Lecture Notes on
Soundness and Correspondence

15-816: Modal Logic
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1 Introduction to This Lecture

In this lecture, we will cover the question how axiomatics and semantics of modal logic fit together. For correspondence theory results we refer to Schmitt [Sch03] and the book by Hughes and Cresswell [HC96].

2 Soundness

In the lecture 5 we have followed an axiomatic and a semantic approach to modal logic. But do these approaches fit together? We should not be using proof rules that make no sense semantically. Recall the axioms and proof rules we had so far Figure 1.

The proof rules are sound iff they can only derive semantical consequences. Note that, unlike in the first lectures, this is an external soundness notion. We do not justify one proof rule by checking compatibility with another proof rule (internal soundness). Instead, we check compatibility of all proof rules with respect to the external mathematical objects of the semantics.

Definition 1 (Soundness) *A system S of proof rules and axioms of modal logic is (externally)* sound *iff, for all formulas ψ and all sets of formulas Φ:*

$$\Phi \vdash_S \psi \text{ implies } \Phi \vDash_g \psi$$

$$
\begin{array}{rl}
\text{(P)} & \text{all propositional tautologies} \\
\text{(K)} & \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi) \\
\text{(T)} & \Box\phi \to \phi \\
\text{(4)} & \Box\phi \to \Box\Box\phi \\
\text{(MP)} & \dfrac{\phi \quad \phi \to \psi}{\psi} \\
\text{(G)} & \dfrac{\phi}{\Box\phi}
\end{array}
$$

Figure 1: Modal logic axioms

In particular, in order to check if a proof rule

$$\frac{\phi_1 \quad \dots \quad \phi_n}{\psi}$$

is sound, we need to check for all instances of the proof rule if

$$\phi_1, \dots, \phi_n \vDash_g \psi$$

In order to check if an axiom

$$\phi$$

is sound, we need to check for all instances of the axiom if they are valid:

$$\vDash \phi$$

Lemma 2 *Axiom K is sound.*

Proof: Let K, s be a Kripke structure and a world. We have to prove that

$$K, s \models \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$$

Suppose the assumptions are true at s as there is nothing to show otherwise. Thus

$$K, s \models \Box(\phi \to \psi) \tag{1}$$
$$K, s \models \Box\phi \tag{2}$$

We thus know for all s with $s\rho t$ that $K, t \models \phi \rightarrow \psi$ by (1) and also that $K, t \models \phi$ by (2).

We have to show that $K, s \models \Box\psi$. For that, we have to show for all t with $s\rho t$ that $K, t \models \psi$. Consider any t with $s\rho t$. Since $s\rho t$, we know by (2) that $K, t \models \phi$. Furthermore, since $s\rho t$, we know by (1) that $K, t \models \phi \rightarrow \psi$. Clearly, this implies $K, t \models \psi$. As t was arbitrary with $s\rho t$, this shows $K, s \models \Box\psi$. □

Lemma 3 *Rule G is sound.*

Proof: We have to prove that $\phi \vDash_g \Box\phi$. Let K be a Kripke structure with $K, t \models \phi$ for all worlds $t \in W$. Now let $s \in W$ be any world. We have to show that $K, s \models \Box\phi$. For that, let $t \in W$ be any world with $s\rho t$. For t we have to show $K, t \models \phi$. But this is simple, because, by assumption, $K, t \models \phi$ for *all* worlds $t \in W$. □

Note that the proof of G crucially depends on knowing the premisses hold globally in all worlds. Thus, the notion of soundness that we are using here is also called *global soundness*, because it for the global consequence relation.

Now when we try to prove soundness of T, we run into problems. The rule just is not sound, although it looks so innocently helpful. What is wrong?

3 Correspondence

We have not been able to prove axiom T to be sound with respect to the semantics. This is a more systematic phenomenon that affects other reasonable modal axioms also, giving rise to the question how the axiomatic and semantic approach to modal logic correspond to one another.

Correspondence theory tries to find connections between properties of Kripke frames and the formulas in modal logic that are true in all Kripke frames with this property.

Lemma 4 *A Kripke frame (W, ρ) is reflexive if and only if $K, s \models \Box q \rightarrow q$ for all Kripke structures $K = (W, \rho, v)$. Here q is a propositional letter.*

Proof: Assume that (W, ρ) is reflexive, i.e., $(W, \rho) \vDash \forall x\, \rho(x, x)$. Now let $K = (W, \rho, v)$ be any Kripke structure, i.e., let v be any truth-value assignment for the Kripke frame (W, ρ). For very world $s \in W$, we have to show

$K, s \models \Box q \to q$. Thus assume $K, s \models \Box q$ as there is nothing to show otherwise. Hence for all t with $s\rho t$ we know $K, t \models q$. By reflexivity of ρ we know $s\rho s$, which implies $K, s \models q$.

Conversely, let (W, ρ) be any Kripke frame that is not reflexive. Let $r \in W$ be a world with $(r, r) \notin \rho$. We can choose any truth-assignment v for the Kripke frame (W, ρ). We choose to let

$$v(s)(q) := \begin{cases} \mathit{true} & \text{if } r\rho s \\ \mathit{false} & \text{otherwise} \end{cases}$$

Consider the Kripke structure $K = (W, \rho, v)$. Then $K, r \models \Box q$ but $K, r \not\models q$. Thus $K, r \not\models \Box q \to q$.

$\Box$

This lemma says that the class of all reflexive frames is characterized by the formula $\Box q \to q$:

Definition 5 (Characterization) *Let C be a class of Kripke frames and ϕ a formula in modal logic. Formula ϕ* characterizes *C, if, for every Kripke frame (W, ρ):*

$$(W, \rho) \in C \textit{ iff for each } v\textit{: } K, s \models \phi \textit{ holds for } K = (W, \rho, v)$$

Lemma 6 *The formula $\Box q \to \Box\Box q$ characterizes the class of all transitive frames.*

Proof: Assume that $K = (W, \rho, v)$ is any Kripke structure with a transitive Kripke frame (W, ρ). Consider any $s \in W$ with $K, s \models \Box q$ as there is nothing to show otherwise. We have to show $K, s \models \Box\Box q$. Let t, r be any worlds with $s\rho t$ and $t\rho r$. We have to show $K, r \models q$. By transitivity of ρ we know that $s\rho t$ and $t\rho r$ imply $s\rho r$. Because of $K, s \models \Box q$, this implies $K, r \models q$.

Conversely, let (W, ρ) be any Kripke frame that is not transitive. Let $r_0, r_1, r_2 \in W$ be worlds with $r_0\rho r_1$ and $r_1\rho r_2$ but $(r_0, r_2) \notin \rho$. We can choose any truth-assignment v for the Kripke frame (W, ρ). We choose

$$v(s)(q) := \begin{cases} \mathit{true} & \text{if } r_0\rho s \\ \mathit{false} & \text{otherwise} \end{cases}$$

Consider the Kripke structure $K = (W, \rho, v)$. Then $K, r_0 \models \Box q$ but also $K, r_0 \not\models \Box\Box q$, because $K, r_2 \not\models q$. Thus $K, r_0 \not\models \Box q \to \Box\Box q$. $\Box$

Combining the above results obtained so far we have:

Theorem 7 (Soundness of S4) *The Kripke proof rules for **S4** in Figure 1 are sound for the class of reflexive and transitive frames.*

Theorem 8 *The conjunction of the following two multimodal formulas*

$$\Box_a p \to (p \land \Box_a \Box_b p)$$
$$\Box_a (p \to \Box_b p) \to (p \to \Box_a p)$$

characterizes the set of all multimodal Kripke frames (W, ρ_a, ρ_b) *such that* ρ_a *is the reflexive, transitive closure of* ρ_b*.*

Proof: If $K = (W, \rho_a, \rho_b, v)$ is a Kripke structure where ρ_a is the reflexive, transitive closure of ρ_b, then it is easy to check that the two formulas are valid in K.

Conversely, consider a Kripke frame (W, ρ_a, ρ_b) in which both formulas are valid, that is, for all Kripke structures (W, ρ_a, ρ_b, v). We have to show that $\rho_a = \rho_b{}^*$.

"$\subseteq$" Fix any $s, t \in W$ with $s \rho_a t$. We show that (s, t) is in the reflexive, transitive closure $\rho_b{}^*$ of ρ_b. We choose the valuation map

$$v(w)(p) = \begin{cases} 1 & \text{if } (s, w) \in \rho_b{}^* \\ 0 & \text{otherwise} \end{cases}$$

First let us show for $K = (W, \rho_a, \rho_b, v)$ that the assumption holds:

$$K, s \models \Box_a (p \to \Box_b p)$$

For that, let $w \in W$ be any world with $s \rho_a w$ and $K, w \models p$. According to our choice of v, this implies that $(s, w) \in \rho_b{}^*$. Now for any world $w' \in W$ with $w \rho_b w'$, we note that $(s, w') \in \rho_b{}^*$ by the definition of a transitive closure. Thus, our choice of v ensures $K, w' \models p$. Consequently, $K, w \models \Box_b p$. Now we have shown $K, s \models \Box_a (p \to \Box_b p)$. We assumed that

$$\Box_a (p \to \Box_b p) \to (p \to \Box_a p)$$

is valid in K, thus $K, s \models p \to \Box_a p$. Since the transitive closure is reflexive, our choice of v implies that $K, s \models p$, hence $K, s \models \Box_a p$. But the world t we fixed in the beginning was one of the worlds with $s \rho_a t$, hence $K, t \models p$. Now our choice of v implies that $(s, t) \in \rho_b{}^*$ is in the reflexive, transitive closure. In summary, $\rho_a \subseteq \rho_b{}^*$.

"$\supseteq$" Fix any $s, t \in W$ with $(s,t) \in {\rho_b}^*$, i.e., that is in the reflexive, transitive closure. We want to show that $s\rho_a t$. We choose the valuation map

$$v(w)(p) = \begin{cases} 1 & \text{if } s\rho_a w \\ 0 & \text{otherwise} \end{cases}$$

By premiss $K, s \models \Box_a p \to (p \land \Box_a \Box_b p)$ for $K = (W, \rho_a, \rho_b, v)$. According to our choice of v, we have the assumption $K, s \models \Box_a p$. Thus the premiss implies

$$K, s \models p \land \Box_a \Box_b p \tag{3}$$

We have assumed $(s,t) \in {\rho_b}^*$. So there are worlds $w_0, w_1, \dots, w_n$ such that $w_i \rho_b w_{i+1}$ for all $i < n$ and $w_0 = s$ and $w_n = t$. Let us show by induction on i that $K, w_i \models p$.

0. For $i = 0$ this is implied by (3).
i. Assume $K, w_i \models p$. According to our choice of v this implies that $s\rho_a w_i$, hence (3) implies $K, w_i \models \Box_b p$ and $K, w_{i+1} \models p$.

Finally for $w_n = t$ this implies $K, t \models p$, which, by our choice of v implies $s\rho_a t$. In summary, $\rho_a \supseteq {\rho_b}^*$.

□

Exercises

Exercise 1 *Give correspondence results for the following cases. That is, for each modal formula identify the class of frames that it characterizes. Explain why and prove correspondence, or explain why there is no correspondence at all.*

1. $\Box q \to \Diamond q$
2. $\Diamond\Box q \to \Diamond q$
3. $\Diamond q \leftrightarrow \Box q$
4. $\Box\Box(a \to b) \to \Box\Box(b \to a)$

Exercise 2 *Give correspondence results for the following cases. That is, for each class of frames find a modal formula that characterizes it. Explain why and prove correspondence, or explain why there is no correspondence at all.*

1. *The class of all symmetric frames.*
2. *The class of all dense frames, i.e., if $s\rho t$ then $s\rho z, z\rho t$ for some z.*
3. *The class of all frames where $s\rho t, s\rho u$ imply $t\rho z, u\rho z$ for some z.*

Exercise 3 *Show correspondence of $\Box(a \land \Box a \to b) \lor \Box(b \land \Box b \to a)$ (weakly connected = sRt and sRu imply tRu or uRt or t=u) with the class of all frames where $s\rho t, s\rho u$ imply $t\rho u$ or $u\rho t$ or $t = u$.*

Exercise 4 *Describe a general correspondence result between modal logic formulas and classes of Kripke frames. That is, to each class of Kripke frames, give one formula in modal logic that characterizes it. Explain!*

Exercise 5 *Prove or disprove: If A and B be two formulas in propositional modal logic that characterize the same class of Kripke structures, then $A \leftrightarrow B$ is valid.*

Exercise 6 *After Definition 1, variations of soundness have been given for proof rules and for axioms. Do these notions coincide with the soundness actually defined in Definition 1? Prove or disprove.*

References

[HC96] G.E. Hughes and M.J. Cresswell. *A New Introduction to Modal Logic*. Routledge, 1996.

[Sch03] Peter H. Schmitt. Nichtklassische Logiken. Vorlesungsskriptum Fakultät für Informatik , Universität Karlsruhe, 2003.