Introduction Rules	Elimination Rules

$$\frac{A\ \mathit{true} \qquad B\ \mathit{true}}{A \wedge B\ \mathit{true}}\wedge I \qquad\qquad \frac{A \wedge B\ \mathit{true}}{A\ \mathit{true}}\wedge E_L \qquad \frac{A \wedge B\ \mathit{true}}{B\ \mathit{true}}\wedge E_R$$

$$\frac{}{\top\ \mathit{true}}\top I \qquad\qquad \mathit{no}\ \top E\ \mathit{rule}$$

$$\frac{\begin{array}{c}\overline{A\ \mathit{true}}\,u \\ \vdots \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\supset I^u \qquad\qquad \frac{A \supset B\ \mathit{true} \qquad A\ \mathit{true}}{B\ \mathit{true}}\supset E$$

$$\frac{A\ \mathit{true}}{A \vee B\ \mathit{true}}\vee I_L \qquad \frac{B\ \mathit{true}}{A \vee B\ \mathit{true}}\vee I_R \qquad\qquad \frac{A \vee B\ \mathit{true} \quad \begin{array}{c}\overline{A\ \mathit{true}}\,u \\ \vdots \\ C\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{B\ \mathit{true}}\,w \\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\vee E^{u,w}$$

$$\mathit{no}\ \bot I\ \mathit{rule} \qquad\qquad \frac{\bot\ \mathit{true}}{C\ \mathit{true}}\bot E$$

Notational Definitions. We use the following notational definitions.

$\neg A$	$=$	$A \supset \bot$	not A
$A \equiv B$	$=$	$(A \supset B) \wedge (B \supset A)$	A if and only if B

2.8 A Linear Notation for Proofs

The two-dimensional format for rules of inference and deductions is almost universal in the literature on logic. Unfortunately, it is not well-suited for writing actual proofs of complex propositions, because deductions become very unwieldy. Instead with use a linearized format explained below. Furthermore, since logical symbols are not available on a keyboard, we use the following concrete syntax for propositions:

<table>
<tr><td>$A \equiv B$</td><td><code>A <=> B</code></td><td>A if and only if B</td></tr>
<tr><td>$A \supset B$</td><td><code>A => B</code></td><td>A implies B</td></tr>
<tr><td>$A \vee B$</td><td><code>A | B</code></td><td>A or B</td></tr>
<tr><td>$A \wedge B$</td><td><code>A & B</code></td><td>A and B</td></tr>
<tr><td>$\neg A$</td><td><code>~ A</code></td><td>not A</td></tr>
</table>

The operators are listed in order of increasing binding strength, and implication (`=>`), disjunction (`|`), and conjunction (`&`) associate to the right, just like the corresponding notation from earlier in this chapter.

The linear format is mostly straightforward. A proof is written as a sequence of judgments separated by semi-colon '`;`'. Later judgements must follow from earlier ones by simple applications of rules of inference. Since it can easily be verified that this is the case, explicit justifications of inferences are omitted. Since the only judgment we are interested in at the moment is the truth of a proposition, the judgment "A *true*" is abbreviated simply as "A".

The only additional notation we need is for hypothetical proofs. A hypothetical proof

$$\begin{array}{c} A\ \mathit{true} \\ \vdots \\ C\ \mathit{true} \end{array}$$

is written as `[A;...;C]`.

In other words, the hypothesis A is immediately preceded by a square bracket ('`[`'), followed by the lines representing the hypothetical proof of C, followed by a closing square bracket ('`]`'). So square brackets are used to delimit the scope of an assumption. If we need more than hypothesis, we nest this construct as we will see in the example below.

As an example, we consider the proof of $(A \supset B) \wedge (B \supset C) \supset (A \supset C)$ *true*. We show each stage in the proof during its natural construction, showing both the mathematical and concrete syntax, except that we omit the judgment "*true*" to keep the size of the derivation manageable. We write '`...`' to indicate that the following line has not yet been justified.

$$\begin{array}{c} \vdots \\ (A \supset B) \wedge (B \supset C) \supset (A \supset C) \end{array}$$

```
...
(A => B) & (B => C) => (A => C);
```

The first bottom-up step is an implication introduction. In the linear form, we use our notation for hypothetical judgments.

$$\dfrac{\begin{array}{c} \overline{(A \supset B) \wedge (B \supset C)}^{\,u} \\ \vdots \\ A \supset C \end{array}}{(A \supset B) \wedge (B \supset C) \supset (A \supset C)} \supset I^u$$

```
[ (A => B) & (B => C);
  ...
  A => C ];
(A => B) & (B => C) => (A => C);
```

Again, we proceed via an implication introduction. In the mathematical notation, the hypotheses are shown next to each other. In the linear notation, the second hypothesis `A` is nested inside the first, also making both of them available to fill the remaining gap in the proof.

$$
\dfrac{\dfrac{\begin{array}{c}\dfrac{}{(A\supset B)\wedge(B\supset C)}u \quad \dfrac{}{A}w \\ \vdots \\ C\end{array}}{A\supset C}\supset I^w}{(A\supset B)\wedge(B\supset C)\supset(A\supset C)}\supset I^u
$$

```
[ (A => B) & (B => C);
  [ A;
    ...
    C ];
  A => C ];
(A => B) & (B => C) => (A => C);
```

Now that the conclusion is atomic and cannot be decomposed further, we reason downwards from the hypotheses. In the linear format, we write the new line `A => B;` immediately below the hypothesis, but we could also have inserted it directly below `A;`. In general, the requirement is that the lines representing the premise of an inference rule must all come before the conclusion. Furthermore, lines cannot be used outside the hypothetical proof in which they appear, because their proof could depend on the hypothesis.

$$
\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{}{(A\supset B)\wedge(B\supset C)}u}{A\supset B}\wedge E_L \quad \dfrac{}{A}w \\ \vdots \\ C\end{array}}{A\supset C}\supset I^w}{(A\supset B)\wedge(B\supset C)\supset(A\supset C)}\supset I^u
$$

```
[ (A => B) & (B => C);
  A => B;
  [ A;
    ...
    C ];
  A => C ];
(A => B) & (B => C) => (A => C);
```

Nex we apply another straightforward top-down reasoning step. In this case, there is no choice on where to insert `B;`.

$$
\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{}{(A\supset B)\wedge(B\supset C)}u}{A\supset B}\wedge E_L \quad \dfrac{}{A}w}{B}\supset E \\ \vdots \\ C\end{array}}{A\supset C}\supset I^w}{(A\supset B)\wedge(B\supset C)\supset(A\supset C)}\supset I^u
$$

```
[ (A => B) & (B => C);
  A => B;
  [ A;
    B;
    ...
    C ];
  A => C ];
(A => B) & (B => C) => (A => C);
```

For the last two steps, we align the derivations vertically. The are both top-down steps (conjunction elimination followed by implication elimination).

$$
\dfrac{\dfrac{\begin{array}{c} \dfrac{\dfrac{}{(A\supset B)\wedge(B\supset C)}\,u}{B\supset C}\,\wedge E_R \qquad \dfrac{\dfrac{\dfrac{}{(A\supset B)\wedge(B\supset C)}\,u}{A\supset B}\,\wedge E_L \quad \dfrac{}{A}\,w}{B}\,\supset E \\ \vdots \\ C \end{array}}{A\supset C}\,\supset I^w}{(A\supset B)\wedge(B\supset C)\supset(A\supset C)}\,\supset I^u
$$

```
[ (A => B) & (B => C);
  A => B;
  B => C;
  [ A;
    B;
    ...
    C ];
  A => C ];
(A => B) & (B => C) => (A => C);
```

In the step above we notice that subproofs may be shared in the linearized format, while in the tree format they appear more than once. In this case it is only the hypothesis $(A\supset B)\wedge(B\supset C)$ which is shared.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A\supset B)\wedge(B\supset C)}\,u}{B\supset C}\,\wedge E_R \qquad \dfrac{\dfrac{\dfrac{}{(A\supset B)\wedge(B\supset C)}\,u}{A\supset B}\,\wedge E_L \quad \dfrac{}{A}\,w}{B}\,\supset E}{C}\,\supset E}{A\supset C}\,\supset I^w}{(A\supset B)\wedge(B\supset C)\supset(A\supset C)}\,\supset I^u
$$

```
[ (A => B) & (B => C);
  A => B;
  B => C;
  [ A;
    B;
    C ];
  A => C ];
(A => B) & (B => C) => (A => C);
```

In the last step, the linear derivation only changed in that we noticed that `C` already follows from two other lines and is therefore justified.

For other details of concrete syntax and usage of the proof-checking program available for this course, please refer to the on-line documentation available through the course home page.

2.9 Normal Deductions

The strategy we have used so far in proof search is easily summarized: we reason with introduction rules from the bottom up and with elimination rules from the top down, hoping that the two will meet in the middle. This description is somewhat vague in that it is not obvious how to apply it to complex rules such as disjunction elimination which involve formulas other than the principal one whose connective is eliminated.

To make this precise we introduce two new judgments

$A\uparrow$	A has a normal proof
$A\downarrow$	A has a neutral proof

We are primarily interest in normal proofs, which are those that our strategy can find. Neutral proofs represent an auxiliary concept (sometimes called an *extraction proof*) necessary for the definition of normal proofs.

We will define these judgments via rules, trying to capture the following intuitions:

1. A normal proof is either neutral, or proceeds by applying introduction rules to other normal proofs.

2. A neutral proof proceeds by applying elimination rules to hypotheses or other neutral proofs.

By construction, every A which has a normal (or neutral) proof is true. The converse, namely that every true A has a normal proof also holds, but is not at all obvious. We may prove this property later on, at least for a fragment of the logic.

First, a general rule to express that every neutral proof is normal.

$$\frac{A\downarrow}{A\uparrow}\downarrow\uparrow$$

Conjunction. The rules for conjunction are easily annotated.

$$\frac{A\uparrow \qquad B\uparrow}{A\wedge B\uparrow}\wedge I \qquad \frac{A\wedge B\downarrow}{A\downarrow}\wedge E_L \qquad \frac{A\wedge B\downarrow}{B\downarrow}\wedge E_R$$

Truth. Truth only has an introduction rule and therefore no neutral proof constructor.

$$\frac{}{\top\uparrow}\top I$$

Implication. Implication first fixes the idea that hypotheses are neutral, so the introduction rule refers to both normal and neutral deductions.

$$
\dfrac{\begin{array}{c}\overline{A\downarrow}^{\,u}\\ \vdots \\ B\uparrow\end{array}}{A \supset B \uparrow} \supset I^u
\qquad
\dfrac{A \supset B \downarrow \qquad A \uparrow}{B \downarrow} \supset E
$$

The elimination rule is more difficult to understand. The principal premise (with the connective "$\supset$" we are eliminating) should have a neutral proof. The resulting derivation will once again be neutral, but we can only require the second premise to have a normal proof.

Disjunction. For disjunction, the introduction rules are straightforward. The elimination rule requires again the requires the principal premise to have a neutral proof. An the assumptions introduced in both branches are also neutral. In the end we can conclude that we have a normal proof of the conclusion, if we can find a normal proof in each premise.

$$
\dfrac{A\uparrow}{A \vee B \uparrow} \vee I_L
\qquad
\dfrac{B\uparrow}{A \vee B \uparrow} \vee I_R
\qquad
\dfrac{A \vee B \downarrow \quad \begin{array}{c}\overline{A\downarrow}^{\,u}\\ \vdots \\ C\uparrow\end{array} \quad \begin{array}{c}\overline{B\downarrow}^{\,w}\\ \vdots \\ C\uparrow\end{array}}{C \uparrow} \vee E^{u,w}
$$

Falsehood. Falsehood is analogous to the rules for disjunction. But since there are no introduction rules, there are no cases to consider in the elimination rule.

$$
\dfrac{\bot\downarrow}{C\uparrow} \bot E
$$

All the proofs we have seen so far in these notes are normal: we can easily annotate them with arrows using only the rules above. The following is an example of a proof which is not normal.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A\ \mathit{true}}\,w \qquad \dfrac{\dfrac{\dfrac{}{A\ \mathit{true}}\,u \quad \dfrac{}{\neg A\ \mathit{true}}\,w}{A \wedge \neg A\ \mathit{true}}\,\wedge I}{A\ \mathit{true}}\,\wedge E_L}{\bot\ \mathit{true}}\,\supset E}{\neg A \supset \bot\ \mathit{true}}\,\supset I^w}{A \supset \neg A \supset \bot\ \mathit{true}}\,\supset I^u$$

If we follow the process of annotation, we fail at only one place as indicated below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A \downarrow}\,w \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \downarrow}\,u \quad \dfrac{}{\neg A \downarrow}\,w}{A \wedge \neg A\ ?}\,\wedge I}{A \downarrow}\,\wedge E_L}{A \uparrow}\,\downarrow\uparrow}{\bot \downarrow}\,\supset E}{\bot \uparrow}\,\downarrow\uparrow}{\neg A \supset \bot \uparrow}\,\supset I^w}{A \supset \neg A \supset \bot \uparrow}\,\supset I^u$$

The situation that prevents this deduction from being normal is that we introduce a connective (in this case, $A \wedge \neg A$) and then immediately eliminate it. This seems like a detour—why do it at all? In fact, we can just replace this little inference with the hypothesis $A \downarrow$ and obtain a deduction which is now normal.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A \downarrow}\,w \qquad \dfrac{\dfrac{}{A \downarrow}\,u}{A \uparrow}\,\downarrow\uparrow}{\bot \downarrow}\,\supset E}{\bot \uparrow}\,\downarrow\uparrow}{\neg A \supset \bot \uparrow}\,\supset I^w}{A \supset \neg A \supset \bot \uparrow}\,\supset I^u$$

It turns out that the only reason a deduction may not be normal is an introduction followed by an elimination, and that we can always simplify such a derivation to (eventually) obtain a normal one. This process of simplification is directly connected to computation in a programming language. We only need to fix a particular simplification strategy. Under this interpretation, a proof corresponds to a program, simplification of the kind above corresponds to computation, and a normal proof corresponds to a value. It is precisely this correspondence which is the central topic of the next chapter.

2.10 Exercises

Exercise 2.1 *Show the derivations for the rules* $\equiv I$, $\equiv E_L$ *and* $\equiv E_R$ *under the definition of* $A \equiv B$ *as* $(A \supset B) \wedge (B \supset A)$.

Bibliography

[Gen35] Gerhard Gentzen. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1935. English translation in M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pages 68–131, North-Holland, 1969.

[ML96] Per Martin-Löf. On the meanings of the logical constants and the justifications of the logical laws. *Nordic Journal of Philosophical Logic*, 1(1):11–60, 1996.