Abstract Syntax

$$\begin{array}{llll}
\text{Types} & \tau & ::= & \tau_1 \to \tau_2 \mid \forall \alpha.\, \tau \mid \alpha \mid \mathsf{bool} \mid \tau_1 \times \tau_2 \\
\text{Terms} & e & ::= & x \mid \lambda x.\, e \mid e_1\, e_2 & (\to) \\
& & \mid & \Lambda \alpha.\, e \mid e\,[\tau] & (\forall) \\
& & \mid & \mathsf{true} \mid \mathsf{false} \mid \mathsf{if}\ e_1\ e_2\ e_3 & (\mathsf{bool}) \\
& & \mid & \langle e_1, e_2 \rangle \mid \mathsf{case}\ e\ (\langle x_1, x_2 \rangle \Rightarrow e') & (\times) \\
\text{Contexts} & \Gamma & ::= & \cdot \mid \Gamma, \alpha\ \mathit{type} \mid \Gamma, x : \tau & (\text{all variables distinct})
\end{array}$$

Judgments

$\Gamma\ \mathit{ctx}$	Γ is a valid context	
$\Gamma \vdash \tau\ \mathit{type}$	τ is a valid type	presupposes $\Gamma\ \mathit{ctx}$
$\Gamma \vdash e : \tau$	expression e has type τ	presupposes $\Gamma\ \mathit{ctx}$, ensures $\Gamma \vdash \tau\ \mathit{type}$
$e\ \mathit{value}$	expression e is a value	presupposes $\cdot \vdash e : \tau$ for some τ
$e \mapsto e'$	expression e steps to e'	presupposes $\cdot \vdash e : \tau$ for some τ

Contexts Γ

$$\frac{}{(\cdot)\ \mathit{ctx}}\ \mathsf{ctx/emp} \qquad \frac{\Gamma\ \mathit{ctx}}{(\Gamma, \alpha\ \mathit{type})\ \mathit{ctx}}\ \mathsf{ctx/tpvar} \qquad \frac{\Gamma\ \mathit{ctx} \quad \Gamma \vdash \tau\ \mathit{type}}{(\Gamma, x : \tau)\ \mathit{ctx}}\ \mathsf{ctx/var}$$

Functions $\tau_1 \to \tau_2$

$$\frac{\Gamma \vdash \tau_1\ \mathit{type} \quad \Gamma \vdash \tau_2\ \mathit{type}}{\Gamma \vdash \tau_1 \to \tau_2\ \mathit{type}}\ \mathsf{tp/arrow}$$

$$\frac{\Gamma, x_1 : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \lambda x_1{:}\tau_1.\, e_2 : \tau_1 \to \tau_2}\ \mathsf{tp/lam} \qquad \frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}\ \mathsf{tp/var} \qquad \frac{\Gamma \vdash e_1 : \tau_2 \to \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1\, e_2 : \tau_1}\ \mathsf{tp/app}$$

$$\frac{}{\lambda x.\, e\ \mathit{value}}\ \mathsf{val/lam} \qquad \frac{e_2\ \mathit{value}}{(\lambda x.\, e_1)\, e_2 \mapsto [e_2/x]e_1}\ \mathsf{step/app/lam}$$

$$\frac{e_1 \mapsto e_1'}{e_1\, e_2 \mapsto e_1'\, e_2}\ \mathsf{step/app_1} \qquad \frac{e_1\ \mathit{value} \quad e_2 \mapsto e_2'}{e_1\, e_2 \mapsto e_1\, e_2'}\ \mathsf{step/app_2}$$

Polymorphic Types $\forall\alpha.\,\tau$

$$\frac{\alpha\ \textit{type} \in \Gamma}{\Gamma \vdash \alpha\ \textit{type}}\ \mathsf{tp/tpvar} \qquad \frac{\Gamma, \alpha\ \textit{type} \vdash \tau\ \textit{type}}{\Gamma \vdash \forall\alpha.\,\tau\ \textit{type}}\ \mathsf{tp/forall}$$

$$\frac{\Gamma, \alpha\ \textit{type} \vdash e : \tau}{\Gamma \vdash \Lambda\alpha.\,e : \forall\alpha.\,\tau}\ \mathsf{tp/tplam} \qquad \frac{\Gamma \vdash e : \forall\alpha.\,\tau \quad \Gamma \vdash \sigma\ \textit{type}}{\Gamma \vdash e\,[\sigma] : [\sigma/\alpha]\tau}\ \mathsf{tp/tpapp}$$

$$\frac{}{\Lambda\alpha.\,e\ \textit{value}}\ \mathsf{val/tplam} \qquad \frac{}{(\Lambda\alpha.\,e)\,[\tau] \mapsto [\tau/\alpha]e}\ \mathsf{step/tpapp/tplam} \qquad \frac{e \mapsto e'}{e\,[\tau] \mapsto e'\,[\tau]}\ \mathsf{step/tpapp}$$

Booleans bool

$$\frac{}{\Gamma \vdash \mathsf{bool}\ \textit{type}}\ \mathsf{tp/bool}$$

$$\frac{}{\Gamma \vdash \mathsf{true} : \mathsf{bool}}\ \mathsf{tp/true} \qquad \frac{}{\Gamma \vdash \mathsf{false} : \mathsf{bool}}\ \mathsf{tp/false} \qquad \frac{\Gamma \vdash e_1 : \mathsf{bool} \quad \Gamma \vdash e_2 : \tau \quad \Gamma \vdash e_3 : \tau}{\Gamma \vdash \mathsf{if}\ e_1\ e_2\ e_3 : \tau}\ \mathsf{tp/if}$$

$$\frac{}{\mathsf{true}\ \textit{value}}\ \mathsf{val/true} \qquad \frac{}{\mathsf{false}\ \textit{value}}\ \mathsf{val/false}$$

$$\frac{}{\mathsf{if}\ \mathsf{true}\ e_2\ e_3 \mapsto e_2}\ \mathsf{step/if/true} \qquad \frac{}{\mathsf{if}\ \mathsf{false}\ e_2\ e_3 \mapsto e_3}\ \mathsf{step/if/false} \qquad \frac{e_1 \mapsto e_1'}{\mathsf{if}\ e_1\ e_2\ e_3 \mapsto \mathsf{if}\ e_1'\ e_2\ e_3}\ \mathsf{step/if}$$

Pairs $\tau_1 \times \tau_2$

$$\frac{\Gamma \vdash \tau_1\ \mathit{type} \quad \Gamma \vdash \tau_2\ \mathit{type}}{\Gamma \vdash \tau_1 \times \tau_2\ \mathit{type}}\ \mathsf{tp/times}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2}\ \mathsf{tp/pair} \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2 \quad \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash e' : \tau'}{\Gamma \vdash \mathsf{case}\ e\ (\langle x_1, x_2 \rangle \Rightarrow e') : \tau'}\ \mathsf{tp/casep}$$

$$\frac{e_1\ \mathit{value} \quad e_2\ \mathit{value}}{\langle e_1, e_2 \rangle\ \mathit{value}}\ \mathsf{val/pair} \qquad \frac{v_1\ \mathit{value} \quad v_2\ \mathit{value}}{\mathsf{case}\ \langle v_1, v_2 \rangle\ (\langle x_1, x_2 \rangle \Rightarrow e_3) \mapsto [v_1/x_2][v_2/x_2]e_3}\ \mathsf{step/casep/pair}$$

$$\frac{e_1 \mapsto e_1'}{\langle e_1, e_2 \rangle \mapsto \langle e_1', e_2 \rangle}\ \mathsf{step/pair}_1 \qquad \frac{v_1\ \mathit{value} \quad e_2 \mapsto e_2'}{\langle v_1, e_2 \rangle \mapsto \langle v_1, e_2' \rangle}\ \mathsf{step/pair}_2$$

$$\frac{e_0 \mapsto e_0'}{\mathsf{case}\ e_0\ (\langle x_1, x_2 \rangle \Rightarrow e_3) \mapsto \mathsf{case}\ e_0'\ (\langle x_1, x_2 \rangle \Rightarrow e_3)}\ \mathsf{step/casep}_0$$

Theorems

Canonical Forms Assume $\cdot \vdash e : \tau$ and e *value*. Then

(i) If $\tau = \tau_1 \to \tau_2$ then $v = \lambda x.\, e_2$ for some e_2.

(ii) If $\tau = \forall \alpha.\, \tau'$ then $e = \Lambda \alpha.\, e'$ for some e'.

(iii) If $\tau = \mathsf{bool}$ then $e = \mathsf{true}$ or $e = \mathsf{false}$.

(iv) If $\tau = \tau_1 \times \tau_2$ then $e = \langle e_1, e_2 \rangle$ for some e_1 *value* and e_2 *value*.

Preservation. If $\cdot \vdash e : \tau$ and $e \mapsto e'$ then $\cdot \vdash e' : \tau$.

Progress. If $\cdot \vdash e : \tau$ then either $e \mapsto e'$ for some e' or e *value*.

Finality of Values. There is no $\cdot \vdash e : \tau$ such that $e \mapsto e'$ for some e' and e *value*.

Determinacy. If $e \mapsto e_1$ and $e \mapsto e_2$ then $e_1 = e_2$.