Lecture Notes on
Judgments and Propositions

15-816: Modal Logic
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Lecture 1
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1 Introduction to This Course

Logic is the study of reasoning. Since most mathematicians believe they are working on the discovery of objective and absolute truths, mathematical logic has focused on how to justify judgments *A is true* and *A is false*, where A denotes propositions about mathematical objects such as integers or real numbers. Every proposition is objectively either true or false, and logical connectives are functions on truth values.

Philosophical logic takes a broader view and investigates how to reason about complex judgments such as *A is possible*, *A is necessary*, *K knows A*, *K believes A*, *K affirms A* and even *A is obligatory* or *A is permitted*, where K denotes agents or principals and A denotes propositions. Modal operators such as "*K knows*" can not be merely functions on truth values. For example, whether K knows A does not depend solely on whether A is true or false. Judgments have a subjective quality which is generally denied in mathematical logic.

In computer science, both objective and subjective views on logic have numerous applications. The objective or *classical* approach, most influenced by mathematics, is exemplified by Hoare logic to reason about imperative programs. Classical program logic defines programs as a new kind of mathematical object, their meaning being given explicitly as functions from states to states. The classical approach then develops inference rules for reasoning about properties of programs. The subjective or *intuitionistic* approach is exemplified by constructive type theory where the very definition of the logical connectives is tied to their computational interpretation.

For example, the meaning of the implication $A \supset B$ might be given as a (computable) function that maps proofs of A to proofs of B.

In my opinion it is extremely unfortunate that, historically, the study of modal operators has been carried out almost exclusively with classical means. Rather than embracing a subjective, intuitionistic point of view which is in harmony with the meaning of the modal operators (which are not truth-functional in nature, after all), researchers have attempted to reduce meaning to truth values anyway, taking an underlying classical logic as axiomatic. This enterprise has been only partially successful, and many problems remain in particular in first-order and higher-order modal logic.

In this course we will pursue both: We will study classical modal logic with classical means, and intuitionistic modal logic with intuitionistic means.[1] But we hope to deliver more than two separate interleaved courses by elucidating the many deep connections between these schools of thought. For example, Gödel's interpretation of intuitionistic logic in classical modal logic, and Kolmogorov's interpretation of classical logic in intuitionistic logic provide means for a classical mathematician to understand intuitionistic logic and vice versa. To make this course feasible we focus on systems that are particularly relevant to computer science.

We hope that students will come away from this course with a working knowledge of modal logic and its applications in computer science. They should be able to confidently apply techniques from modal logic to problems in their area of research, be it in the use of classical modal logic for verification, or intuitionistic modal logic to capture interesting computational phenomena. They should be able to apply existing modal logics where appropriate and design new logical systems when necessary and rigorously analyze their properties.

2 Introduction to This Lecture

The goal of this first lecture is to develop the two principal notions of logic, namely *propositions* and *proofs*. There is no universal agreement about the proper foundations for these notions. One approach, which has been particularly successful for applications in computer science, is to understand the meaning of a proposition by understanding its proofs. In the words of Martin-Löf [ML96, Page 27]:

[1] Not coincidentally, this course is co-taught by practitioners steeped in these distinct traditions.

> *The meaning of a proposition is determined by [...] what counts as a verification of it.*

A *verification* may be understood as a certain kind of proof that only examines the constituents of a proposition. This is analyzed in greater detail by Dummett [Dum91] although with less direct connection to computer science. The system of inference rules that arises from this point of view is *natural deduction*, first proposed by Gentzen [Gen35] and studied in depth by Prawitz [Pra65].

In this lecture we apply Martin-Löf's approach to explain the basic propositional connectives. We will see later that universal and existential quantifiers and, in particular, modal operators naturally fit into the same framework.

3 Judgments and Propositions

The cornerstone of Martin-Löf's foundation of logic is a clear separation of the notions of judgment and proposition. A *judgment* is something we may know, that is, an object of knowledge. A judgment is *evident* if we in fact know it.

We make a judgment such as "*it is raining*", because we have evidence for it. In everyday life, such evidence is often immediate: we may look out the window and see that it is raining. In logic, we are concerned with situation where the evidence is indirect: we deduce the judgment by making correct inferences from other evident judgments. In other words: a judgment is evident if we have a proof for it.

The most important judgment form in logic is "A *is true*", where A is a proposition. There are many others that have been studied extensively. For example, "A *is false*", "A *is true at time* t" (from temporal logic), "A *is necessarily true*" (from modal logic), "*program* M *has type* τ" (from programming languages), etc.

Returning to the first judgment, let us try to explain the meaning of conjunction. We write A *true* for the judgment "A *is true*" (presupposing that A is a proposition). Given propositions A and B, we can form the compound proposition "A *and* B", written more formally as $A \wedge B$. But we have not yet specified what conjunction *means*, that is, what counts as a verification of $A \wedge B$. This is accomplished by the following inference rule:

$$\frac{A\ \mathit{true} \quad B\ \mathit{true}}{A \wedge B\ \mathit{true}} \wedge I$$

Here the name $\wedge I$ stands for "conjunction introduction", since the conjunction is introduced in the conclusion.

This rule allows us to conclude that $A \wedge B$ *true* if we already know that A *true* and B *true*. In this inference rule, A and B are *schematic variables*, and $\wedge I$ is the name of the rule. The general form of an inference rule is

$$\frac{J_1 \ \ldots \ J_n}{J} \ \textit{name}$$

where the judgments $J_1, \ldots, J_n$ are called the *premises*, the judgment J is called the *conclusion*. In general, we will use letters J to stand for judgments, while A, B, and C are reserved for propositions.

We take conjunction introduction as specifying the meaning of $A \wedge B$ completely. So what can be deduced if we know that $A \wedge B$ is true? By the above rule, to have a verification for $A \wedge B$ means to have verifications for A and B. Hence the following two rules are justified:

$$\frac{A \wedge B \ \textit{true}}{A \ \textit{true}} \ \wedge E_L \qquad\qquad \frac{A \wedge B \ \textit{true}}{B \ \textit{true}} \ \wedge E_R$$

The name $\wedge E_L$ stands for "left conjunction elimination", since the conjunction in the premise has been eliminated in the conclusion. Similarly $\wedge E_R$ stands for "right conjunction elimination".

We will see in Section 8 what precisely is required in order to guarantee that the formation, introduction, and elimination rules for a connective fit together correctly. For now, we will informally argue the correctness of the elimination rules, as we did for the conjunction elimination rules.

As a second example we consider the proposition "*truth*" written as $\top$. Truth should always be true, which means its introduction rule has no premises.

$$\frac{}{\top \ \textit{true}} \ \top I$$

Consequently, we have no information if we know $\top$ *true*, so there is no elimination rule.

A conjunction of two propositions is characterized by one introduction rule with two premises, and two corresponding elimination rules. We may think of truth as a conjunction of zero propositions. By analogy it should then have one introduction rule with zero premises, and zero corresponding elimination rules. This is precisely what we wrote out above.

4 Hypothetical Judgments

Consider the following derivation, for some arbitrary propositions A, B, and C:

$$\cfrac{\cfrac{A \wedge (B \wedge C)\ \mathit{true}}{B \wedge C\ \mathit{true}}\ \wedge E_R}{B\ \mathit{true}}\ \wedge E_L$$

Have we actually proved anything here? At first glance it seems that cannot be the case: B is an arbitrary proposition; clearly we should not be able to prove that it is true. Upon closer inspection we see that all inferences are correct, but the first judgment $A \wedge (B \wedge C)$ *true* has not been justified. We can extract the following knowledge:

> *From the assumption that $A \wedge (B \wedge C)$ is true, we deduce that B must be true.*

This is an example of a *hypothetical judgment*, and the figure above is an *hypothetical deduction*. In general, we may have more than one assumption, so a hypothetical deduction has the form

$$\begin{array}{ccc} J_1 & \cdots & J_n \\ & \vdots & \\ & J & \end{array}$$

where the judgments $J_1, \ldots, J_n$ are unproven assumptions, and the judgment J is the conclusion. Note that we can always substitute a proof for any hypothesis J_i to eliminate the assumption. We call this the *substitution principle* for hypotheses.

Many mistakes in reasoning arise because dependencies on some hidden assumptions are ignored. When we need to be explicit, we will write $J_1, \ldots, J_n \vdash J$ for the hypothetical judgment which is established by the hypothetical deduction above. We may refer to $J_1, \ldots, J_n$ as the antecedents and J as the succedent of the hypothetical judgment.

One has to keep in mind that hypotheses may be used more than once, or not at all. For example, for arbitrary propositions A and B,

$$\cfrac{\cfrac{A \wedge B\ \mathit{true}}{B\ \mathit{true}}\ \wedge E_R \qquad \cfrac{A \wedge B\ \mathit{true}}{A\ \mathit{true}}\ \wedge E_L}{B \wedge A\ \mathit{true}}\ \wedge I$$

can be seen a hypothetical derivation of $A \wedge B$ *true* $\vdash B \wedge A$ *true*.

With hypothetical judgments, we can now explain the meaning of implication "*A implies B*" or "*if A then B*" (more formally: $A \supset B$). The introduction rule reads: $A \supset B$ is true, if B is true under the assumption that A is true.

$$\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}\ u \\ \vdots \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\ {\supset}I^u$$

The tricky part of this rule is the label u. If we omit this annotation, the rule would read

$$\dfrac{\begin{array}{c}A\ \mathit{true} \\ \vdots \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\ {\supset}I$$

which would be incorrect: it looks like a derivation of $A \supset B$ *true* from the hypothesis A *true*. But the assumption A *true* is introduced in the process of proving $A \supset B\ true$; the conclusion should not depend on it! Therefore we label uses of the assumption with a new name u, and the corresponding inference which introduced this assumption into the derivation with the same label u.

As a concrete example, consider the following proof of $A \supset (B \supset (A \wedge B))$.

$$\dfrac{\dfrac{\dfrac{\overline{A\ \mathit{true}}\ u \quad \overline{B\ \mathit{true}}\ w}{A \wedge B\ \mathit{true}}\ {\wedge}I}{B \supset (A \wedge B)\ \mathit{true}}\ {\supset}I^w}{A \supset (B \supset (A \wedge B))\ \mathit{true}}\ {\supset}I^u$$

Note that this derivation is not hypothetical (it does not depend on any assumptions). The assumption A *true* labeled u is discharged in the last inference, and the assumption B *true* labeled w is discharged in the second-to-last inference. It is critical that a discharged hypothesis is no longer available for reasoning, and that all labels introduced in a derivation are distinct.

Finally, we consider what the elimination rule for implication should say. By the only introduction rule, having a proof of $A \supset B$ *true* means that we have a hypothetical proof of B *true* from A *true*. By the substitution principle, if we also have a proof of A *true* then we get a proof of B *true*.

$$\frac{A \supset B\ \mathit{true} \quad A\ \mathit{true}}{B\ \mathit{true}}\supset\! E$$

This completes the rules concerning implication.

With the rules so far, we can write out proofs of simple properties concerning conjunction and implication. The first expresses that conjunction is commutative—intuitively, an obvious property.

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A \wedge B\ \mathit{true}}\,u}{B\ \mathit{true}}\wedge E_R \quad \cfrac{\cfrac{}{A \wedge B\ \mathit{true}}\,u}{A\ \mathit{true}}\wedge E_L}{B \wedge A\ \mathit{true}}\wedge I}{(A \wedge B) \supset (B \wedge A)\ \mathit{true}}\supset\! I^u$$

When we construct such a derivation, we generally proceed by a combination of bottom-up and top-down reasoning. The next example is a distributivity law, allowing us to move implications over conjunctions. This time, we show the partial proofs in each step. Of course, other sequences of steps in proof constructions are also possible.

$$\begin{array}{c}\vdots \\ (A \supset (B \wedge C)) \supset ((A \supset B) \wedge (A \supset C))\ \mathit{true}\end{array}$$

First, we use the implication introduction rule bottom-up.

$$\cfrac{\begin{array}{c}\cfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \\ \vdots \\ (A \supset B) \wedge (A \supset C)\ \mathit{true}\end{array}}{(A \supset (B \wedge C) \supset ((A \supset B) \wedge (A \supset C))\ \mathit{true}}\supset\! I^u$$

Next, we use the conjunction introduction rule bottom-up.

$$\cfrac{\cfrac{\begin{array}{c}\cfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \\ \vdots \\ A \supset B\ \mathit{true}\end{array} \quad \begin{array}{c}\cfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \\ \vdots \\ A \supset C\ \mathit{true}\end{array}}{(A \supset B) \wedge (A \supset C)\ \mathit{true}}\wedge I}{(A \supset (B \wedge C)) \supset ((A \supset B) \wedge (A \supset C))\ \mathit{true}}\supset\! I^u$$

We now pursue the left branch, again using implication introduction bottom-up.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\overline{A\supset(B\wedge C)\ \mathit{true}}\;u \quad \overline{A\ \mathit{true}}\;w\\ \vdots \\ B\ \mathit{true}\end{array}}{A\supset B\ \mathit{true}}\supset I^w \qquad \begin{array}{c}\overline{A\supset(B\wedge C)\ \mathit{true}}\;u\\ \vdots\\ A\supset C\ \mathit{true}\end{array}}{(A\supset B)\wedge(A\supset C)\ \mathit{true}}\wedge I}{(A\supset(B\wedge C))\supset((A\supset B)\wedge(A\supset C))\ \mathit{true}}\supset I^u$$

Note that the hypothesis A *true* is available only in the left branch and not in the right one: it is discharged at the inference $\supset I^w$. We now switch to top-down reasoning, taking advantage of implication elimination.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{\overline{A\supset(B\wedge C)\ \mathit{true}}\;u \quad \overline{A\ \mathit{true}}\;w}{B\wedge C\ \mathit{true}}\supset E\\ \vdots \\ B\ \mathit{true}\end{array}}{A\supset B\ \mathit{true}}\supset I^w \qquad \begin{array}{c}\overline{A\supset(B\wedge C)\ \mathit{true}}\;u\\ \vdots\\ A\supset C\ \mathit{true}\end{array}}{(A\supset B)\wedge(A\supset C)\ \mathit{true}}\wedge I}{(A\supset(B\wedge C))\supset((A\supset B)\wedge(A\supset C))\ \mathit{true}}\supset I^u$$

Now we can close the gap in the left-hand side by conjunction elimination.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A\supset(B\wedge C)\ \mathit{true}}\;u \quad \overline{A\ \mathit{true}}\;w}{B\wedge C\ \mathit{true}}\supset E}{B\ \mathit{true}}\wedge E_L}{A\supset B\ \mathit{true}}\supset I^w \qquad \begin{array}{c}\overline{A\supset(B\wedge C)\ \mathit{true}}\;u\\ \vdots\\ A\supset C\ \mathit{true}\end{array}}{(A\supset B)\wedge(A\supset C)\ \mathit{true}}\wedge I}{(A\supset(B\wedge C))\supset((A\supset B)\wedge(A\supset C))\ \mathit{true}}\supset I^u$$

The right premise of the conjunction introduction can be filled in analogously. We skip the intermediate steps and only show the final derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \qquad \dfrac{}{A\ \mathit{true}}\,w}{B \wedge C\ \mathit{true}}\,{\supset}E}{B\ \mathit{true}}\,{\wedge}E_L}{A \supset B\ \mathit{true}}\,{\supset}I^w \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \supset (B \wedge C)\ \mathit{true}}\,u \qquad \dfrac{}{A\ \mathit{true}}\,v}{B \wedge C\ \mathit{true}}\,{\supset}E}{C\ \mathit{true}}\,{\wedge}E_R}{A \supset C\ \mathit{true}}\,{\supset}I^v}{(A \supset B) \wedge (A \supset C)\ \mathit{true}}\,{\wedge}I}{(A \supset (B \wedge C)) \supset ((A \supset B) \wedge (A \supset C))\ \mathit{true}}\,{\supset}I^u$$

5 Disjunction and Falsehood

So far we have explained the meaning of conjunction, truth, and implication. The disjunction "*A or B*" (written as $A \vee B$) is more difficult, but does not require any new judgment forms. Disjunction is characterized by two introduction rules: $A \vee B$ is true, if either A or B is true.

$$\dfrac{A\ \mathit{true}}{A \vee B\ \mathit{true}}\,{\vee}I_L \qquad \dfrac{B\ \mathit{true}}{A \vee B\ \mathit{true}}\,{\vee}I_R$$

Now it would be incorrect to have an elimination rule such as

$$\dfrac{A \vee B\ \mathit{true}}{A\ \mathit{true}}\,{\vee}E_L?$$

because even if we know that $A \vee B$ is true, we do not know whether the disjunct A or the disjunct B is true. Concretely, with such a rule we could derive the truth of *every* proposition A as follows:

$$\dfrac{\dfrac{\dfrac{}{\top\ \mathit{true}}\,{\top}I}{A \vee \top\ \mathit{true}}\,{\vee}I_R}{A\ \mathit{true}}\,{\vee}E_L?$$

Thus we take a different approach. If we know that $A \vee B$ is true, we must consider two cases: $A\ \mathit{true}$ and $B\ \mathit{true}$. If we can prove a conclusion $C\ \mathit{true}$ in both cases, then C must be true! Written as an inference rule:

$$\dfrac{A \vee B\ \mathit{true} \qquad \begin{array}{c}\dfrac{}{A\ \mathit{true}}\,u \\ \vdots \\ C\ \mathit{true}\end{array} \qquad \begin{array}{c}\dfrac{}{B\ \mathit{true}}\,w \\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\,{\vee}E^{u,w}$$

Note that we use once again the mechanism of hypothetical judgments. In the proof of the second premise we may use the assumption A *true* labeled u, in the proof of the third premise we may use the assumption B *true* labeled w. Both are discharged at the disjunction elimination rule.

Let us justify the conclusion of this rule more explicitly. By the first premise we know $A \lor B$ *true*. The premises of the two possible introduction rules are A *true* and B *true*. In case A *true* we conclude C *true* by the substitution principle and the second premise: we substitute the proof of $A\ \mathit{true}$ for any use of the assumption labeled u in the hypothetical derivation. The case for B *true* is symmetric, using the hypothetical derivation in the third premise.

Because of the complex nature of the elimination rule, reasoning with disjunction is more difficult than with implication and conjunction. As a simple example, we prove the commutativity of disjunction.

$$\begin{array}{c} \vdots \\ (A \lor B) \supset (B \lor A)\ \mathit{true} \end{array}$$

We begin with an implication introduction.

$$\dfrac{\begin{array}{c} \overline{A \lor B\ \mathit{true}}^{\,u} \\ \vdots \\ B \lor A\ \mathit{true} \end{array}}{(A \lor B) \supset (B \lor A)\ \mathit{true}} \supset I^u$$

At this point we cannot use either of the two disjunction introduction rules. The problem is that neither B nor A follow from our assumption $A \lor B$! So first we need to distinguish the two cases via the rule of disjunction elimination.

$$\dfrac{\dfrac{\overline{A \lor B\ \mathit{true}}^{\,u} \quad \begin{array}{c} \overline{A\ \mathit{true}}^{\,v} \\ \vdots \\ B \lor A\ \mathit{true} \end{array} \quad \begin{array}{c} \overline{B\ \mathit{true}}^{\,w} \\ \vdots \\ B \lor A\ \mathit{true} \end{array}}{B \lor A\ \mathit{true}} \lor E^{v,w}}{(A \lor B) \supset (B \lor A)\ \mathit{true}} \supset I^u$$

The assumption labeled u is still available for each of the two proof obligations, but we have omitted it, since it is no longer needed.

Now each gap can be filled in directly by the two disjunction introduction rules.

$$\dfrac{\dfrac{}{A \vee B\ \mathit{true}}\,u \quad \dfrac{\dfrac{}{A\ \mathit{true}}\,v}{B \vee A\ \mathit{true}}\,{\vee}I_R \quad \dfrac{\dfrac{}{B\ \mathit{true}}\,w}{B \vee A\ \mathit{true}}\,{\vee}I_L}{\dfrac{B \vee A\ \mathit{true}}{(A \vee B) \supset (B \vee A)\ \mathit{true}}\,{\supset}I^u}\,{\vee}E^{v,w}$$

This concludes the discussion of disjunction. Falsehood (written as $\bot$, sometimes called absurdity) is a proposition that should have no proof! Therefore there are no introduction rules.

Since there cannot be a proof of $\bot$ *true*, it is sound to conclude the truth of any arbitrary proposition if we know $\bot$ *true*. This justifies the elimination rule

$$\dfrac{\bot\ \mathit{true}}{C\ \mathit{true}}\,\bot E$$

We can also think of falsehood as a disjunction between zero alternatives. By analogy with the binary disjunction, we therefore have zero introduction rules, and an elimination rule in which we have to consider zero cases. This is precisely the $\bot E$ rule above.

From this is might seem that falsehood it useless: we can never prove it. This is correct, except that we might reason from contradictory hypotheses! We will see some examples when we discuss negation, since we may think of the proposition "*not* A" (written $\neg A$) as $A \supset \bot$. In other words, $\neg A$ is true precisely if the assumption A *true* is contradictory because we could derive $\bot$ *true*.

6 Natural Deduction

The judgments, propositions, and inference rules we have defined so far collectively form a system of *natural deduction*. It is a minor variant of a system introduced by Gentzen [Gen35] and studied in depth by Prawitz [Pra65]. One of Gentzen's main motivations was to devise rules that model mathematical reasoning as directly as possible, although clearly in much more detail than in a typical mathematical argument.

The specific interpretation of the truth judgment underlying these rules is *intuitionistic* or *constructive*. This differs from the *classical* or *Boolean* interpretation of truth. For example, classical logic accepts the proposition $A \vee (A \supset B)$ as true for arbitrary A and B, although in the system we have

Introduction Rules | **Elimination Rules**

$$\frac{A\ \mathit{true} \quad B\ \mathit{true}}{A \wedge B\ \mathit{true}}\ \wedge I \qquad\qquad \frac{A \wedge B\ \mathit{true}}{A\ \mathit{true}}\ \wedge E_L \quad \frac{A \wedge B\ \mathit{true}}{B\ \mathit{true}}\ \wedge E_R$$

$$\frac{}{\top\ \mathit{true}}\ \top I \qquad\qquad \mathit{no}\ \top E\ \mathit{rule}$$

$$\frac{\begin{array}{c}\overline{A\ \mathit{true}}\ u\\ \vdots \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\ {\supset} I^u \qquad\qquad \frac{A \supset B\ \mathit{true} \quad A\ \mathit{true}}{B\ \mathit{true}}\ {\supset} E$$

$$\frac{A\ \mathit{true}}{A \vee B\ \mathit{true}}\ \vee I_L \quad \frac{B\ \mathit{true}}{A \vee B\ \mathit{true}}\ \vee I_R \qquad \frac{A \vee B\ \mathit{true} \quad \begin{array}{c}\overline{A\ \mathit{true}}\ u\\ \vdots \\ C\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{B\ \mathit{true}}\ w\\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\ \vee E^{u,w}$$

$$\mathit{no}\ \bot I\ \mathit{rule} \qquad\qquad \frac{\bot\ \mathit{true}}{C\ \mathit{true}}\ \bot E$$

Figure 1: Rules for intuitionistic natural deduction

presented so far this would have no proof. Classical logic is based on the principle that every proposition must be true or false. If we distinguish these cases we see that $A \vee (A \supset B)$ should be accepted, because in case that A is true, the left disjunct holds; in case A is false, the right disjunct holds. In contrast, intuitionistic logic is based on explicit evidence, and evidence for a disjunction requires evidence for one of the disjuncts. We will return to classical logic and its relationship to intuitionistic logic later; for now our reasoning remains intuitionistic since, as we will see in Lecture 2, it has a direct connection to functional computation, which classical logic lacks.

We summarize the rules of inference for the truth judgment introduced so far in Figure 1.

7 Notational Definition

So far, we have defined the meaning of the logical connectives by their introduction rules, which is the so-called *verificationist* approach. Another common way to define a logical connective is by a *notational definition*. A notational definition gives the meaning of the general form of a proposition in terms of another proposition whose meaning has already been defined. For example, we can define *logical equivalence*, written $A \equiv B$ as $(A \supset B) \wedge (B \supset A)$. This definition is justified, because we already understand implication and conjunction.

As mentioned above, another common notational definition in intuitionistic logic is $\neg A = (A \supset \bot)$. Several other, more direct definitions of intuitionistic negation also exist, and we will see some of them later in the course. Perhaps the most intuitive one is to say that $\neg A$ *true* if A *false*, but this requires the new judgment of falsehood.

Notational definitions can be convenient, but they can be a bit cumbersome at times. We sometimes give a notational definition and then derive introduction and elimination rules for the connective. It should be understood that these rules, even if they may be called introduction or elimination rules, have a different status from those that define a connective.

8 Harmony

In the verificationist definition of the logical connectives via their introduction rules we have briefly justified the elimination rules. In this section we study the balance between introduction and elimination rules more closely. In order to show that the two are in harmony we establish two properties: *local soundness* and *local completeness*.

Local soundness shows that the elimination rules are not too strong: no matter how we apply elimination rules to the result of an introduction we cannot gain any new information. We demonstrate this by showing that we can find a more direct proof of the conclusion of an elimination than one that first introduces and then eliminates the connective in question. This is witnessed by a *local reduction* of the given introduction and the subsequent elimination.

Local completeness shows that the elimination rules are not too weak: there is always a way to apply elimination rules so that we can reconstitute a proof of the original proposition from the results by applying introduction rules. This is witnessed by a *local expansion* of an arbitrary given derivation into one that introduces the primary connective.

Connectives whose introduction and elimination rules are in harmony in the sense that they are locally sound and complete are properly defined from the verificationist perspective. If not, the proposed connective should be viewed with suspicion. Another criterion we would like to apply uniformly is that both introduction and elimination rules do not refer to other propositional constants or connectives (besides the one we are trying to define), which could create a dangerous dependency of the various connectives on each other. As we present correct definitions we will occasionally also give some counterexamples to illustrate the consequences of violating the principles behind the patterns of valid inference.

In the discussion of each individual connective below we use the notation

$$\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \Longrightarrow_R \begin{array}{c}\mathcal{D}'\\ A\ \mathit{true}\end{array}$$

for the local reduction of a deduction $\mathcal{D}$ to another deduction $\mathcal{D}'$ of the same judgment A *true*. In fact, $\Longrightarrow_R$ can itself be considered a higher level judgment relating two proofs, $\mathcal{D}$ and $\mathcal{D}'$, although we will not directly exploit this point of view. Similarly,

$$\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \Longrightarrow_E \begin{array}{c}\mathcal{D}'\\ A\ \mathit{true}\end{array}$$

is the notation of the local expansion of $\mathcal{D}$ to $\mathcal{D}'$.

Conjunction. We start with local soundness. Since there are two elimination rules and one introduction, it turns out we have two cases to consider. In either case, we can easily reduce.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \quad \begin{array}{c}\mathcal{E}\\ B\ \mathit{true}\end{array}}{A \wedge B\ \mathit{true}}\wedge I}{A\ \mathit{true}}\wedge E_L \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \quad \begin{array}{c}\mathcal{E}\\ B\ \mathit{true}\end{array}}{A \wedge B\ \mathit{true}}\wedge I}{B\ \mathit{true}}\wedge E_R \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{E}\\ B\ \mathit{true}\end{array}$$

Local completeness requires us to apply eliminations to an arbitrary proof of $A \wedge B$ *true* in such a way that we can reconstitute a proof of $A \wedge B$

from the results.

$$\begin{array}{c}\mathcal{D}\\ A \wedge B \; \mathit{true}\end{array} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B \; \mathit{true}\end{array}}{A \; \mathit{true}} \wedge E_L \quad \dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B \; \mathit{true}\end{array}}{B \; \mathit{true}} \wedge E_R}{A \wedge B \; \mathit{true}} \wedge I$$

As an example where local completeness might fail, consider the case where we "forget" the right elimination rule for conjunction. The remaining rule is still locally sound, but not locally complete because we cannot extract a proof of B from the assumption $A \wedge B$. Now, for example, we cannot prove $(A \wedge B) \supset (B \wedge A)$ even though this should clearly be true.

Substitution Principle. We need the defining property for hypothetical judgments before we can discuss implication. Intuitively, we can always substitute a deduction of A *true* for any use of a hypothesis A *true*. In order to avoid ambiguity, we make sure assumptions are labelled and we substitute for all uses of an assumption with a given label. Note that we can only substitute for assumptions that are not discharged in the subproof we are considering. The substitution principle then reads as follows:

If

$$\begin{array}{c}\overline{A \; \mathit{true}}\; u\\ \mathcal{E}\\ C \; \mathit{true}\end{array}$$

is a hypothetical proof of C *true* under the undischarged hypothesis A *true* labelled u, and

$$\begin{array}{c}\mathcal{D}\\ A \; \mathit{true}\end{array}$$

is a proof of A *true* then

$$\begin{array}{c}\dfrac{\mathcal{D}}{A \; \mathit{true}}\; u\\ \mathcal{E}\\ C \; \mathit{true}\end{array}$$

is our notation for substituting $\mathcal{D}$ for all uses of the hypothesis labelled u in $\mathcal{E}$. This deduction, also sometime written as $[\mathcal{D}/u]\mathcal{E}$ no longer depends on u.

Implication. To witness local soundness, we reduce an implication introduction followed by an elimination using the substitution operation.

$$\dfrac{\dfrac{\begin{array}{c}\overline{A\ true}^{\,u}\\ \mathcal{E}\\ B\ true\end{array}}{A \supset B\ true}\supset I^u \quad \begin{array}{c}\mathcal{D}\\ A\ true\end{array}}{B\ true}\supset E \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{D}\\ \overline{A\ true}^{\,u}\\ \mathcal{E}\\ B\ true\end{array}$$

The conditions on the substitution operation is satisfied, because u is introduced at the $\supset I^u$ inference and therefore not discharged in $\mathcal{E}$.

Local completeness is witnessed by the following expansion.

$$\begin{array}{c}\mathcal{D}\\ A \supset B\ true\end{array} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \supset B\ true\end{array} \quad \overline{A\ true}^{\,u}}{B\ true}\supset E}{A \supset B\ true}\supset I^u$$

Here u must be chosen fresh: it only labels the new hypothesis *A true* which is used only once.

Disjunction. For disjunction we also employ the substitution principle because the two cases we consider in the elimination rule introduce hypotheses. Also, in order to show local soundness we have two possibilities for the introduction rule, in both situations followed by the only elimination rule.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ true\end{array}}{A \vee B\ true}\vee I_L \quad \begin{array}{c}\overline{A\ true}^{\,u}\\ \mathcal{E}\\ C\ true\end{array} \quad \begin{array}{c}\overline{B\ true}^{\,w}\\ \mathcal{F}\\ C\ true\end{array}}{C\ true}\vee E^{u,w} \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{D}\\ \overline{A\ true}^{\,u}\\ \mathcal{E}\\ C\ true\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ B\ true\end{array}}{A \vee B\ true}\vee I_R \quad \begin{array}{c}\overline{A\ true}^{\,u}\\ \mathcal{E}\\ C\ true\end{array} \quad \begin{array}{c}\overline{B\ true}^{\,w}\\ \mathcal{F}\\ C\ true\end{array}}{C\ true}\vee E^{u,w} \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{D}\\ \overline{B\ true}^{\,w}\\ \mathcal{F}\\ C\ true\end{array}$$

An example of a rule that would not be locally sound is

$$\dfrac{A \vee B\ true}{A\ true}\vee E_L?$$

and, indeed, we would not be able to reduce

$$\dfrac{\dfrac{B\ \mathit{true}}{A \vee B\ \mathit{true}}\ \vee I_R}{A\ \mathit{true}}\ \vee E_L?$$

In fact, as noted before, we can now derive a contradiction from no assumption, which means the whole system is incorrect.

$$\dfrac{\dfrac{\dfrac{}{\top\ \mathit{true}}\ \top I}{\bot \vee \top\ \mathit{true}}\ \vee I_R}{\bot\ \mathit{true}}\ \vee E_L?$$

Local completeness of disjunction distinguishes cases on the known $A \vee B\ \mathit{true}$, using $A \vee B\ \mathit{true}$ as the conclusion.

$$\begin{array}{c}\mathcal{D}\\ A \vee B\ \mathit{true}\end{array} \Longrightarrow_E \dfrac{\begin{array}{c}\mathcal{D}\\ A \vee B\ \mathit{true}\end{array} \quad \dfrac{\dfrac{}{A\ \mathit{true}}\ u}{A \vee B\ \mathit{true}}\ \vee I_L \quad \dfrac{\dfrac{}{B\ \mathit{true}}\ w}{A \vee B\ \mathit{true}}\ \vee I_R}{A \vee B\ \mathit{true}}\ \vee E^{u,w}$$

Visually, this looks somewhat different from the local expansions for conjunction or implication. It looks like the elimination rule is applied last, rather than first. Mostly, this is due to the notation of natural deduction: the above represents the step from using the knowledge of $A \vee B\ \mathit{true}$ and eliminating it to obtain the hypotheses $A\ \mathit{true}$ and $B\ \mathit{true}$ in the two cases.

Truth. The local constant $\top$ has only an introduction rule, but no elimination rule. Consequently, there are no cases to check for local soundness: any introduction followed by any elimination can be reduced.

However, local completeness still yields a local expansion: Any proof of $\top\ \mathit{true}$ can be trivially converted to one by $\top I$.

$$\begin{array}{c}\mathcal{D}\\ \top\ \mathit{true}\end{array} \Longrightarrow_E \dfrac{}{\top\ \mathit{true}}\ \top I$$

Falsehood. As for truth, there is no local reduction because local soundness is trivially satisfied since we have no introduction rule.

Local completeness is slightly tricky. Literally, we have to show that there is a way to apply an elimination rule to any proof of $\bot\ \mathit{true}$ so that we can reintroduce a proof of $\bot$ *true* from the result. However, there will be zero cases to consider, so we apply no introductions. Nevertheless, the following is the right local expansion.

$$\begin{array}{c}\mathcal{D}\\ \bot\ \mathit{true}\end{array} \quad \Longrightarrow_E \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \bot\ \mathit{true}\end{array}}{\bot\ \mathit{true}}\ \bot E$$

Reasoning about situation when falsehood is true may seem vacuous, but is common in practice because it corresponds to reaching a contradiction. In intuitionistic reasoning, this occurs when we prove $A \supset \bot$ which is often abbreviated as $\neg A$. In classical reasoning it is even more frequent, due to the rule of proof by contradiction.

9 Verifications and Uses

The verificationist point of view on the meaning of a proposition is that it is determined by its *verifications*. Intuitively, a verification should be a proof that only analyzes the constituents of a proposition. This restriction of the space of all possible proofs is necessary so that the definition is well-founded. For example, if in order to understand the meaning of A, we would have to understand the meaning of $B \supset A$ and B, the whole program of understanding the meaning of the connectives by their proofs is in jeopardy because B could be a proposition containing, say, A. But the meaning of A would then in turn depend on the meaning of A, creating a vicious cycle.

In this section we will make the structure of verifications more explicit. We write $A\uparrow$ for the judgment "*A has a verification*". Naturally, this should mean that A is true, and that the evidence for that has a special form. Eventually we will also establish the converse: if A is true than A has a verification.

Conjunction is easy to understand. A verification of $A \wedge B$ should consist of a verification of A and a verification of B.

$$\dfrac{A\uparrow \quad B\uparrow}{A \wedge B\uparrow}\ \wedge I$$

We reuse here the names of the introduction rule, because this rule is strictly analogous to the introduction rule for the truth of a conjunction.

Implication, however, introduces a new hypothesis which is not explicitly justified by an introduction rule but just a new label. For example, in the proof

$$\dfrac{\dfrac{\overline{A \wedge B\ \mathit{true}}\ u}{A\ \mathit{true}}\ \wedge E_L}{(A \wedge B) \supset A\ \mathit{true}}\ \supset I^u$$

the conjunction $A \wedge B$ is not justified by an introduction.

The informal discussion of proof search strategies earlier, namely to use introduction rules from the bottom up and elimination rules from the top down contains the answer. We introduce a second judgment, $A{\downarrow}$ which means "*A may be used*". $A{\downarrow}$ should be the case when either A *true* is a hypothesis, or A is deduced from a hypothesis via elimination rules. Our local soundness arguments provide some evidence that we cannot deduce anything incorrect in this manner.

We now go through the connectives in turn, defining verifications and uses.

Conjunction. In summary of the discussion above, we obtain:

$$\dfrac{A{\uparrow} \quad B{\uparrow}}{A \wedge B{\uparrow}}\ \wedge I \qquad \dfrac{A \wedge B{\downarrow}}{A{\downarrow}}\ \wedge E_L \qquad \dfrac{A \wedge B{\downarrow}}{B{\downarrow}}\ \wedge E_R$$

The left elimination rule can be read as: "*If we can use $A \wedge B$ we can use A*", and similarly for the right elimination rule.

Implication. The introduction rule creates a new hypothesis, which we may use in a proof. The assumption is therefore of the judgment $A{\downarrow}$

$$\dfrac{\begin{array}{c}\overline{A{\downarrow}}\ u\\ \vdots\\ B{\uparrow}\end{array}}{A \supset B{\uparrow}}\ \supset^u$$

In order to use an implication $A \supset B$ we require a verification of A. Just requiring that A may be used would be too weak, as can be seen when trying to prove $((A \supset A) \supset B) \supset B{\uparrow}$. It should also be clear from the fact that we are not eliminating a connective from A.

$$\dfrac{A \supset B{\downarrow} \quad A{\uparrow}}{B{\downarrow}}\ \supset E$$

Disjunction. The verifications of a disjunction immediately follow from their introduction rules.

$$\frac{A\uparrow}{A \vee B\uparrow}\ \vee I_L \qquad \frac{B\uparrow}{A \vee B\uparrow}\ \vee I_R$$

A disjunction is used in a proof by cases, called here $\vee E$. This introduces two new hypotheses, and each of them may be used in the corresponding subproof. Whenever we set up a hypothetical judgment we are trying to find a verification of the conclusion, possibly with uses of hypotheses. So the conclusion of $\vee E$ should be a verification.

$$\cfrac{A \vee B\downarrow \quad \begin{array}{c}\overline{A\downarrow}\ u \\ \vdots \\ C\uparrow\end{array} \quad \begin{array}{c}\overline{B\downarrow}\ w \\ \vdots \\ C\uparrow\end{array}}{C\uparrow}\ \vee E^{u,w}$$

Truth. The only verification of truth is the trival one.

$$\frac{}{\top\uparrow}\ \top I$$

A hypothesis $\top\downarrow$ cannot be used because there is no elimination rule for $\top$.

Falsehood. There is no verification of falsehood because we have no introduction rule.

We can use falsehood, signifying a contradiction from our current hypotheses, to verify any conclusion. This is the zero-ary case of a disjunction.

$$\frac{\bot\downarrow}{C\uparrow}\ \bot E$$

Atomic propositions. How do we construct a verification of an atomic proposition P? We cannot break down the structure of P because there is none, so we can only proceed if we already know P is true. This can only come from a hypothesis, so we have a rule that lets us use the knowledge of an atomic proposition to construct a verification.

$$\frac{P\downarrow}{P\uparrow}\ \downarrow\uparrow$$

This rule has a special status in that it represents a change in judgments but is not tied to a particular local connective. We call this a *judgmental rule* in order to distinguish it from the usual introduction and elimination rules that characterize the connectives.

Global soundness. Local soundness is an intrinsic property of each connective, asserting that the elimination rules for it are not too strong given the introduction rules. Global soundness is its counterpart for the whole system of inference rules. It says that if an arbitrary proposition A has a verification then we may use A without gaining any information. That is, for arbitrary propositions A and C:

$$\textit{If} \quad A\uparrow \quad \textit{and} \quad \begin{array}{c} A\downarrow \\ \vdots \\ C\uparrow \end{array} \quad \textit{then} \quad C\uparrow.$$

We would want to prove this using a substitution principle, except that the judgment $A\uparrow$ and $A\downarrow$ do not match.

Global completeness. Local completeness is also an intrinsic property of each connective. It asserts that the elimination rules are not too weak, given the introduction rule. Global completeness is its counterpart for the whole system of inference rules. It says that if we may use A then we can construct from this a verification of A. That is, for arbitrary propositions A:

$$\begin{array}{c} A\downarrow \\ \vdots \\ A\uparrow. \end{array}$$

Global completeness follows from local completeness rather directly by induction on the structure of A.

Global soundness and completeness are properties of whole deductive systems. Their proof must be carried out in a mathematical *metalanguage* which makes them a bit different than the formal proofs that we have done so far within natural deduction. Of course, we would like them to be correct as well, which means they should follow the same principles of valid inference that we have laid out so far.

There are two further properties we would like, relating truth, verifications, and uses. The first is that if A has a verification or A may be used, then A is true. This is rather evident since we have just specialized the introduction and elimination rules, except for the judgmental rule $\downarrow\uparrow$. But under the interpretation of verification and use as truth, this inference becomes redundant.

Significantly more difficult is the property that if A is true then A has a verification. Since we justified the meaning of the connectives from their verifications, a failure of this property would be devastating to the verificationist program. Fortunately it holds and can be proved by exhibiting a process of *proof normalization* that takes an arbitrary proof of A true and constructs a verification of A.

All these properties in concert show that our rules are well constructed, locally as well as globally. Experience with many other logical systems indicates that this is not an isolated phenomenon: we can employ the verificationist point of view to give coherent sets of rules not just for constructive logic, but for classical logic, temporal logic, spatial logic, modal logic, and many other logics that are of interest in computer science. Taken together, these constitute strong evidence that separating judgments from propositions and taking a verificationist point of view in the definition of the logical connectives is indeed a proper and useful foundation for logic.

10 Derived Rules of Inference

One popular device for shortening derivations is to introduce *derived rules of inference*. For example,

$$\frac{A \supset B\ \mathit{true} \quad B \supset C\ \mathit{true}}{A \supset C\ \mathit{true}}$$

is a derived rule of inference. Its derivation is the following:

$$\dfrac{\dfrac{B \supset C\ \mathit{true} \qquad \dfrac{A \supset B\ \mathit{true} \quad \dfrac{}{A\ \mathit{true}}\,u}{B\ \mathit{true}}\supset\!E}{C\ \mathit{true}}\supset\!E}{A \supset C\ \mathit{true}}\supset\!I^u$$

Note that this is simply a hypothetical deduction, using the premises of the derived rule as assumptions. In other words, a derived rule of inference is nothing but an evident hypothetical judgment; its justification is a hypothetical deduction.

We can freely use derived rules in proofs, since any occurrence of such a rule can be expanded by replacing it with its justification.

11 Logical Equivalences

We now consider several classes of logical equivalences in order to develop some intuitions regarding the truth of propositions. Each equivalence has the form $A \equiv B$, but we consider only the basic connectives and constants ($\wedge$, $\supset$, $\vee$, $\top$, $\bot$) in A and B. Later on we consider negation as a special case. We use some standard conventions that allow us to omit some parentheses while writing propositions. We use the following operator precedences

$$\neg > \wedge > \vee > \supset > \equiv$$

where $\wedge$, $\vee$, and $\supset$ are right associative. For example

$$\neg A \supset A \vee \neg\neg A \supset \bot$$

stands for

$$(\neg A) \supset ((A \vee (\neg(\neg A))) \supset \bot)$$

In ordinary mathematical usage, $A \equiv B \equiv C$ stands for $(A \equiv B) \wedge (B \equiv C)$; in the formal language we do not allow iterated equivalences without explicit parentheses in order to avoid confusion with propositions such as $(A \equiv A) \equiv \top$.

Commutativity. Conjunction and disjunction are clearly commutative, while implication is not.

(C1) $A \wedge B \equiv B \wedge A$ *true*

(C2) $A \vee B \equiv B \vee A$ *true*

(C3) $A \supset B$ is not commutative

Idempotence. Conjunction and disjunction are idempotent, while self-implication reduces to truth.

(I1) $A \wedge A \equiv A$ *true*

(I2) $A \vee A \equiv A$ *true*

(I3) $A \supset A \equiv \top$ *true*

Interaction Laws. These involve two interacting connectives. In principle, there are left and right interaction laws, but because conjunction and disjunction are commutative, some coincide and are not repeated here.

(L1) $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$ *true*

(L2) $A \wedge \top \equiv A$ *true*

(L3) $A \wedge (B \supset C)$ do not interact

(L4) $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ *true*

(L5) $A \wedge \bot \equiv \bot$ *true*

(L6) $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ *true*

(L7) $A \vee \top \equiv \top$ *true*

(L8) $A \vee (B \supset C)$ do not interact

(L9) $A \vee (B \vee C) \equiv (A \vee B) \vee C$ *true*

(L10) $A \vee \bot \equiv A$ *true*

(L11) $A \supset (B \wedge C) \equiv (A \supset B) \wedge (A \supset C)$ *true*

(L12) $A \supset \top \equiv \top$ *true*

(L13) $A \supset (B \supset C) \equiv (A \wedge B) \supset C$ *true*

(L14) $A \supset (B \vee C)$ do not interact

(L15) $A \supset \bot$ do not interact

(L16) $(A \wedge B) \supset C \equiv A \supset (B \supset C)$ *true*

(L17) $\top \supset C \equiv C$ *true*

(L18) $(A \supset B) \supset C$ do not interact

(L19) $(A \vee B) \supset C \equiv (A \supset C) \wedge (B \supset C)$ *true*

(L20) $\bot \supset C \equiv \top$ *true*

Exercises

Exercise 1 *Prove interaction laws (L4) and (L6) using natural deduction.*

Exercise 2 *Prove interaction laws (L13), (L19), and (L20) using natural deduction.*

Exercise 3 *Using the definition of $\neg A$ as $A \supset \bot$, derive interaction laws of negation with conjunction, implication, disjunction, truth, falsehood, and negation. If no interaction law exists, indicate which direction of a potential equivalence holds and which does not.*

Exercise 4 *Prove global completeness of the elimination rules with respect to the introduction rules, that is, $A{\downarrow} \vdash A{\uparrow}$ for any proposition A using atomic propositions, conjunction, implication, disjunction, truth, and falsehood.*

Exercise 5 *Proofs of judgments A true are* parametric *in their schematic propositional variables, denoted in this exercise by P, Q, R. For example, if we have a proof*

$$\begin{array}{c} \mathcal{D} \\ P \wedge (P \supset Q) \supset Q \textit{ true} \end{array}$$

then we can substitute other propositions for P and Q in $\mathcal{D}$ and obtain other valid proofs. For example:

$$\begin{array}{c} [\top/P][(R \vee S)/Q]\mathcal{D} \\ \top \wedge (\top \supset (R \vee S)) \supset (R \vee S) \textit{ true} \end{array}$$

In general, the substitution $[B/P]\mathcal{D}$ proceeds by replacing every occurrence of the propositional variable P in any judgment in $\mathcal{D}$ by the proposition B. If $\mathcal{D}$ proves A true, then $[B/P]\mathcal{D}$ proves $[B/P]A$ true.

Now consider verifications $A{\uparrow}$ and uses $A{\downarrow}$.

(i) Show by counterexample that we cannot simply replace a propositional variable P by an arbitrary proposition B in a verification of $A{\uparrow}$ and obtain a verification of $[B/P]A{\uparrow}$.

(ii) Correct this deficiency by defining a new substitution operation $[\![B/P]\!]\mathcal{D}$ which works correctly on proofs $\mathcal{D}$ of $A{\uparrow}$ and $A{\downarrow}$.

(iii) Prove that your new substitution operation is correct.

Exercise 6 *Logical equivalence, $A \equiv B$, can also be defined as a primitive rather than as a notational definition.*

- *Give introduction and elimination rules for $A \equiv B$. Make sure they are* pure*, that is, they do not use any other logical connectives.*
- *Show that your rules are locally sound and complete by exhibiting local reductions and expansions.*
- *Define verifications and uses.*
- *Carefully state in which way the two definitions of logical equivalence are equivalent and prove it.*

Exercise 7 *An alternative way to define negation as a primitive connective is to use a judgment that is not only hypothetical in a new assumption A true but also parametric in a new propositional variable p.*

$$\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \vdots \\ p\ \mathit{true}\end{array}}{\neg A\ \mathit{true}}\ \neg I^{p,u} \qquad \dfrac{\neg A\ \mathit{true} \quad A\ \mathit{true}}{C\ \mathit{true}}\ \neg E$$

(i) *Show that the elimination rule is locally sound and complete with respect to the introduction rule, exhibiting local reductions and expansions.*

(ii) *Define verifications and uses for $\neg A$.*

(iii) *Prove that the two definitions of negation, notationally and via the introduction rule above, agree. State carefully in which sense they are equivalent.*

Exercise 8 *Explore a possible definition of exclusive or, $A \oplus B = (A \wedge \neg B) \vee (\neg A \wedge B)$, by introduction and elimination rules that are locally sound and complete. Make sure your rules are* pure*, that is, do not mention other constant or connectives. If successful, show local soundness and completeness as well as verifications and uses. If not, discuss the reason for the failure.*

Exercise 9 *The rule governing uses of $A \vee B$ requires the conclusion to be a verification $C\uparrow$. Explore an additional potential rule where the conclusion is permitted to be $C\downarrow$.*

$$\dfrac{A \vee B\downarrow \quad \begin{array}{c}\overline{A\downarrow}^{\,u}\\ \vdots \\ C\downarrow\end{array} \quad \begin{array}{c}\overline{B\downarrow}^{\,w}\\ \vdots \\ C\downarrow\end{array}}{C\downarrow}\ \vee E^{u,w}$$

Give arguments for and against such a rule, using examples and counterexamples as appropriate. Which properties of verifications may be jeopardized with this rule, if any? Are their new judgments we can now establish that we could not without this rule, or does it only affect the structure of the verifications themselves? Would a pragmatics feel differently about this rule than a verificationist?

Exercise 10 *The connectives were defined from the verificationist perspective. Take the pragmatist perspective, interpreting the elimination rules as the definition of the connective and justify the introduction rules from them. Explain the rule of local reduction and expansion (or whatever analogue you devise).*

Exercise 11 *Natural deduction elegantly expresses intuitionistic logic, but does not work as well for classical logic. There are three commonly used additional rules to obtain classical reasoning: the law of* exluded middle *(*xm*),* indirect proof *(*ip*), and* double negation elimination *(¬¬e).*

$$\frac{}{A \vee \neg A\ \mathit{true}}\ \mathsf{xm} \qquad \frac{\begin{matrix}\overline{A\ \mathit{true}}^{\,u} \\ \vdots \\ \bot\ \mathit{true}\end{matrix}}{A\ \mathit{true}}\ \mathsf{ip}^u \qquad \frac{\neg\neg A\ \mathit{true}}{A\ \mathit{true}}\ \neg\neg e$$

Carefully state in which way they are equivalent and then prove it.

Exercise 12 *Reconsider the missing interaction laws (L3), (L8), (L14), (L15), (L18). In each case, find a proposition, not using negation, that is* classically *equivalent and prove it. Try to find one that might reasonably be considered an interaction law. For example,* $A \supset \bot \equiv A \supset \bot \vee \bot$*, but that would hardly be considered an interaction law. You may use any of the rules in Exercise 11, whatever is convenient.*

Exercise 13 *Write out* classical *interaction laws of negation with conjunction, disjunction, implication, truth, falsehood, and negation. Your laws should be strong enough that every proposition can be shown to be equivalent to one where negation is only applied to atomic propositions. The interaction laws of negation with conjunction and disjunction are usually called De Morgan's laws. Prove one direction each of the laws for conjunction, disjunction, and implication using natural deduction with any of the rules in Exercise 11.*

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