

Lecture Notes on Recursive Types

15-814: Types and Programming Languages
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Lecture 9
Tuesday, October 1, 2019

1 Introduction

Using type structure to capture common constructions available in programming languages, we have built a rich set of primitives in our programming language (see the summary in Section L8.7). Booleans turned out to be representable using generic constructions, since $bool = 1 + 1$. However, natural numbers would be

$$nat = 1 + (1 + (1 + \dots))$$

which cannot be expressed already. However, we can observe that the tail of the sum is equal to the whole sum. That is,

$$nat = 1 + nat$$

We won't be able to achieve such an equality, but we *can* achieve an *isomorphism*

$$nat \cong 1 + nat$$

with two functions to witness the isomorphism.

$$nat \begin{array}{c} \xrightarrow{\text{unfold}} \\ \cong \\ \xleftarrow{\text{fold}} \end{array} 1 + nat$$

Actually, `unfold` and `fold` will not be functions but language primitives because we want them to apply to a large class of recursively defined types.

2 Recursive Types

The more general type constructor that solves recursive type equations is written as $\rho\alpha.\tau$. Rho (ρ) here stands for “recursive”, α is a type variable with scope τ . The general picture to keep in mind is that a recursive type $\rho\alpha.\tau$ should be isomorphic to its *unfolding* $[\rho\alpha.\tau/\alpha]\tau$.

$$\rho\alpha.\tau \begin{array}{c} \xrightarrow{\text{unfold}} \\ \cong \\ \xleftarrow{\text{fold}} \end{array} [\rho\alpha.\tau/\alpha]\tau$$

Once we have defined the fold and unfold expressions with their statics and dynamics, we will have to check that these two types are indeed isomorphic.

As an example, consider

$$\text{nat} = \rho\alpha.1 + \alpha$$

Does this give us the desired isomorphism? Let’s check:

$$\begin{aligned} \text{nat} &= \rho\alpha.1 + \alpha \\ &\cong [\rho\alpha.1 + \alpha/\alpha](1 + \alpha) \\ &= 1 + (\rho\alpha.1 + \alpha) \\ &= 1 + \text{nat} \end{aligned}$$

So, yes, we get the desired isomorphism. Here are some other examples of types with recursive definitions we’d like to represent in a similar manner.

Lists	$list\ \tau \cong 1 + (\tau \times list\ \tau)$
Binary Trees	$tree \cong 1 + (tree \times nat \times tree)$
Binary Numbers	$bin \cong list\ (1 + 1)$

For example, binary trees of natural numbers would then be explicitly defined as

$$\begin{aligned} tree &= \rho\alpha.1 + (\alpha \times nat \times \alpha) \\ &\cong 1 + (tree \times nat \times tree) \end{aligned}$$

and satisfy the desired isomorphism.

3 Fold and Unfold

Let’s recall the principal isomorphism we would like to have:

$$\rho\alpha.\tau \begin{array}{c} \xrightarrow{\text{unfold}} \\ \cong \\ \xleftarrow{\text{fold}} \end{array} [\rho\alpha.\tau/\alpha]\tau$$

Each new type we have comes with some constructors for values of the new type and some destructors. Computation arises when a destructor meets a constructor. According to the display above, fold should be the *constructor* (because it results in something of type $\rho\alpha.\tau$), while unfold is a destructor. Reading the types off the above desired isomorphism:

$$\frac{\Gamma \vdash e : [\rho\alpha.\tau/\alpha]\tau}{\Gamma \vdash \text{fold } e : \rho\alpha.\tau} \text{ fold} \quad \frac{\Gamma \vdash e : \rho\alpha.\tau}{\Gamma \vdash \text{unfold } e : [\rho\alpha.\tau/\alpha]\tau} \text{ unfold}$$

We decide that fold e is a value only if e is a value. This is so that, for example, when we write $v : \text{nat}$, the value v will actually directly represent a natural number instead of some expression that might result in a natural number (see Exercise 1)

$$\frac{e \text{ val}}{\text{fold } e \text{ val}} \text{ val/fold}$$

The interesting rule for stepping (usually the first one to write) is the one where a destructor meets a constructor.

$$\frac{v \text{ val}}{\text{unfold } (\text{fold } v) \mapsto v} \text{ step/unfold}$$

Does this rule preserve types? Let's say we have

$$\cdot \vdash \text{unfold } (\text{fold } v) : \sigma$$

By inversion (only the unfold rule could have this conclusion), we obtain

$$\cdot \vdash \text{fold } v : \rho\alpha.\tau$$

where $\sigma = [\rho\alpha.\tau/\alpha]\tau$. Applying inversion again, we get

$$\cdot \vdash v : [\rho\alpha.\tau/\alpha]\tau$$

which is also the type of unfold (fold v). Therefore, the rule step/unfold satisfies type preservation.

We now only need to add rules to reach values and redices, so-called *congruence rules*.

$$\frac{e \mapsto e'}{\text{fold } e \mapsto \text{fold } e'} \text{ step/fold} \quad \frac{e \mapsto e'}{\text{unfold } e \mapsto \text{unfold } e'} \text{ step/unfold}_0$$

It is a matter of checking the progress theorem and also verifying the desired isomorphism to ensure that we now have enough rules. A student suggested

$$\frac{}{\text{fold}(\text{unfold } e) \mapsto e} ?$$

which is eminently reasonable, but turned out to be unnecessary.

4 Examples

Before we check our desired properties, let's write some examples on natural numbers (in our unary representation).

$$\begin{aligned} \text{nat} &= \rho\alpha. 1 + \alpha \\ &\cong 1 + \text{nat} \\ \text{zero} &: \text{nat} \\ \text{zero} &= \text{fold}(\ell \cdot \langle \rangle) \\ \text{one} &: \text{nat} \\ \text{one} &= \text{fold}(r \cdot \text{zero}) \\ &= \text{fold}(r \cdot \text{fold}(\ell \cdot \langle \rangle)) \\ \text{succ} &: \text{nat} \rightarrow \text{nat} \\ \text{succ} &= \lambda n. \text{fold}(r \cdot n) \\ \text{pred} &: \text{nat} \rightarrow \text{nat} \\ \text{pred} &= \lambda n. \text{case}(\text{unfold } n) (\ell \cdot x_1 \Rightarrow \text{zero} \mid r \cdot x_2 \Rightarrow x_2) \end{aligned}$$

At this point we realize that we cannot write any function that recurses over a natural number. Unlike the λ -calculus, the representation here as a sum and a recursive types only allows us to implement a case construct. This is not a significant obstacle, since we will shortly add general recursion to our language and then functions like addition, multiplication, exponentiation, and greatest common divisor can be implemented simply and uniformly.

5 Preservation and Progress

We have already seen the key idea in the preservation theorem; all other cases are simple and follow familiar patterns.

For progress, we first need a canonical form theorem. We get the new case

(vi) If $\cdot \vdash v : \rho\alpha. \tau$ and v *val* then $v = \text{fold } v'$ for a value v' .

This follows, as before, by analyzing the cases for typing and values.

The critical case in the proof of progress (by rule induction on the given typing derivation) is

$$\frac{\cdot \vdash e_1 : \rho\alpha. \tau}{\cdot \vdash \text{unfold } e_1 : [\rho\alpha. \tau/\alpha]\tau} \text{ fold}$$

If $e_1 \mapsto e'_1$ then, by rule, $\text{unfold } e_1 \mapsto \text{unfold } e'_1$. If e_1 is a value, then the canonical forms theorem tells us that $e_1 = \text{fold } v_2$ for some value v_2 . Therefore, the step/unfold applies and $\text{unfold } (\text{fold } v_2) \mapsto v_2$.

6 Isorecursive Types

The new type constructor $\rho\alpha. \tau$ we have defined is called an *isorecursive type*, because we have an isomorphism

$$\rho\alpha. \tau \begin{array}{c} \xrightarrow{\text{unfold}} \\ \cong \\ \xleftarrow{\text{fold}} \end{array} [\rho\alpha. \tau/\alpha]\tau$$

rather than an equality between the two types (which would be *equirecursive*). But is it really an isomorphism? Let's check the two directions.

First, we need to check that $\text{unfold } (\text{fold } v) = v$ for any value $v : [\rho\alpha. \tau/\alpha]\tau$. But immediately (by rule step/unfold) we have

$$\text{unfold } (\text{fold } v) \mapsto v$$

so the two are certainly equal.

In the other direction, we need to verify that

$$\text{fold } (\text{unfold } v) \stackrel{?}{=} v \quad \text{for any value } v : \rho\alpha. \tau$$

By the canonical forms theorem, $v = \text{fold } v'$ for some value v' . Then we reason

$$\begin{aligned} & \text{fold } (\text{unfold } v) \\ &= \text{fold } (\text{unfold } (\text{fold } v')) \\ &\mapsto \text{fold } v' \\ &= v \end{aligned}$$

So, an isorecursive type is indeed isomorphic to its unfolding.

7 Excursion: Embedding the Untyped λ -Calculus

As one of you suspected during lecture, now that we have recursive types, perhaps we can type $\lambda x. x x$, which we previously proved to have no type. And if that works, why stop there? Why not type the Y combinator itself? In Lecture 5 (pages L5.2-3) we convinced ourselves that $\lambda x. x x : \tau_3 \rightarrow \tau_2$ for any types τ_3 and τ_2 satisfying $\tau_3 = \tau_3 \rightarrow \tau_2$. That's because x needs to take itself as an argument.

This does not seem promising, since we still cannot solve this equation! But we may be able to approximate it by an *isomorphism*. Can we find a type U such that $U \cong U \rightarrow \tau_2$. The unspecified type τ_2 gets in the way, so let's try it with $\tau_2 = U$. So, we have to solve

$$U \begin{array}{c} \xrightarrow{\text{unfold}} \\ \cong \\ \xleftarrow{\text{fold}} \end{array} U \rightarrow U$$

In our language, any recursive type equation has a solution (perhaps degenerate), so we just set

$$U = \rho\alpha. \alpha \rightarrow \alpha \cong U \rightarrow U$$

Let's try to type self-application at type $U \rightarrow U$.

$$\frac{x : U \vdash x x : U}{\cdot \vdash \lambda x. x x : U \rightarrow U} \text{ lam} \quad ?$$

This still does not work, but we can *unfold* the type of the first occurrence of x so it matches the type of its argument!

$$\frac{\frac{\frac{}{x : U \vdash x : U} \text{ var}}{x : U \vdash \text{unfold } x : U \rightarrow U} \text{ unfold} \quad \frac{}{x : U \vdash x : U} \text{ var}}{x : U \vdash (\text{unfold } x) x : U} \text{ app}}{\cdot \vdash \lambda x. (\text{unfold } x) x : U \rightarrow U} \text{ lam}$$

So, lo and behold, if we are willing to insert an unfold we can now type-check self-application.

Curious: can we do the same with the Y combinator? The answer is yes, but let's be even more ambitious: let's translate the whole untyped

λ -calculus into our language! We write M for untyped expressions to distinguish them from the target language expressions e .

Untyped Exps $M ::= x \mid \lambda x. M \mid M_1 M_2$

We try to devise a translation $\lceil - \rceil$ such that

$$\lceil M \rceil : U$$

for any untyped expression M . To be more precise, assume the untyped expression has free variables x_1, \dots, x_n , then we aim for

$$x_1 : U, \dots, x_n : U \vdash \lceil M \rceil : U$$

The reason all variables have type U because in the source they stand for an arbitrary untyped expression. We define

$$\begin{aligned} \lceil x \rceil &= x \\ \lceil \lambda x. M \rceil &= \text{fold } (\lambda x. \lceil M \rceil) \\ \lceil M_1 M_2 \rceil &= (\text{unfold } \lceil M_1 \rceil) \lceil M_2 \rceil \end{aligned}$$

We suggest you go through these definitions and type-check them, keeping in mind the all-important

$$U \begin{array}{c} \xrightarrow{\text{unfold}} \\ \cong \\ \xleftarrow{\text{fold}} \end{array} U \rightarrow U$$

The type-correctness of this translation means we have a very direct representation of the whole *untyped λ -calculus* in our language, using only a single type U (but exploiting recursive types). Therefore, the untyped λ -calculus is sometimes referred to as the *untyped λ -calculus* because it can be represented with a single universal type U .

Since the Y combinator is only a particular untyped λ -expression, we can also translate it into the target.

However, there is still a fly in the ointment: even though we know the target is well-typed, we don't know if it behaves correctly, operationally. Under some definitions it does not. For example, $\lambda x. \Omega$ has no normal form, but $\lceil \lambda x. \Omega \rceil = \text{fold } (\lambda x. \lceil \Omega \rceil)$ is a value and does not take a step. We will discuss at a later point how to bridge this gap, which is not straightforward.

8 Fixed Point Expressions

We have added recursive types that solve recursive type equations. But in order to write all the programs we want (for example, on natural numbers all the recursive functions) we also need recursively defined expressions. The Y combinator is not directly available to us in the needed generality, even though it can be defined at type U . Instead we add a primitive, $\text{fix } f. e$, where f is a variable. It is not a value, and it steps by *unrolling* the fixed point:

$$\frac{}{\text{fix } f. e \mapsto [\text{fix } f. e/f]e} \text{ step/fix}$$

This “unrolling” is quite similar to unfolding a recursive type, but at the level of expressions. However, it is independent of recursive types and can be applied in full generality. One particular example is $\text{fix } f. f \mapsto \text{fix } f. f$ so in this language we can define $\perp = \text{fix } f. f$ (see Exercise L6.2). Emboldened by this property, we imagine we might have in general

$$\frac{\Gamma, f : \boxed{} \vdash e : \boxed{}}{\Gamma \vdash \text{fix } f. e : \tau} \text{ fix}$$

but there are still some holes in this typing rule.

We want preservation to hold (progress is trivial to extend, because a fixed point always steps) so we need that

$$\cdot \vdash \text{fix } f. e : \tau \text{ implies } \cdot \vdash [\text{fix } f. e/f]e : \tau$$

From this we can deduce two things: first, $e : \tau$ because that is the result of substitution. And, second, for the substitution property to hold we need that $f : \tau$ so we can substitute $[\text{fix } f. e/f]e$. Filling in this information:

$$\frac{\Gamma, f : \tau \vdash e : \tau}{\Gamma \vdash \text{fix } f. e : \tau} \text{ fix}$$

Now we have settled both statics and dynamic and have fixed point expressions available to us. For example

$$\begin{aligned} \text{plus} & : \text{nat} \rightarrow (\text{nat} \rightarrow \text{nat}) \\ \text{plus} & = \text{fix } p. \lambda n. \lambda k. \text{if } (\text{is_zero } n) \ k \ (\text{succ } (p \ (\text{pred } n) \ k)) \end{aligned}$$

where we have seen succ and pred , and is_zero is easy to define (see Exercise 3). We see that after one step of unrolling we have

$$\text{plus} \mapsto \lambda n. \lambda k. \text{if } (\text{is_zero } n) \ k \ (\text{succ } (\text{plus } (\text{pred } n) \ k))$$

which is the intended recursive definition. We can treat for example gcd as in Section L2.4 and Exercise L2.3.

There are a few unpleasant things about fixed point expressions. One is that it is neither a constructor nor a destructor of any particular type, but is applicable at any type τ . It thus violates one of the design principles of our language that we have followed so far. We may interpret this as an indication that recursion is a fundamental computational principle separate from any particular typing construct, but this is not a universally held view.

The second one is that in $\text{fix } f. e$ the variable f does not stand for a value (like all other variables x we have used so far) but an expression (we substitute $\text{fix } f. e$ for f , and that's not a value). To avoid this latter issue, in call-by-value languages sometimes the fixed point expression is limited to functions, as in $\text{fun } f(x) = e$ where e can depend on both x and f .

Exercises

Exercise 1 Prove adequacy of natural number encodings in type nat .

1. Define a (mathematical) function $\ulcorner n \urcorner$ on natural numbers n such that $\cdot \vdash \ulcorner n \urcorner : \text{nat}$ and $\ulcorner n \urcorner \text{ val}$.
2. Define a (mathematical) function $\llbracket v \rrbracket$ on values v with $\cdot \vdash v : \text{nat}$ returning the number represented by v .
3. Prove that the pair of functions $\ulcorner - \urcorner$ and $\llbracket - \rrbracket$ witness an isomorphism between the usual (mathematical) natural numbers and closed values of type nat .

Exercise 2 Consider the combinators Y and Z . Here Z , the call-by-value fixed point combinator, is defined as

$$Z = \lambda f. (\lambda x. f (\lambda v. x x v)) (\lambda x. f (\lambda v. x x v))$$

1. Exhibit a difference between Y and Z under that assumption that the pure untyped λ -calculus follows a call-by-value evaluation strategy.
2. Give the translation $\ulcorner Z \urcorner : U$ into the universal type.

Exercise 3 Define $\text{is_zero} : \text{nat} \rightarrow \text{bool}$, where you may take $\text{bool} = 1 + 1$ or as a primitive with true , false , and if .