Lecture Notes on Categorical Judgments

15-816: Modal Logic
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Lecture 3
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1 Introduction

The main basic judgments we have considered so far are:

A *true*	A is true
$A\!\uparrow$	A has a verification
$A\!\downarrow$	A may be used
$M : A$	M is a proof term for A, *or, equivalently,* M has type A

In addition, we have considered hypothetical judgments $J_1, \ldots, J_n \vdash J$ in general, and $x_1{:}A_1, \ldots, x_n{:}A_n \vdash M : C$ in particular.

A few crucial properties of these systems are still outstanding. In particular, we still need to prove global versions of the local soundness and completeness properties. We call them here *internal* soundess and completeness to remind us that they refer to properties of proofs and verifications, rather than to any *external* semantics in terms of mathematical structures.

Internal Soundness. If $A\!\uparrow$ and $A\!\downarrow \vdash C\!\uparrow$ then $C\!\uparrow$.

Internal soundness licenses us to use A in verifications of C, if we have a verification of A. This means that, globally, the elimination rules are not too strong: we cannot abuse the assumption A to deduce something that does not have a verification already.

Internal Completeness. $A\!\downarrow \vdash A\!\uparrow$.

Internal completeness show that the elimination rules are strong enough so that we can construct a verification of A, if given the license to use A.

In addition, we would like to relate verifications to the notion of truth, viewing (as verificationists like myself tend to do) uses as an auxiliary judgment form.

Truth and Verifications. $A\ \mathit{true}$ if and only if $A\!\uparrow$.

The proof that A is true exactly if it has a verification is rather straightforward, once we have established internal soundness and completeness.

We will return to these properties in a later lecture. In this lecture we present a generic and frequent construction of a new kind of judgment from a given one, by insisting that it be proved without using assumptions. The presentation in this lecture is based closely on [PD01].

2 Validity

We say that A is valid if A *true* has a proof that does not require any truth assumptions. The latter is an example of a *categorical judgment*, a judgment which does not depend on assumptions. We write $\bullet \vdash A\ \mathit{true}$ for a categorical judgment, emphasizing that there are no truth hypotheses. Formally, '$\bullet$' is synonymous with '$\cdot$', our prior notation for an empty collection of hypotheses.

Definition of Validity. A *valid* if $\bullet \vdash A\ \mathit{true}$.

Now we would like to ask questions such as: *If* $A \supset B$ *and* A *are both valid, is* B *also valid?* In order to be able to ask the question we would like to reflect the notion of validity as a proposition. This proposition is traditionally written as $\Box A$. A first introduction rule for this could simply be

$$\frac{A\ \mathit{valid}}{\Gamma \vdash \Box A\ \mathit{true}}\ \Box I_1$$

Note that we lose the assumptions Γ, since a proof of validity is not permitted to depend on truth assumptions, which are collected in Γ.

It turns out to be convenient for the judgment A *valid* never to appear as the conclusion of a hypothetical judgment. In order to achieve this, we just replace it with its definition.

$$\frac{\bullet \vdash A\ \mathit{true}}{\Gamma \vdash \Box A\ \mathit{true}}\ \Box I_2$$

As an introduction rule, this is rather straightforward. In our judgmental formulation, the complexity of $\Box A$ comes from the elimination rule.

A first attempt would be to simply reverse the introduction rule.

$$\frac{\Gamma \vdash \Box A\ \mathit{true}}{\bullet \vdash A\ \mathit{true}}\ \Box E_1?$$

This would seem to be locally sounds, but it misinterprets the notion of hypothetical judgment. Clearly, we use Γ to deduce $\Box A$; dropping this list of assumptions is not justified. For example,

$$\frac{\dfrac{}{\Box A\ \mathit{true} \vdash \Box A\ \mathit{true}}\ \mathsf{hyp}}{\bullet \vdash A\ \mathit{true}}\ \Box E_1?$$

would mean that we can derive an arbitrary proposition from no assumption, which renders the system meaningless. The rule above is unsound.

Another attempt would be to simply extract A *true*.

$$\frac{\Gamma \vdash \Box A\ \mathit{true}}{\Gamma \vdash A\ \mathit{true}}\ \Box E_2?$$

This is clearly locally sound, but is not locally complete:

$$\begin{array}{c}\mathcal{D}\\ \Gamma \vdash \Box A\ \mathit{true}\end{array} \quad \Longrightarrow_E \quad \frac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash \Box A\ \mathit{true}\end{array}}{\Gamma \vdash A\ \mathit{true}}\ \Box E_2?}{\Gamma \vdash \Box A\ \mathit{true}}\ \Box I??$$

The problem is of course that the introduction rule does not apply because the hypotheses are not empty.

The solution is to introduce $A\,\mathit{valid}$ as a new form of hypothesis. The judgment we consider then has the general form

$$\underbrace{u_1{::}B_1\ \mathit{valid}, \ldots, u_k{::}B_k\ \mathit{valid}}_{\Delta}\ ;\ \underbrace{x_1{:}A_1\ \mathit{true}, \ldots, x_n{:}A_n\ \mathit{true}}_{\Gamma} \vdash C\ \mathit{true}$$

As usual, we assume all the u_j and x_i are distinct and maintain this invariant through renaming of bound variables as needed.

The elimination form now just makes the validity of A available as an assumption.

$$\frac{\Delta; \Gamma \vdash \Box A\ \mathit{true} \quad (\Delta, u{::}A\,\mathit{valid}); \Gamma \vdash C\ \mathit{true}}{\Delta; \Gamma \vdash C\ \mathit{true}}\ \Box E^u$$

We also have to reexamine the introduction rule, now that we have additional hypotheses. The idea is that the assumptions in Δ are valid, and should therefore be available in the proof of A *valid*, while the truth assumptions in Γ must be erased. In short:

$$\frac{\Delta; \bullet \vdash A\ \mathit{true}}{\Delta; \Gamma \vdash \Box A\ \mathit{true}}\ \Box I$$

Of course, there also must be some way to actually use hypotheses A *valid*. The nature of hypothetical judgments yields $(\Delta, A\ \mathit{valid}); \Gamma \vdash A\ \mathit{valid}$, except that A *valid* is not a permissable conclusion. So instead we infer that A is true, which is correct by the definition of validity.

$$\frac{u{::}A\ \mathit{valid} \in \Delta}{\Delta; \Gamma \vdash A\ \mathit{true}}\ u$$

All other rules of natural deduction are extended systematically, following the nature of hypothetical judgments. This kind of extension is completely implicit in the usual two-dimensional form of natural deduction, which is inherently open-ended as to what kind of other hypotheses might be needed to explain new propositions. In the localized form, which is convenient for the analysis of the rules, we just add valid hypotheses Δ to the premises and conclusions of all rules.

3 Local Soundness and Completeness

With a new form of hypothesis, we obtain a new substitution principle.

> **Substitution Principle for Validity, v.1:** If $\Delta \vdash A\ \mathit{valid}$ and $(\Delta, u{::}A\ \mathit{valid}); \Gamma \vdash C\ \mathit{true}$ then $\Delta; \Gamma \vdash C\ \mathit{true}$.

However, we decided to use validity only in assumptions and never explicitly as a conclusion, so we replace the first assumption by its definition.

> **Substitution Principle for Validity, v.2:** If $\Delta; \bullet \vdash A\ \mathit{true}$ and $(\Delta, u{::}A\ \mathit{valid}); \Gamma \vdash C\ \mathit{true}$ then $\Delta; \Gamma \vdash C\ \mathit{true}$.

Note that it is crucial that there are no truth assumptions in the proof of A *true*; otherwise A would not be valid!

Now local soundness is easy to check.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta; \bullet \vdash A\ \mathit{true}\end{array}}{\Delta;\Gamma \vdash \Box A\ \mathit{true}}\,\Box I \quad \begin{array}{c}\mathcal{E}\\ (\Delta, u{::}A\ \mathit{valid});\Gamma \vdash C\ \mathit{true}\end{array}}{\Delta;\Gamma\vdash C\ \mathit{true}}\,\Box E^u \quad \Longrightarrow_R \quad \begin{array}{c}[\![\mathcal{D}/u]\!]\mathcal{E}\\ \Delta;\Gamma\vdash C\ \mathit{true}\end{array}$$

where $[\![\mathcal{D}/u]\!]\mathcal{E}$ is our notation of substituting $\mathcal{D}$ for uses of the assumption $u{::}A\ \mathit{valid}$ in $\mathcal{E}$. This deduction exists by the substitution principle for validity.

Local expansion is also easy to check.

$$\begin{array}{c}\mathcal{D}\\ \Delta;\Gamma\vdash\Box A\ \mathit{true}\end{array} \quad \Longrightarrow_E \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Delta;\Gamma\vdash\Box A\ \mathit{true}\end{array} \quad \dfrac{\dfrac{}{(\Delta,u{::}A\ \mathit{valid});\bullet\vdash A\ \mathit{true}}\,u}{(\Delta,u{::}A\ \mathit{valid});\Gamma\vdash \Box A\ \mathit{true}}\,\Box I}{\Delta;\Gamma\vdash\Box A\ \mathit{true}}\,\Box E^u$$

4 Sample Proofs

The following theorems characterize the □ modality on the implicational fragment of intuitionistic logic, together with the rule of necessitation:

$$\dfrac{\vdash^H\ A\ \mathit{true}}{\vdash^H\ \Box A\ \mathit{true}}\,\mathsf{nec}$$

Here we write $\vdash^H$ for derivability in an axiomatic Hilbert system which will be discussed further in a later lecture.

$\vdash \Box A \supset A$.

$$\dfrac{\dfrac{\dfrac{}{\cdot; x{:}\Box A\ \mathit{true}\vdash \Box A\ \mathit{true}}\,x \quad \dfrac{}{u{::}A\ \mathit{valid}; x{:}\Box A\ \mathit{true}\vdash A\ \mathit{true}}\,u}{\cdot;x{:}\Box A\ \mathit{true}\vdash A\ \mathit{true}}\,\Box E^u}{\cdot;\cdot\vdash \Box A\supset A\ \mathit{true}}\,\supset I^x$$

$\vdash \square A \supset \square\square A$.

$$\dfrac{\dfrac{\dfrac{}{\cdot; x{:}\square A\ \mathit{true} \vdash \square A\ \mathit{true}}\,x \qquad \dfrac{\dfrac{\dfrac{}{u{::}A\ \mathit{true}; \bullet \vdash A\ \mathit{true}}\,u}{u{::}A\ \mathit{true}; \bullet \vdash \square A\ \mathit{true}}\,\square I}{u{::}A\ \mathit{true}; x{:}\square A\ \mathit{true} \vdash \square\square A\ \mathit{true}}\,\square I}{\cdot; x{:}\square A\ \mathit{true} \vdash \square\square A\ \mathit{true}}\,\square E^{u}}{\cdot; \cdot \vdash \square A \supset \square\square A\ \mathit{true}}\,{\supset} I^{x}$$

$\vdash \square(A \supset B) \supset \square A \supset \square B$. For the sake of brevity, we omit here the explicit judgments A *true* and A *valid*, which can be inferred from the position in the hypothetical judgment. We have also elided unused hypotheses, writing '$-$' for irrelevant hypotheses which may not be empty.

$$\dfrac{\dfrac{\dfrac{}{\cdot; x{:}\square(A \supset B) \vdash \square(A \supset B)}\,x \qquad \dfrac{\dfrac{}{-; y{:}\square A \vdash \square A}\,y \qquad \dfrac{\dfrac{\dfrac{}{u{::}A \supset B \vdash A \supset B}\,u \quad \dfrac{}{w{::}A \vdash A}\,w}{u{::}A \supset B, w{:}A; \bullet \vdash A}\,{\supset} E}{u{::}A \supset B, w{:}A; - \vdash \square A}\,\square I}{u{::}A \supset B; y{:}\square A \vdash \square B}\,\square E}{\cdot; x{:}\square(A \supset B), y{:}\square A \vdash \square B}\,\square E}{\cdot; \cdot \vdash \square(A \supset B) \supset \square A \supset \square B}\,{\supset} I \times 2$$

5 Proof Terms

Next we assign proof terms in preparation for a computational interpretation in the next lecture.

$$\dfrac{u{::}A \in \Delta}{\Delta; \Gamma \vdash u : A}\,u$$

$$\dfrac{\Delta; \bullet \vdash M : A}{\Delta; \Gamma \vdash \mathbf{box}\, M : \square A}\,\square I \qquad\qquad \dfrac{\Delta; \Gamma \vdash M : \square A \quad (\Delta, u{::}A); \Gamma \vdash N : C}{\Delta; \Gamma \vdash \mathbf{let\,box}\, u = M\ \mathbf{in}\ N : C}\,\square E$$

We can now revisit the proofs of the characteristic axioms.

$$\begin{array}{rl} & \lambda x{:}\Box A.\, \textbf{let box}\, u = x \;\textbf{in}\; u \\ : & \Box A \supset A \\[1ex] & \lambda x{:}\Box A.\, \textbf{let box}\, u = x \;\textbf{in}\; \textbf{box}\,\textbf{box}\, u \\ : & \Box A \supset \Box\Box A \\[1ex] & \lambda x{:}\Box(A \supset B).\, \lambda y{:}\Box A.\, \textbf{let box}\, u = x \;\textbf{in}\; \textbf{let box}\, w = y \;\textbf{in}\; \textbf{box}\,(u\, w) \\ : & \Box(A \supset B) \supset \Box A \supset \Box B \end{array}$$

The local reductions and expansions can be easily expressed on proof terms, but we need to define a new form of substitution that realizes the substitution property. We refer to variables u that label hypotheses $u{::}A$ *valid* as *valid variables*. We view them as syntactically distinct from ordinary variables labeling $x{:}A$ *true*. We write $[\![M/u]\!]N$ for (capture-avoiding) substitution of M for u in N, were u is a variable labeling a valid assumption. Note that M does not contain any free (ordinary) variables, only other valid variables. This is because the operation will only be applied in cases where $\Delta; \bullet \vdash M : A$ and $(\Delta, u{::}A); \Gamma \vdash N : C$ with result $\Delta; \Gamma \vdash [\![M/u]\!]N : C$.

Modal substitution $[\![M/u]\!]N$ is defined compositionally on the structure of N. We show only a few cases.

$$\begin{array}{lcl} [\![M/u]\!]u & = & M \\ [\![M/u]\!]w & = & w \qquad \text{for } w \neq u \\ [\![M/u]\!]x & = & x \\ [\![M/u]\!](\lambda x{:}A.\, N) & = & \lambda x{:}A.\, [\![M/u]\!]N \\ [\![M/u]\!](N_1\, N_2) & = & ([\![M/u]\!]N_1)\, ([\![M/u]\!]N_2) \\ [\![M/u]\!](\textbf{box}\, N) & = & \textbf{box}\, ([\![M/u]\!]N) \\ [\![M/u]\!](\textbf{let box}\, w = N_1 \;\textbf{in}\; N_2) & = & \textbf{let box}\, w = [\![M/u]\!]N_1 \;\textbf{in}\; ([\![M/u]\!]N_2) \\ & & \qquad \text{provided } w \neq u,\, w \notin FV(M) \end{array}$$

Note that in the case for λ-abstraction, the ordinary bound variable x cannot appear free in M, so no proviso is necessary. in the case for **let box**, the proviso can always be satisfied by renaming the bound valid variable w.

One interesting effect is that the extension of ordinary substitution $[M/x]N$ may be slightly unexpected. Because x cannot occur underneath a **box**-operator, substitution does not descend under a **box**.

$$\begin{array}{lcl} [M/x]u & = & u \\ [M/x](\textbf{box}\, N) & = & \textbf{box}\, N \\ [M/x](\textbf{let box}\, u = N_1 \;\textbf{in}\; N_2) & = & \textbf{let box}\, u = [M/x]N_1 \;\textbf{in}\; ([M/x]N_2) \end{array}$$

As usual, we take weakening and tacit renaming of bound variables for granted, but we state explicitly the substitution theorem on proof terms.

Theorem 1 (Substitution)

(i) If $\Delta; \Gamma \vdash M : A$ *and* $\Delta; (\Gamma, x{:}A, \Gamma') \vdash N : C$ *then* $\Delta; \Gamma \vdash [M/x]N : C$.

(ii) If $\Delta; \bullet \vdash M : A$ *and* $(\Delta, u{::}A, \Delta'); \Gamma \vdash N : C$ *then* $(\Delta, \Delta'); \Gamma \vdash [\![M/u]\!]N$.

Proof: By separate inductions on the structure of N for each part. □

With substitution in hand, local reductions and expansions are easy to write down.

$$\begin{array}{lll} \mathbf{let\,box}\,u = \mathbf{box}\,M\ \mathbf{in}\ N & \Longrightarrow_R & [\![M/u]\!]N \\ M : \Box A & \Longrightarrow_E & \mathbf{let\,box}\,u = M\ \mathbf{in}\ \mathbf{box}\,u \end{array}$$

The proofs of subject reduction and expansion easily extend to encompass the new cases, relying on the substitution property.

Verifications and Uses

The verificationist approach is quite easy to sustain. Introduction rules construct verifications, and elimination rules use hypotheses. Valid hypothesis play a slight special role in that they cannot be used directly except in the modal hypothesis rule, so we write $A\Downarrow$ for valid hypotheses.

$$\frac{u{::}A\Downarrow \in \Delta}{\Delta; \Gamma \vdash A\downarrow}\, u$$

$$\frac{\Delta; \bullet \vdash A\uparrow}{\Delta; \Gamma \vdash \Box A\uparrow}\, \Box I \qquad \frac{\Delta; \Gamma \vdash \Box A\downarrow \quad (\Delta, u{::}A\Downarrow); \Gamma \vdash C\uparrow}{\Delta; \Gamma \vdash C\uparrow}\, \Box E^u$$

Again, we do not show it in this lecture, but the arguments for global (internal) soundness and completeness regarding verifications and uses carry over to this logic.

We can now check that $A \supset \Box A$ does not have a verification in general. Assume $\bullet; \bullet \vdash A \supset \Box A\uparrow$. By inversion, $\bullet; A\downarrow \vdash \Box A\uparrow$. Again, only one inference rule could have been applied to conclude this (namely $\Box I$), with the premise $\bullet; \bullet \vdash A\uparrow$. We cannot infer this parametrically in A and we have reached a contradication.

Of course there are a number of propositions for which, indeed, $A \supset \Box A$. For example, $A = \Box A'$, because $\vdash \Box A' \supset \Box(\Box A')$. Also, $A = \top$ or $A = \bot$ have this property (see Exercise 5).

6 A Counterexample

In the literature one often find formulations of modal logics where inference rules restrict contexts to have the form $\Box\Gamma$, meaning that every proposition in Γ has the form $\Box A$ for some A. While these presentations are usually adequate with respect to provability, the fine structure of proofs is often flawed. The separation of judgments from propositions underlying our approach solves these kinds of problems elegantly, and also give a clearly well-founded approach to the semantic explanation of the modalities via their verification.

As an example, we take a system adapted from Prawitz's seminal volume on natural deduction [Pra65]. We have the two rules

$$\frac{\Box\Gamma \vdash M : A}{\Box\Gamma, \Gamma' \vdash \mathbf{box}\, M : \Box A}\ \Box I_P \qquad\qquad \frac{\Gamma \vdash M : \Box A}{\Gamma \vdash \mathbf{unbox}\, M : A}\ \Box E_P$$

decorated already with proof terms. Unfortunately, this system does not satisfy subject reduction. Consider the following proof, written as a term for brevity:

$$f{:}B \supset \Box A, y{:}B \vdash (\lambda x{:}\Box A.\, \mathbf{box}\, x)\,(f\ y) : \Box\Box A$$

This proof is correct, because the assumption $x{:}\Box A$ has the right form to be available above the $\Box I_P$ rule. However, once we perform a reduction and substitute

$$[(f\ y)/x](\mathbf{box}\, x) = \mathbf{box}\,(f\ y)$$

we see that the resulting term is no longer well-typed. This is because the assumptions $f{:}B \supset \Box A$ and $y{:}B$ do not have the form $\Box(-)$ and are therefore not available underneath the **box** operator.

The failure can be traced to a failure of the substitution principle, violating the fundamental nature of hypothetical proofs. Several fixes have been proposed: Prawitz himself suggest a non-local correctness criterion for proofs [Pra65] which can be made explicit in the term structure via a notion of explicit substitution [PW95]. These also fix a second problem with the above rules, namely the fact that the rules are not locally complete. Nevertheless, I feel the judgmental formulation [PD01] gives a much better foundation and proof theory than either of these closely related approaches.

Exercises

Exercise 1 *Consider whether appropriate interaction equivalences exists for the $\Box$-modality in each of the following cases. Provide a proof where it exists, either in natural deduction form or as a proof term.*

(i) $\vdash \Box(A \wedge B) \equiv ?$

(ii) $\vdash \Box(A \supset B) \equiv ?$

(iii) $\vdash \Box(A \vee B) \equiv ?$

(iv) $\vdash \Box\top \equiv ?$

(v) $\vdash \Box\bot \equiv ?$

(vi) $\vdash \Box(\Box A) \equiv ?$

Exercise 2 *For those cases in Exercise 1 where no equivalence exist, propose and prove an implication instead.*

Exercise 3 *In intuitionistic logic, it is the case that if $\bullet \vdash A \vee B$ true then either $\bullet \vdash A$ true or $\bullet \vdash B$ true. This can be seen by inversion on $\bullet \vdash A \vee B\uparrow$, using that every true proposition has a verification.*

Rendering this into modal logic, we might expect

$$\vdash \Box(A \vee B) \supset (\Box A \vee \Box B)?$$

which is not the case for arbitrary A and B. Explain this discrepancy.

Exercise 4 *Demonstrate that*

> ***Incorrect Substitution Principle:*** *If $\Delta; \Gamma \vdash A$ true and $(\Delta, A$ valid$); \Gamma \vdash C$ true then $\Delta; \Gamma \vdash C$ true*

is indeed incorrect. Which property fails? Give a counterexample.

Exercise 5 *Characterize the propositions A such that $\vdash A \equiv \Box A$.*

Exercise 6 *The construction of validity can be iterated. For example, we can introduce A universal if its proof does not depend on any assumptions about the truth or validity of any propositions. In other words, A universal if $\bullet; \bullet \vdash A$ true. Develop the proposition $\blacksquare A$ which internalizes universality: give introduction rules, elimination rules, substitution principles, local reductions, local expansions. Also investigate how $\blacksquare$ interacts with ordinary logical operators ($\wedge$, $\supset$, $\vee$, $\top$, $\bot$) and $\Box$.*

Exercise 7 *As explained in Exercise 6, the construction of categorical judgments can be iterated. Consider judgments valid*^{n} *where* A *valid*^{0} *means* A *true, and* A *valid*$^{n+1}$ *means categorical with respect to assumptions* A *valid*$^n, \ldots,$ A *valid*^{0}.

Give a uniform system of natural deduction for iterated validity, including introduction and eliminations for $\Box^n A$ *for arbitrary* n*, internalizing* A *valid*^{n}*. State the substitution principle and show local soundness and completeness.*

We would expect that $\Box^0 A \equiv A$ *and* $\Box^1 A \equiv \Box A$ *where* $\Box$ *is necessity as defined in this lecture.*

How do iterated $\Box^n$ *operators interact with each other? Is there always simpler proposition equivalent to* $\Box^n \Box^m A$*?*

Exercise 8 *An alternative way to internalize the concept of validity is derived from the observation that* A *valid need only appear as a hypothesis. We define*

$$\frac{(\Delta, u{::}A\,\textit{valid}); \Gamma \vdash B\ \textit{true}}{\Delta; \Gamma \vdash A \Rightarrow B\ \textit{true}} \Rightarrow\!I$$

(i) Give the corresponding elimination rule(s).

(ii) Show local soundness and completeness.

(iii) Assign proof terms and show local reductions and expansions on proof terms.

(iv) If both $\Box A$ *and* $A \Rightarrow B$ *are present, can we define* $\Box$ *in terms of* $\Rightarrow$ *and vice versa? You may use any other (non-modal) logical connectives in your proposed definitions.*

(v) How do $\Rightarrow$ *and* $\supset$ *interact? Can we find equivalences for* $A \Rightarrow (B \Rightarrow C)$*,* $A \Rightarrow (B \supset A)$*, and* $A \supset (B \Rightarrow C)$*?*

References

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