) = \overline{d}_k(t) + \frac{k-1}{2t}\overline{d}_{k-1}(t) - \frac{k}{2t}\overline{d}_k(t) + 1_{k=m} + O\left(t^{-\gamma}\right) \tag{5}$$

We consider the recurrence given by $f_{m-1} = 0$ and for $k \geq m$,

$$f_k = 1_{k=m} + \frac{k-1}{2}f_{k-1} - \frac{k}{2}f_k$$

which has solution

$$\begin{aligned}
f_k &= f_m\prod_{i=m+1}^{k}\frac{i-1}{i+2} \\
&= f_m\frac{m(m+1)(m+2)}{k(k+1)(k+2)}.
\end{aligned}$$

Let $t_0 = n^{1-\alpha}$. We finish the proof of Theorem 1(a) by showing that there exists a constant $M > 0$ such that

$$|\overline{d}_k(t) - f_k t| \le M(t_0 + t^{1-\gamma}) \tag{6}$$

for all k with $m \le k \le n^{1-\alpha}$ and all $t > 0$. For $k > n^{1-\alpha}$ we use the fact that $\overline{d}_k(t) \le 2mt/k$.

Let $\Theta_k(t) = \overline{d}_k(t) - f_k t$. Then for $m \le k \le n^{\alpha}$ and $t \ge t_0$,

$$\Theta_k(t+1) = \frac{k-1}{2t}\Theta_{k-1}(t) - \frac{k}{2t}\Theta_k(t) + O(t^{-\gamma}). \tag{7}$$

Let L denote the hidden constant in $O(t^{-\gamma})$ of (7). Our inductive hypothesis $\mathcal{H}_t$ is that $|\Theta_k(t)| \le M(t_0 + t^{1-\gamma})$ for every $m \le k \le n^{1-\alpha}$. It is trivially true for $t \le t_0$. So assume that $t \ge t_0$. Then, from (7),

$$\begin{aligned} |\Theta_k(t+1)| &\le M(t_0 + t^{1-\gamma}) + Lt^{-\gamma} \\ &\le M(t_0 + (t+1)^{1-\gamma}). \end{aligned}$$

This verifies $\mathcal{H}_{t+1}$ and completes the proof by induction.

3.2 Expected Value of $d_k(t,u)$

Lemma 1 *Let $u \in S$ and let k and t be positive integers. Then* $\mathbf{E}[d_k(t,u)] = A_r \overline{d}_k(t)$

Proof By symmetry for any $w \in S$, $d_k(t,u)) \sim d_k(t,w)$. Then

$$\begin{aligned} \mathbf{E}[d_k(t,u)] &= \int_S \mathbf{E}[d_k(t,u)]\,dw = \int_S \mathbf{E}[d_k(t,w)]\,dw \\ &= \mathbf{E}\left[\int_S d_k(t,w)dw\right] = \mathbf{E}\left[\int_S \sum_{v\in V_t} 1_{\deg v=k} 1_{v\in B_r(w)} dw\right] \\ &= \mathbf{E}\left[\sum_{v\in V_t} 1_{\deg v=k} \int_S 1_{w\in B_r(v)} dw\right] = \mathbf{E}\left[\sum_{v\in V_t} 1_{\deg v=k} A_r\right] \\ &= A_r \mathbf{E}[d_k(t)] \end{aligned}$$

□

Lemma 2 *Let $u \in S$ and $t > 0$ then* $\mathbf{E}[D_t(u)] = 2A_r mt$

Proof

$$\mathbf{E}[D_t(u)] = \sum_{k>0} \mathbf{E}[d_k(t,u)] = A_r \sum_{k>0} \mathbf{E}[d_k(t)] = A_r \mathbf{E}\left[\sum_{k>0} d_k(t)\right] = 2A_r mt$$

□

3.3 Concentration of $D_t(u)$

In this section we prove

Lemma 3 *Let $\alpha = 1/400$ and $\gamma = \frac{\alpha^2}{2(1-\alpha^2)}$ and $n_0 = n^{1-2\alpha}$. If $t > n^{1-\alpha}$ and $u \in S$ then*

$$\mathbf{Pr}\left[|D_t(u) - \mathbf{E}\left[D_t(u)\right]| \geq A_r m t^{1-\gamma}\right] = O\left(n^{-2}\right).$$

Proof We think of every edge added as two directed edges. Then choosing a vertex by preferential attachment is equivalent to choosing one of these directed edges uniformly, and taking the vertex pointed to by this edge as the chosen vertex. So the ith step of the process is defined by a tuple of random variables $T = (X, Y_1, \ldots, Y_m) \in S \times E_i^m$ where X is the location of the new vertex, a randomly chosen point in S, and Y_j is an edge chosen u.a.r. among the edges directed into $B_r(X)$ in G_{i-1}. The process G_t is then defined by a sequence $\langle T_1, \ldots, T_t \rangle$, where each $T_i \in S \times E_i^m$.

Let s be a sequence $s = \langle s_1, \ldots, s_t \rangle$ where $s_i = (x_i, y_{(i-1)m+1}, \ldots, y_{im})$ with $x_i \in S$ and $y_j \in E_{\lceil t/j \rceil}$. We say s is *acceptable* if for every j, y_j is an edge entering $B_r(x_{\lceil t/j \rceil})$. Notice that non-acceptable sequences have probability 0 of being observed.

In what follows we condition on the event

$$\mathcal{E} = \{\text{for all } s \text{ with } n_0 < s \leq n \text{ we have } D_s(u) > (1+\alpha)A_r m s\},$$

where $\alpha > 0$ is an appropriate constant that will be chosen later.

Fix $t > 0$. Fix an acceptable sequence $s = \langle s_1, \ldots, s_t \rangle$, and let $A_k(s) = \{z \in S \times E_k^m : \langle s_1, \ldots, s_{k-1}, z \rangle \text{ is acceptable}\}$. For any k with $1 \leq k \leq t$ and any $z \in A_k(s)$ let

$$g_k(z) = \mathbf{E}\left[D_t(u) \mid T_1 = s_1, \ldots, T_{k-1} = s_{k-1}, T_k = z, \mathcal{E}\right],$$

let $r_k(s) = \sup\{|g_k(z_1) - g_k(z_2)| : z_1, z_2 \in A_k(s)\}$ and let $R^2(s) = \sum_{k=1}^{t} r_k(s)^2$. Finally define $\hat{r}^2 = \sup_s R^2(s)$, where the supremum is taken over all acceptable sequences.

By Thm 3.7 of [26] we know that for all $\lambda > 0$,

$$\mathbf{Pr}\left[|D_t(u) - \mathbf{E}\left[D_t(u)\right]| \geq \lambda\right] < 2e^{-2\lambda^2/\hat{r}^2} + \mathbf{Pr}\left[\neg\mathcal{E}\right]. \tag{8}$$

Now, fix k, with $1 \leq k \leq t$. Our goal now is to bound $r_k(s)$ for any acceptable sequence s.

Fix $z, z' \in A_k(s)$. We define $\Omega(G_t, G'_t)$, the following coupling between $G_t = G_t(s_1, \ldots, s_{k-1}, z)$ and $G'_t = G_t(s_1, \ldots, s_{k-1}, z')$

- Step k: Start with the graph $G_k(s_1, \ldots, s_{k-1}, z)$ and $G'_k(s_1, \ldots, s_{k-1}, z')$ respectively.

- Step τ ($\tau > k$): Choose the same point $x_\tau \in S$ in both processes. Let E_τ (resp. E'_τ) be the edges pointing to the vertices in $B_r(x_\tau)$ in $G_{\tau-1}$ (resp. $G'_{\tau-1}$). Let $C_\tau = E_\tau \cap E'_\tau$, $R_\tau = E_\tau \setminus E'_\tau$, and $L_\tau = E'_\tau \setminus E_\tau$

  Let $D_\tau = |E_\tau|$ and $D'_\tau = |E'_\tau|$. Without loss of generality assume that $D_\tau \leq D'_\tau$. Note that $D_\tau = 0$ iff $V_{\tau-1} \cap B_r(x_\tau) = \emptyset$, in which case $D'_\tau = 0$ as well. Assume for now that $D_\tau > 0$. Let $p = 1/D_\tau$ and let $p' = 1/D'_\tau$. Construct G_τ choosing m edges u.a.r. $e_1^\tau, \ldots, e_m^\tau$ in E_τ, and joining x_τ to the end point of them. For each of the m edges $e_i = e_i^\tau$, we define $\hat{e}_i = \hat{e}_i^\tau$ by

- If $e_i \in C_\tau$ then, with probability p'/p, $\hat{e}_i = e_i$. With probability $1 - p'/p$, $\hat{e}_i$ is chosen from L_τ u.a.r.
- If $e_i \in R_\tau$, $\hat{e}_i \in L_\tau$ is chosen u.a.r.

Notice that for every $i = 1, \ldots, m$ and every $e \in E'_\tau$, $\mathbf{Pr}[\hat{e}_i = e] = p'$. To finish, in G'_τ join x_τ to the m vertices pointed to by the edges $\hat{e}_i$.

Lemma 4 *Let $\Delta_\tau = \Delta_\tau(k, s, z, z', u) = |E_{G_\tau}(B_r(u)) \triangle E_{G'_\tau}(B_r(u))|$, the discrepancy between the edge-sets incident to $V_\tau(u)$ in the two coupled graphs. Then $|g_k(z) - g_k(z')| \leq \mathbf{E}[\Delta_t 1_{\mathcal{E}}] / \mathbf{Pr}[\mathcal{E}]$.*

Proof

$$
\begin{aligned}
|g_k(z) - g_k(z')| &= |\mathbf{E}_{G_t}[D_t(u) \mid \mathcal{E}] - \mathbf{E}_{G'_t}[D_t(u) \mid \mathcal{E}]| \\
&\leq \mathbf{E}_{\Omega(G_t, G'_t)}[D'_t(u) - D_t(u) \mid \mathcal{E}] \\
&\leq \mathbf{E}_{\Omega(G_t, G'_t)}[\Delta_t \mid \mathcal{E}] \\
&= \mathbf{E}_{\Omega(G_t, G'_t)}[\Delta_t 1_{\mathcal{E}}] / \mathbf{Pr}[\mathcal{E}].
\end{aligned}
$$

□

Recall that $A_r = Area(B_r(u)) = c_0 n^{2\beta - 1} (\log n)^2$ and $n_0 = n^{1-2\alpha}$ and we have fixed k to be an integer with $1 \leq k \leq t$.

Lemma 5 *Let $k_0 = \max\{k, n_0\}$ and ϵ_1 be a small positive constant, then*

$$
\mathbf{E}[\Delta_t 1_{\mathcal{E}}] \leq 16 m A_r n^{c_0 \epsilon_1} \left(\frac{k_0}{k}\right)^{1/(1-2\epsilon_1)} \left(\frac{t}{k_0}\right)^{1/(1+\alpha)}.
$$

Proof Notice that for $\tau > k$, $\Delta_\tau = \Delta_{\tau-1} + Y_\tau$, where Y_τ is the edge discrepancy created at time τ. $Y_\tau \leq 2 \sum_{i=1}^m 1_{e_i^\tau \neq \hat{e}_i^\tau}$ where $1_{e_i^\tau \neq \hat{e}_i^\tau}$ is the indicator variable of the i^{th} edge chosen at time τ not being the same for both processes. For every $i = 1, \ldots, m$

$$
\mathbf{Pr}\left[e_i^\tau \neq \hat{e}_i^\tau \mid G_{\tau-1}, G'_{\tau-1}, x_\tau\right] = 1 - \frac{|C_\tau|}{|E_\tau|}\frac{p'}{p} = 1 - \frac{|C_\tau|}{|E'_\tau|} = \frac{|L_\tau|}{|E'_\tau|} = \frac{|L_\tau|}{\max\{D_\tau, D'_\tau\}}.
$$

Therefore

$$
\mathbf{E}\left[\Delta_\tau \mid G_{\tau-1}, G'_{\tau-1}, x_\tau\right] = \Delta_{\tau-1} + 2m \frac{|L_\tau|}{\max\{D_\tau, D'_\tau\}}.
$$

For each $e \in E(G'_\tau) \setminus E(G_\tau)$, $e \in L_\tau$ implies $x_{\tau+1}$ is in the ball of radius r centered at the end point of e. Thus $\mathbf{Pr}\left[e \in L_\tau \mid G_{\tau-1}, G'_{\tau-1}\right] \leq A_r$ and therefore

$$
\begin{aligned}
\mathbf{E}\left[|L_\tau| \mid G_{\tau-1}, G'_{\tau-1}\right] &\leq A_r \Delta_{\tau-1}. && (9) \\
\mathbf{E}[|L_\tau|] &\leq \mathbf{E}[A_r \Delta_{\tau-1}/2], && (10)
\end{aligned}
$$

where the factor of $1/2$ in the unconditional expectation comes from symmetry.

Let $k_1 = \epsilon_1 n^{2\beta-1} / \log n$.

Case 1 $\tau \geq k_0$.
If $\mathcal{E}$ holds then $D_\tau > (1+\alpha)mA_r\tau$ and so, by using (10), we have

$$\begin{aligned} \mathbf{E}\left[\Delta_{\tau+1}1_{\mathcal{E}}\right] &\leq \mathbf{E}\left[\Delta_\tau 1_{\mathcal{E}}\right] + 2m\frac{\mathbf{E}\left[|L_\tau|1_{\mathcal{E}}\right]}{(1+\alpha)mA_r\tau} \\ &\leq \mathbf{E}\left[\Delta_\tau 1_{\mathcal{E}}\right]\left(1+\frac{1}{(1+\alpha)\tau}\right). \end{aligned} \tag{11}$$

Case 2 $k \leq \tau < k_1$.
If $|L_\tau| > 0$ then $\max\{D_\tau, D'_\tau\} \geq 1$. So we have

$$\mathbf{E}\left[\Delta_\tau 1_{\mathcal{E}}\right] \leq \mathbf{E}\left[\Delta_{\tau-1}1_{\mathcal{E}}\right](1+mA_r).$$

Case 3 $\max\{k, k_1\} \leq \tau < k_0$.
We write

$$\begin{aligned} &\mathbf{E}\left[\frac{|L_\tau|1_{\mathcal{E}}}{\max\{D_\tau, D'_\tau\}}\right] \leq \\ &\mathbf{E}\left[\frac{|L_\tau|1_{\mathcal{E}}}{m(1-\epsilon_1)A_r\tau}\;\middle|\; |V_\tau(x_{\tau+1})| \geq (1-\epsilon_1)A_r\tau\right]\mathbf{Pr}\left[|V_\tau(x_{\tau+1})| \geq (1-\epsilon_1)A_r\tau\right] \\ &+\mathbf{E}\left[|L_\tau|1_{\mathcal{E}}\;\middle|\; |V_\tau(x_{\tau+1})| < (1-\epsilon_1)A_r\tau\right]\mathbf{Pr}\left[|V_\tau(x_{\tau+1})| < (1-\epsilon_1)A_r\tau\right] \\ &\leq \frac{A_r\mathbf{E}\left[\Delta_{\tau-1}1_{\mathcal{E}}\right]/2}{m(1-\epsilon_1)A_r\tau} + A_r\mathbf{E}\left[\Delta_{\tau-1}1_{\mathcal{E}}\right]n^{-\Omega(\epsilon_1^2\epsilon)} \end{aligned}$$

(After using (9) and the Chernoff bounds)

$$\leq \frac{\mathbf{E}\left[\Delta_{\tau-1}1_{\mathcal{E}}\right]}{2m(1-2\epsilon_1)\tau}.$$

Putting all 3 cases together, we have

$$\begin{aligned} \mathbf{E}\left[\Delta_t\right] &\leq \mathbf{E}\left[\Delta_k\right]e^{mA_rk_1}\prod_{\tau=k+1}^{k_0}\left(1+\frac{1}{(1-2\epsilon_1)\tau}\right)\prod_{\tau=k_0+1}^{t}\left(1+\frac{1}{(1+\alpha)\tau}\right) \\ &\leq \mathbf{E}\left[\Delta_k\right]n^{c_0\epsilon_1}\left(\frac{k_0}{k}\right)^{1/(1-2\epsilon_1)}\left(\frac{t}{k_0}\right)^{\frac{1}{1+\alpha}}. \end{aligned}$$

Now, $\Delta_k = \Delta_k(k, s, z, z') = |E_{G_k}(B_r(u)) \triangle E_{G'_k}(B_r(u))| \leq 2m$, because the graphs G_k and G'_k differ at most in the last m edges.

Also, if $|E_{G_k}(B_r(u)) \triangle E_{G'_k}(B_r(u))| > 0$ then $u \in B_{2r}(x_k) \cup B_{2r}(x'_k)$. So

$$\mathbf{E}\left[\Delta_k 1_{\mathcal{E}}\right] \leq 2m\mathbf{Pr}\left[u \in B_{2r}(x_k) \cup B_{2r}(x'_k)\right] \leq 16mA_r.$$

□

By applying Lemma 5, we have that for any acceptable sequence we have

$$
\begin{aligned}
&R^2(s) = \sum_{k=1}^{t} r_k(s)^2 \le \\
&(16A_r m)^2 n^{2c_0\epsilon_1} \left(\sum_{k=1}^{n_0} \left(\frac{n_0}{k}\right)^{\frac{2}{1-2\epsilon_1}} \left(\frac{t}{n_0}\right)^{\frac{2}{1+\alpha}} + \sum_{k=n_0+1}^{t} \left(\frac{t}{k}\right)^{\frac{2}{1+\alpha}} \right) \mathbf{Pr}\left[\mathcal{E}\right]^{-2} \\
&\le \left(256 A_r^2 m^2 t^{\frac{2}{1+\alpha}} n^{2c_0\epsilon_1}\right) \left(n_0^{\frac{2}{1-2\epsilon_1} - \frac{2}{1+\alpha}} \sum_{k=1}^{\infty} k^{-2/(1-2\epsilon_1)} + \sum_{k=1}^{\infty} k^{-\frac{2}{1+\alpha}} \right) \mathbf{Pr}\left[\mathcal{E}\right]^{-2} \\
&= O\left(A_r^2 m^2 t^{\frac{2}{1+\alpha}} n^{2c_0\epsilon_1} n_0^{\frac{2}{1-2\epsilon_1} - \frac{2}{1+\alpha}} \right),
\end{aligned}
$$

where the final equality relies on the fact that $\mathbf{Pr}\left[\mathcal{E}\right] = 1 - o(1)$, which is proved below.

Therefore, by using Eq. (8), we have

$$
\mathbf{Pr}\left[|D_t(u) - \mathbf{E}\left[D_t(u)\right]| \ge A_r m t^{\frac{1}{1+\alpha}} n^{c_0\epsilon_1/2} n_0^{\frac{1}{1-2\epsilon_1} - \frac{1}{1+\alpha}} \log n \right] \le e^{-\Omega(\log n^2)} + \mathbf{Pr}\left[\neg\mathcal{E}\right]. \quad (12)
$$

Now we concentrate in bounding $\mathbf{Pr}\left[\neg\mathcal{E}\right]$.

Lemma 6 *Let $\alpha = 1/400$. There is $c > 0$ such that $\mathbf{Pr}\left[\neg\mathcal{E}\right] = O\left(e^{-cn_0 A_r}\right)$.*

Proof Let W be a set of points in S such that every point in S is at distance smaller than $r/2$ from W. We can construct W such that $|W| = O(1/r^2)$.

Let $w \in W$. We are going to prove that new vertices that fall in $B_{r/2}(w)$ are likely to choose vertices in $B_{r/2}(w)$ with positive probability and therefore $D_\tau^-(B_{r/2}(w))$ is likely to be a positive proportion of $mr\tau$.

Suppose we are in step $\tau + 1$ of the process. Let $x_{\tau+1}$ be the chosen point in S, and let $y_1, \ldots, y_m = y_1^\tau, \ldots, y_m^\tau$ be the m vertices chosen by $x_{\tau+1}$. Then

$$
\mathbf{Pr}\left[y_i \in B_{r/2}(w) \mid G_{\tau-1}, x_{\tau+1}\right] = \frac{D_\tau(B_{r/2}(w) \cap B_r(x_{\tau+1}))}{D_\tau(B_r(x_{\tau+1}))}. \quad (13)
$$

The key of the proof is the following bound on $D_\tau(B_R(w))$, for any $R > 0$. It follows from (1) and the fact that any neighbor of a vertex in $B_R(w)$ lies in $B_{R+r}(w)$.

$$
\begin{aligned}
m|V_\tau(B_R(w))| &\le D_\tau(B_R(w)) \\
&= m|V_\tau(B_R(w))| + D_\tau^-(B_R(w)) \le 2m|V_\tau(B_{R+r}(w))|. \quad (14)
\end{aligned}
$$

Notice that if $x_{\tau+1} \in B_{r/2}(w)$ then $B_{r/2}(w) \subseteq B_r(w) \cap B_r(x_{\tau+1})$ and $B_r(x_{\tau+1}) \subseteq B_{3r/2}(w)$. So using Eq. (13) and Eq. (14) we have

$$
\begin{aligned}
\mathbf{Pr}\left[y_i \in B_{r/2}(w) \mid G_{\tau-1},\, x_{\tau+1} \in B_{r/2}(w)\right] &\ge \frac{D_\tau(B_{r/2}(w))}{D_\tau(B_{3r/2}(w))} \\
&\ge \frac{|V_\tau(B_{r/2}(w))|}{2|V_\tau(B_{5r/2}(w))|}. \quad (15)
\end{aligned}
$$

Let $n_1 = n_0/100$ and let $\mathcal{E}_1$ be the event "for all s with $n_1 < s \leq n$ we have $|V_s(B_{r/2}(w))| \geq A_r s/6$ and $|V_s(B_{5r/2}(w))| \leq 8A_r s$". Then

$$\mathbf{Pr}\left[y_i \in B_r(w) \mid \mathcal{E}_1,\, x_{\tau+1} \in B_{r/2}(w)\right] \geq \frac{1}{96},$$

and therefore,

$$\mathbf{Pr}\left[y_i \in B_r(w) \mid \mathcal{E}_1\right] \geq \frac{A_r}{384},$$

Writing $D^-_\tau(B_{r/2}(w)) = \sum_{k=1}^{\tau} \sum_{i=1}^{m} 1_{y_i^k \in B_{r/2}(w)}$ we get that, conditioned on $\mathcal{E}_1$, $D^-_\tau(B_{r/2}(w))$ stochastically dominates $X \sim \mathrm{Bi}\left(m(\tau - n_1), \frac{A_r}{384}\right)$. So, if $\tau > n_0 = 100n_1$ then

$$\begin{aligned} \mathbf{Pr}\left[D^-_\tau(B_{r/2}(w)) < \frac{m\tau A_r}{400}\right] &\leq \mathbf{Pr}\left[X < \frac{m\tau A_r}{400}\right] + \mathbf{Pr}\left[\neg\mathcal{E}_1\right] \\ &\leq e^{-cm\tau A_r} + \mathbf{Pr}\left[\neg\mathcal{E}_1\right]. \end{aligned} \tag{16}$$

for some $c > 0$.

Now we are going to bound $\mathbf{Pr}\left[\neg\mathcal{E}_1\right]$. Let s be such that $n_1 < s \leq n$. Let $\tau \leq s$. $\mathbf{Pr}\left[x_\tau \in B_{r/2}(w)\right] = A_r/4$ and $\mathbf{Pr}\left[x_\tau \in B_{5r/2}(w)\right] = 25A_r/4$. Then $V_s(B_{r/2}(w)) \sim \mathrm{Bi}(s, A_r/4)$ and $V_s(B_{5r/2}(w)) \sim \mathrm{Bi}(s, 25A_r/4)$. By Chernoff's bound

$$\begin{aligned} \mathbf{Pr}\left[V_s(B_{r/2}(w)) \leq A_r s/6\right] &\leq e^{-A_r s/72} \\ \mathbf{Pr}\left[V_s(B_{5r/2}(w)) \geq 8A_r s\right] &\leq e^{-A_r s/6} \end{aligned} \tag{17}$$

So,

$$\mathbf{Pr}\left[\neg\mathcal{E}_1\right] \leq \sum_{\tau=n_1}^{n} \left(e^{-A_r s/72} + e^{-A_r s/6}\right) \leq n e^{-A_r n_1/72} = O\left(e^{-A_r n_1/100}\right)$$

Now, by using (16) we see that if $\tau > n_0$ then

$$\mathbf{Pr}\left[D^-_\tau(B_{r/2}(w)) < \frac{m\tau A_r}{400}\right] \leq e^{-c_1 n_0 A_r}, \tag{18}$$

for some $c_1 > 0$

To extend this to *all* points in S, note that for any $u \in S$, there is a $w \in W$ such that $\|u - w\| \leq r/2$. Therefore $B_{r/2}(w) \subset B_r(u)$ and for any τ, $D^-_\tau(u) \geq D^-_\tau(B_{r/2}(w))$. So

$$\begin{aligned} \mathbf{Pr}\left[\neg\mathcal{E}\right] &= \mathbf{Pr}\left[\exists\tau,\, n_0 \leq \tau \leq t,\, \exists u \in S,\, D^-_\tau(u) \leq mr\tau/400\right] \\ &\leq \mathbf{Pr}\left[\exists\tau,\, n_0 \leq \tau \leq t,\, \exists w \in W,\, D^-_\tau(B_{r/2}(w)) \leq mr\tau/400\right] \\ &\leq \sum_{\tau=n_0}^{t} \sum_{w \in W} \mathbf{Pr}\left[D^-_\tau(B_{r/2}(w)) \leq mr\tau/400\right] \\ &= O\left(nr^{-2} e^{-c_1 n_0 A_r}\right) = O\left(e^{-c_2 n_0 A_r}\right), \end{aligned}$$

for some $c_2 > 0$. □

Returning to (12) and taking ϵ_1 sufficiently small, we see that there is $c > 0$ such that

$$\mathbf{Pr}\left[|D_t(u) - \mathbf{E}\left[D_t(u)\right]| \geq A_r m t^{1-\gamma}\right] \leq n^{-2} + O\left(e^{-cn_0 A_r}\right), \tag{19}$$

which completes the proof of Lemma 3.

4 Connectivity

Here we are going to prove that for $r \geq n^{-1/2} \log n$, $m > K \log n$, and K sufficiently large, **whp** G_n is connected and has diameter $O(\log n/r)$. Notice that G_n is a subgraph of the graph $G(n,r)$, the intersection graph of the caps $B_r(x_t)$, $t = 1, 2, \ldots, n$ and therefore it is disconnected for $r = o(n^{-1/2} \log n)$ [30]. We denote the diameter of G by $\mathrm{diam}(G)$, and follow the convention of defining $\mathrm{diam}(G) = \infty$, when G is disconnected. In particular, when we say that a graph has finite diameter this implies it is connected.

Let $T = K_1 \log n / A_r = \Theta(n/\log n)$ where K_1 is sufficiently large, and $K_1 \ll K$.

Lemma 7 *Let $u \in S$ and let $B = B_{r/2}(u)$. Then*

$$\mathbf{Pr}\left[\mathrm{diam}(G_n(B)) \geq 2(K_1 + 1) \log n\right] = O(n^{-3})$$

where $G_n(B)$ is the induced subgraph of G_n in B.

Proof

Given k_0 and N, we consider the following process which generates a sequence of graphs $H_s = (W_s, F_s)$, $s = 1, 2, \ldots, N$. (The meanings of N, k_0 will become apparant soon).

Time step 1

To initialize the process, we start with H_1 consisting of k_0 isolated vertices $y_1, \ldots, y_{k_0}$.

Time step $s \geq 1$: We add vertex y_{s+k_0}. We then add $m/4000$ random edges incident with y_{s+k_0} of the form (y_{s+k_0}, w_i) for $i = 1, 2, \ldots, m/4000$. Here each w_i is chosen uniformly from W_s.

The idea is to couple the construction of G_n with the construction of H_N for $N \sim \mathrm{Bi}(n - T, A_r/4)$ and $k_0 = \mathrm{Bi}(T, A_r/4)$ such that **whp** H_N is a subgraph of G_n with vertex set $V_n(B)$. We are going to show that **whp** $\mathrm{diam}(H_N) \leq 2(K_1 + 1) \log n$, and therefore $\mathrm{diam}(G_n(B)) \leq 2(K_1 + 1) \log n$.

To do the coupling we use two counters, t for the steps in G_n and s for the steps in H_N:

- Given G_n, set $s = 0$. Let $W_0 = V_T(B)$. Notice $k_0 = |W_0| \sim \mathrm{Bi}(T, A_r/4)$ and that $k_0 \leq K_1 \log n$ **whp**.

- For every $t > T$.

  - If $x_t \notin B$, do nothing in H_N.
  - If $x_t \in B$, set $s := s + 1$. Set $y_{s+k_0} = x_t$. As we want H_N to be a subgraph of G_n we must choose the neighbors of y_{s+k_0} among the neighbors of G_n. Let A be the set of vertices chosen by x_t in $V_t(B)$. Notice that $|A|$ stochastically dominates $a_t \sim \mathrm{Bi}\left(m, \frac{D_t(B)}{D_t(x_t)}\right)$. If $\frac{D_t(B)}{D_t(x_t)} \geq \frac{1}{100}$, then a_t stochastically dominates $b_t \sim \mathrm{Bi}(m, \frac{1}{100})$ and so **whp** is at least $m/200$. If $\frac{D_t(B)}{D_t(x_t)} < \frac{1}{100}$ we declare failure, but as we see below this is unlikely to happen — see (20). We can assume that $D_t(B) \leq 3mA_r t$ and $k_t = |V_t(B)| \geq A_r t/5$ and so each vertex of B has probability at least $\frac{m}{3mA_r t} \geq \frac{1}{15k_t}$ of being chosen under preferential attachment. Thus, as insightfully observed by Bollobás and Riordan [10] we can legitimately start the addition of x_t in G_t by choosing $\mathrm{Bi}(m, 1/3000)$ random neighbours uniformly in B. Observe that $\mathrm{Bi}(m, 1/3000) \geq m/4000$ **whp**.

Notice that N, the number of times s is increased, is the number of steps for which $x_t \in B$, and so $N \sim \mathrm{Bi}(n-T, A_r/4)$.

Notice also that, by (14), we have

$$\frac{D_t(B)}{D_t(x_t)} \geq \frac{V_t(B)}{2V_t(B_{2r}(x_t))},$$

and therefore, for $t \geq T$,

$$\mathbf{Pr}\left[\frac{D_t(B)}{D_t(x_t)} \leq \frac{1}{100}\right] \leq \mathbf{Pr}\left[V_t(B) \leq A_r t/6 \text{ or } V_t(B_{2r}(x_t)) \geq 8A_r t\right] \\ \leq 2n^{-K_1/8}, \quad (20)$$

where the final inequality follows from Chernoff's bound (see (17)).

Now we are ready to show that H_N is connected **whp**.

Notice that by Chernoff's bound we get that

$$\mathbf{Pr}\left[\left|k_0 - \frac{K_1}{4}\log n\right| \geq \frac{K_1}{8}\log n\right] \leq 2n^{-K_1/48}$$

and

$$\mathbf{Pr}\left[N \leq \frac{1}{3}(\log n)^2\right] \leq e^{-c(\log n)^2}$$

for some $c > 0$. Therefore, we can assume $\log n \leq k_0 \leq K_1 \log n$ and $N \geq \frac{1}{3}(\log n)^2$.

Let X_s be the number of connected components of H_s. Then

$$X_{s+1} = X_s - Y_s, \qquad X_0 = k_0$$

where Y_s is the number of components (minus one) collapsed into one by y_{s+k_0}. Then

$$\mathbf{Pr}\left[Y_s = 0\right] \leq \sum_{i=1}^{X_s}\left(\frac{c_i}{s+k_0}\right)^{m/4000}$$

where the c_i are the component sizes of H_s. Therefore, if $s < 2K_1 \log n$ then, since $m \geq K\log n$, we have

$$\mathbf{Pr}\left[Y_s = 0 \mid X_s \geq 2\right] \leq 2\left(1 - \frac{1}{s+k_0}\right)^{m/4000} \leq e^{-m/(4000(s+k_0))} \leq 1/10.$$

So X_s is stochastically dominated by the random variable $\max(1, k_0 - Z_s)$ where $Z_s \sim \mathrm{Bi}(s, 9/10)$. We get then

$$\mathbf{Pr}\left[X_{2K_1\log n} > 1\right] \leq \mathbf{Pr}\left[Z_{2K_1\log n} < k_0\right] \leq \mathbf{Pr}\left[Z_{2K_1\log n} < K_1\log n\right] \leq n^{-3}.$$

And therefore

$$\mathbf{Pr}\left[H_{2K_1\log n} \text{ is not connected}\right] \leq n^{-3}.$$

Now, to obtain an upper bound on the diameter, we run the process of construction of H_N by rounds. The first round consists of $2K_1\log n$ steps and in each new round we double the size of the graph, i.e. it consists of as many steps as the total number of steps of all the previous rounds. Notice that we have less than $\log n$ rounds in total. Let $\mathcal{A}$ be the event for all $i > 0$ every vertex created in the $(i+1)^{th}$ round is adjacent to a vertex in $H_{2K_1 \log n+(i-1)\log n}$, the graph at the end of the i^{th} round.

Conditioning in ρA, every vertex in H_N is at distance at most $\log n$ of $H_{2K_1 \log n}$ whose diameter is not greater than $2K_1 \log n$. Thus, the diameter of H_N is smaller than $2(K_1 + 1)\log n$.

Now, we have that if v is created in the $(i+1)^{th}$ round,

$$\mathbf{Pr}\left[v \text{ is not adjacent to } H_{2K_1 \log n+(i-1)\log n}\right] \leq \left(\frac{1}{2}\right)^m.$$

Therefore

$$\mathbf{Pr}\left[\neg\mathcal{A}\right] \leq \left(\frac{1}{2}\right)^m n(\log n) \leq \frac{\log n}{n^{K\log 2-1}}.$$

□

To finish the proof of connectivity and the diameter, let u, v be two vertices of G_n. Let $C_1, C_2, \ldots, C_M$, $M = O(1/r)$ be a sequence of spherical caps of radius $r/4$ such that u is the center of C_1, v is the center of v and such that the centers of C_i, C_{i+1} are distance $\leq r/2$ apart. The intersections of C_i, C_{i+1} have area at least $A_r/40$ and so **whp** each intersection contains a vertex. Using Lemma 7 we deduce that **whp** there is a path from u to v in G_n of size at most $O(\log n/r)$.

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