Assignment 2
The Simply Typed λ-Calculus

15-814: Types and Programming Languages
Frank Pfenning

Due Tuesday, September 24, 2019

This assignment is due on the above date and it must be submitted electronically as a PDF file on Canvas. Please use the attached template to typeset your assignment and make sure to include your full name and Andrew ID.

Please carefully read the policies on collaboration and credit on the course web pages at http://www.cs.cmu.edu/~fp/courses/15814-f19/assignments.html.

1   Simple types

Task 1 (L3.1, 10 points) Fill in the blanks in the following typing judgments so the resulting judgment holds, or indicate there is no way to do so. You do not need to justify your answer or supply a typing derivation, and the types do not need to be "most general" in any sense. Remember that the function type constructor associates to the right, so that $\tau \to \sigma \to \rho = \tau \to (\sigma \to \rho)$.

(i) $\boxed{} \vdash y\,x : \alpha$

(ii) $\boxed{} \vdash x\,x : \boxed{}$

(iii) $\cdot \vdash \boxed{} : (\alpha \to \alpha) \to \alpha$

(iv) $\cdot \vdash (\lambda z.\,z)\,(\lambda x.\,\lambda y.\,\lambda p.\,p\,x\,y) : \boxed{}$

(v) $\cdot \vdash \lambda f.\,\lambda g.\,\lambda x.\,(f\,x)\,(g\,x) : (\alpha \to \boxed{}) \to (\alpha \to \boxed{}) \to (\alpha \to \boxed{})$

2   Proof by Rule Induction

Since this is the first time we (that is, you) are proving theorems about judgments defined by rules, we ask you to be very explicit, as we were in the lectures and lecture notes. In particular:

- Explicitly state the overall structure of your proof: whether it proceeds by rule induction, and, if so, on the derivation of which judgment, or by structural induction, or by inversion, or just directly. If you need to split out a lemma for your proof, state it clearly and prove it separately. If you need to generalize your induction hypothesis, clearly state the generalized form.

- Explicitly list all cases in an induction proof. If a case is impossible, prove that is is impossible. Often, that's just inversion, but sometimes it is more subtle.

- Explicitly note any appeals to the induction hypothesis.

- Any appeals to inversion should be noted as such, as well as the rules that could have inferred the judgment we already know. This could lead to zero cases (a contradiction—the judgment could not have been derived), one case (there is exactly one rule whose conclusion matches our knowledge), or multiple cases, in which case your proof now splits into multiple cases.

Task 2 (L4.1.1 & L4.1.2, 30 points) If we have two judgments defined simultaneously (like e *normal* and e *neutral* we often need to prove properties about them by *simultaneous induction*. In simultaneous induction you have multiple induction hypotheses and if the premise of a rule comes from a different judgment, you may apply the appropriate induction hypothesis to it. In proving property 2 below, make a note if you needed a simple or a simultaneous induction.

1. In each case below, give an example of an expressions e and type τ with $\cdot \vdash e : \tau$ and also the stated property, or indicate no such expression and type exist. You do not need to justify your answer further (no need for typing derivations or proofs).

   (i) $\cdot \vdash e \Leftarrow \tau$ and also $\cdot \vdash e \Rightarrow \tau$

   (ii) $\cdot \vdash e \Leftarrow \tau$ but not $\cdot \vdash e \Rightarrow \tau$

   (iii) $\cdot \vdash e \Rightarrow \tau$ but not $\cdot \vdash e \Leftarrow \tau$

   (iv) Neither $\cdot \vdash e \Leftarrow \tau$ nor $\cdot \vdash e \Rightarrow \tau$

2. Prove that the bidirectional typing rules are *sound*, that is, we verify or synthesize only correct types.

   (i) If $\Gamma \vdash e \Leftarrow \tau$ then $\Gamma \vdash e : \tau$ and e *normal*.

   (ii) If $\Gamma \vdash e \Rightarrow \tau$ then $\Gamma \vdash e : \tau$ and e *neutral*.

3. **You do not need to prove Part L4.1.3**

Task 3 (L4.2, 20 points) Prove the following theorems.

1. If e *nf* then e *normal*.

2. If e *normal* then e *nf*.

Because the judgment e *normal* is defined simultaneously with e *neutral*, you may have to generalize some of the statements before you can prove them by simultaneous induction.