

Assignment 8

Parametricity

15-814: Types and Programming Languages
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Due Tuesday, November 12, 2019

Task 1 (L16.1, 15 points) Prove that $\forall \alpha. \alpha \rightarrow \alpha \cong 1$. You may use the results of Sections L16.3 and L16.5.

Task 2 (L16.2, 15 points) Prove, using parametricity, that if we have $f : \forall \alpha. \alpha \rightarrow \alpha \rightarrow \alpha$ for a value f then either $f \sim \Lambda \alpha. \lambda x. \lambda y. x \in [\forall \alpha. \alpha \rightarrow \alpha \rightarrow \alpha]$ or $f \sim \Lambda \alpha. \lambda x. \lambda y. y \in [\forall \alpha. \alpha \rightarrow \alpha \rightarrow \alpha]$.

Task 3 (L18.3, 30 points) In this exercise we pursue two different implementations of an integer counter, which can become negative (unlike the natural number counter in Lecture 18). The functions are simpler than the ones in Exercises L18.1 and L18.2 so that the logical equality argument is more manageable. We specify a signature

```
INTCTR = {  
  type ictr  
  zero : ictr  
  inc  : ictr -> ictr  
  dec  : ictr -> ictr  
  is0  : ictr -> bool  
}
```

where *zero*, *inc*, *dec* and *is0* have their obvious specification with respect to integers.

1. Write out the definition of *INTCTR* as an existential type.
2. Define the constants and functions *d_zero*, *d_inc*, *d_dec* and *d_is0* for the implementation where type *ictr* = *diff* from Exercise L18.1, using pattern matching as explained in Exercise L18.1.
3. Define the constants and functions *s_zero*, *s_inc*, *s_dec* and *s_is0* for the implementation where type *ictr* = *sign* from Exercise L18.2, using pattern matching as explained in Exercise L18.2.

Now consider the two definitions

$$DCTR : INTCTR = \langle diff, \langle d_zero, \langle d_inc, \langle d_dec, d_is0 \rangle \rangle \rangle \rangle$$
$$SCTR : INTCTR = \langle sign, \langle s_zero, \langle s_inc, \langle s_dec, s_is0 \rangle \rangle \rangle \rangle$$

4. Prove that $DCTR \sim SCTR$ in $[INTCTR]$ by defining a suitable relation $R : diff \leftrightarrow sign$ and proving that

$$\langle d_zero, \langle d_inc, \langle d_dec, d_is0 \rangle \rangle \rangle \sim \langle s_zero, \langle s_inc, \langle s_dec, s_is0 \rangle \rangle \rangle$$
$$\in [R \times (R \rightarrow R) \times (R \rightarrow R) \times (R \rightarrow bool)]$$