Lecture Notes on Soundness of Modal Tableaux

15-816: Modal Logic
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Lecture 11
Februrary 23, 2010

1 Introduction to This Lecture

In the last lecture, we have seen the modal tableau calculus. Now we see why it is sound. Again, we refer to Fitting [Fit83, Fit88] and Schmitt [Sch03].

α	α_1	α_2
$T A \land B$	$T A$	$T B$
$F A \lor B$	$F A$	$F B$
$F A \to B$	$T A$	$F B$
$F \neg A$	$T A$	$T A$

β	β_1	β_2
$T A \lor B$	$T A$	$T B$
$F A \land B$	$F A$	$F B$
$T A \to B$	$F A$	$T B$
$T \neg A$	$F A$	$F A$

For the following cases of formulas we define one successor formula

ν	ν_0
$T \Box A$	$T A$
$F \Diamond A$	$F A$

π	π_0
$T \Diamond A$	$T A$
$F \Box A$	$F A$

$$(\alpha)\ \frac{\sigma\alpha}{\begin{array}{c}\sigma\alpha_1\\ \sigma\alpha_2\end{array}} \qquad (\beta)\ \frac{\sigma\beta}{\sigma\beta_1 \quad \sigma\beta_2} \qquad (\nu^*)\ \frac{\sigma\nu}{\sigma'\nu_0}\,[1] \qquad (\pi)\ \frac{\sigma\pi}{\sigma'\pi_0}\,[2]$$

[1] σ' accessible from σ and σ' occurs on the branch already

[2] σ' is a simple unrestricted extension of σ, i.e., σ' is accessible from σ and no other prefix on the branch starts with σ'

Figure 1: Tableau proof rules for QML

Every combination of top-level operator and sign occurs in one of the above cases. Tableau proof rules by those classes are shown in Figure 1. A tableau is *closed* if every branch contains some pair of formulas of the form σTA and σFA. A *proof* for modal logic formula consists of a closed tableau starting with the root $1FA$.

2 Tableaux and Models

Let us try to prove $\Box(A \vee B) \to \Box A \vee \Box B$:

1	$F\Box(A \vee B) \to \Box A \vee \Box B$	(1)
1	$T\Box(A \vee B)$	(2) from 1
1	$F\Box A \vee \Box B$	(3) from 1
1	$F\Box A$	(4) from 3
1	$F\Box B$	(5) from 3
1.1	FA	(6) from 4
1.2	FB	(7) from 5
1.1	$TA \vee B$	(8) from 2
1.2	$TA \vee B$	(9) from 2

1.1 TA (10) from 8	1.1 TB (11) from 8	1.2 TA (12) from 9	1.2 TB (13) from 9
$*$ 10 and 6	open	open	$*$ 13 and 7

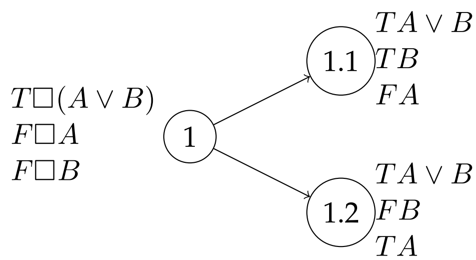
This tableau does not close but remains open, which is good news because the formula we set out to prove is not valid in **K**.

Definition 1 (Satisfiability) *A tableau is satisfiable in* ***K*** *if it has a branch P, for which there is a Kripke structure $K = (W, \rho, M)$ for the modal logic* ***K*** *and a mapping m from prefixes of P to W such that*

1. *$m(\sigma)\rho m(\sigma')$ for all prefixes σ' that are accessible from any prefix σ; and*
2. *$K, m(\sigma) \models A$ for every formula σTA on branch P.*
3. *$K, m(\sigma) \models \neg A$ for every formula σFA on branch P.*

In the sequel we will just abbreviate the last two cases to: $K, m(\sigma) \models A$ for every (signed) formula σA on branch P.

The last tableau is satisfiable, because the second branch from the left is satisfiable and we can read off the Kripke structure belonging to the following model:



3 Soundness

Theorem 2 (Soundness) *If there is a closed **K**-tableau with root $1FA$, then A is valid in all Kripke structures of **K**.*

Proof: Let $\mathcal{T}$ be the closed **K**-tableau. Suppose there was a Kripke structure $K = (W, \rho, M)$ in **K** and a world $s \in W$ such that $K, s \models \neg A$. Then the subtableau of $\mathcal{T}$ consisting only of the root note $1FA$ is a satisfiable tableau in **K** using the mapping $m(1) = s$. By induction on the rules (i.e., induction on the structure of the proof with a case distinction by the proof rules used at each step), we will show that tableau-expansions of satisfiable tableaux remain satisfiable (both in **K**). Once we have shown this, the full tableau $\mathcal{T}$ has to be satisfiable in **K**, which contradicts the fact that it is closed.

Now we prove by induction that if $\mathcal{T}_0$ is a satisfiable tableau and $\mathcal{T}$ emerged from $\mathcal{T}_0$ by expansion with one of the tableau rules from Figure 1, then $\mathcal{T}$ is also satisfiable. Let $K = (W, \rho, M)$ be the Kripke structure for $\mathcal{T}_0$ according to the definition of satisfiability.

ν Consider the case where $\mathcal{T}$ results from $\mathcal{T}_0$ by applying a ν expansion rule to a prefix formula $\sigma\nu$ on a branch P, which will thus be expanded to $P \cup \{\sigma'\nu_0\}$ with a prefix σ' that is accessible from σ and already occurred on P. Since $\mathcal{T}_0$ was satisfiable, we have

$$m(\sigma)\rho m(\sigma') \quad \text{and} \quad K, m(\sigma) \models \nu$$

Now the semantics immediately implies that $K, m(\sigma') \models \nu'$ (consider $\nu = T\Box A$ and the like).

π Consider the case where $\mathcal{T}$ results from $\mathcal{T}_0$ by applying a π expansion rule to a prefix formula $\sigma\pi$ on a branch P, which will thus be expanded to $P \cup \{\sigma'\pi_0\}$ with a prefix σ' that is a simple unrestricted extension of σ. Since $\mathcal{T}_0$ was satisfiable, we know that $K, m(\sigma) \models \pi$.

Thus, there is a world $s \in W$ such that (consider $\pi = T\Diamond A$ and the like):

$$K, s \models \pi_0 \tag{1}$$

Now we extend the mapping m by defining $m(\sigma') := s$. Because σ' is a simple extension of σ, every prefix σ_2 on the branch satisfies

$$\sigma' \text{ is accessible from } \sigma_2 \quad \text{iff} \quad \sigma_2 = \sigma$$

Because σ' is an unrestricted extension of σ, every prefix σ_2 on the branch satisfies

$$\sigma_2 \text{ is not accessible from } \sigma'$$

Putting these together we have for all prefixes σ_1, σ_2 on the extended branch $P \cup \{\sigma'\pi_0\}$ that:

$$m(\sigma_1)\rho m(\sigma_2) \quad \text{if } \sigma_2 \text{ is accessible from } \sigma_1$$

Together with (1), this implies that the tableau extension $\mathcal{T}$ is satisfiable.

- The other cases are standard for propositional logic.

□

So far, we have modal tableaux for proving validities. Tableaux can be extended easily for proving local and global consequences. For proving a local consequence $\Phi \vDash_l \psi$, we start with the initial tableau root $1F\psi$ and add the tableau rule

$$\frac{}{1T\phi} \quad \text{where } 1F\psi \text{ was the root of the tableau and } \phi \in \Phi$$

For proving global consequence $\Phi \vDash_g \psi$, we start with the initial tableau root $1F\psi$ and add the tableau rule

$$\frac{}{\sigma T\phi} \quad \text{for any } \phi \in \Phi \text{ and any prefix } \sigma \text{ occurring on the path}$$

4 Prefix Variable Tableaux

The disadvantage of the ν-rule in the tableau calculus is that we have to guess the right prefix without knowing which one will succeed. To overcome this, we consider a tableau calculus with free variables in the prefix.

Definition 3 (Prefix with variables) *A prefix σ is a finite sequence of natural numbers and prefix variables $U, V, W, \dots$. That is, we define the set $\mathcal{S}$ of prefixes inductively as:*

1. *The empty sequence is in $\mathcal{S}$.*
2. *If $\sigma \in \mathcal{S}$ and $n \in \mathbb{N}$ then $\sigma n \in \mathcal{S}$.*
3. *If $\sigma \in \mathcal{S}$ and $U \in PV := \{U, V, W, \dots\}$ is a prefix variable then $\sigma U \in \mathcal{S}$.*

A prefix without prefix variables is called variable-free or ground. A prefix substitution *is a function $PV \to \mathbb{N}^*$ that associates a sequence of natural numbers with each prefix variable.*

Depending on the actual modal logic under consideration, prefix substitutions have to satisfy additional properties. For **K** the sequence associated to every prefix variable must consist of exactly one natural number. Prefix substitution $\Phi : PV \to \mathbb{N}^*$ *unifies* branches σ and σ' iff $\Phi(\sigma) = \Phi(\sigma')$.

Now the tableau calculus with free prefix variables is shown in Figure 2. For rule π we define the set of all *ground prefix instances* of prefix σ from branch B as

$$gpi(\sigma, B) = \{\Phi(\sigma) \mid \Phi : PV \to \mathbb{N}^* \text{ and } \Phi(\sigma) \text{ is a ground prefix on } B\}$$

We say that a free prefix variable tableau closes if there is a prefix substitution Φ such that all branches close, i.e., on each branch, σA and $\tau \neg A$ occur such that $\Phi(\sigma) = \Phi(\tau)$.

$$(\alpha)\ \frac{\sigma\alpha}{\begin{array}{c}\sigma\alpha_1\\ \sigma\alpha_2\end{array}} \qquad (\beta)\ \frac{\sigma\beta}{\sigma\beta_1 \quad \sigma\beta_2} \qquad (\nu^*)\ \frac{\sigma\nu}{\sigma U \nu_0}\,^{1} \qquad (\pi)\ \frac{\sigma\pi}{\sigma n \pi_0}\,^{2}$$

[1] prefix variable U is new and does not occur on the branch
[2] $n \in \mathbb{N}$ is such that $\sigma_0 n$ does not occur on the branch B for all $\sigma' \in gpi(\sigma, B)$

Figure 2: Tableau proof rules for QML

Exercises

Exercise 1 *Give a tableau calculus for modal logic **T**, show an example of a proof in this calculus that illustrates the difference to **K**. Prove soundness.*

Exercise 2 *Give a tableau calculus for modal logic **S4**, show an example of a proof in this calculus that illustrates the difference to **T**. Prove soundness.*

Exercise 3 *Consider tableaux for propositional modal logic **K**. Let $len(A)$ denote the number of symbols in A, i.e., propositional letters and logical operators. For a set of formulas Φ, let further $subfor(\Phi)$ be the set of subformulas of formulas in Φ. We also use $subfor(\Phi)$ to denote the set of signed subformulas of signed formulas in Φ. Finally, for a branch P of a tableau and a given prefix σ, we define the following set of prefix formulas*

$$\sigma^-(P) := \{A | \sigma A \in P\}$$

Show that

1. *For every branch P of a tableau with root $1FA$, and every prefix σ:*

$$subfor(\sigma^-(P)) \subseteq subfor(FA)$$

2. *For every branch P of a **K**-tableau and any prefixes σ_1, σ_2 such that σ_2 is accessible from σ_1:*

$$\max_{A \in \sigma_2^-(P)} len(A) < \max_{A \in \sigma_1^-(P)} len(A)$$

*Which property about tableau constructions for **K** can you show with these properties and how?*

References

[Fit83] Melvin Fitting. *Proof Methods for Modal and Intuitionistic Logic*. Reidel, 1983.

[Fit88] Melvin Fitting. First-order modal tableaux. *J. Autom. Reasoning*, 4(2):191–213, 1988.

[Sch03] Peter H. Schmitt. Nichtklassische Logiken. Vorlesungsskriptum Fakultät für Informatik , Universität Karlsruhe, 2003.