Lecture Notes on
Intuitionistic Kripke Semantics

15-816: Modal Logic
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1 Introduction

In this lecture we present an intuitionistic approach to describing a multiple-world semantics for modal logic in the style of Kripke. This is done by providing judgments and inference rules that reason about truths in multiple worlds. A thorough analysis of intuitionistic modal logic in this style has been carried out by Simpson [Sim94]. This presentation of the logic suggests an operational interpretation via distributed computation, where worlds correspond to loci of computation (hosts or processes), with accessibility being connectivity between hosts. Variations of this operational interpretation go back to Jia and Walker [JW04] Murphy et al. [MCHP04, Mur08], although we will not be able to sustain this interpretation in its full generality.

This is very different from providing a Kripke semantics for intuitionistic logic; something that can also be done (see Lecture 14) but is somehow the inverse. It would describe intuitionistic logic from a classical point of view, while we want to give an intuitionistic perspective on the typically classical Kripke structures. In the next lecture we will make the connection to the intuitionistic modal logic of validity and possibility described in earlier lectures.

2 Basic Judgments

Rather than adding new judgments as we did when we moved from intuitionistic to (judgmental) modal logic, we revise the basic judgment from

A *true* to $A \mathrel{@} w$ which we read as "*A is true at world w*". All the usual logical inference rules are now relativized to an arbitrary world w. This expresses that the meaning of the usual connectives is *local* to a world, and that we reason with the same rules at all worlds.

$$\frac{A \mathrel{@} w \quad B \mathrel{@} w}{A \wedge B \mathrel{@} w} \wedge I \qquad \frac{A \wedge B \mathrel{@} w}{A \mathrel{@} w} \wedge E_1 \quad \frac{A \wedge B \mathrel{@} w}{B \mathrel{@} w} \wedge E_2$$

$$\frac{\begin{array}{c}\overline{A \mathrel{@} w}^{\,x} \\ \vdots \\ B \mathrel{@} w\end{array}}{A \supset B \mathrel{@} w} \supset I^x \qquad \frac{A \supset B \mathrel{@} w \quad A \mathrel{@} w}{B \mathrel{@} w} \supset E$$

$$\frac{}{\top \mathrel{@} w} \top I \qquad \text{no } \top E$$

$$\frac{A \mathrel{@} w}{A \vee B \mathrel{@} w} \vee I_1 \quad \frac{B \mathrel{@} w}{A \vee B \mathrel{@} w} \vee I_2 \qquad \frac{A \vee B \mathrel{@} w \quad \begin{array}{c}\overline{A \mathrel{@} w}^{\,x} \\ \vdots \\ C \mathrel{@} w''\end{array} \quad \begin{array}{c}\overline{B \mathrel{@} w}^{\,y} \\ \vdots \\ C \mathrel{@} w''\end{array}}{C \mathrel{@} w''} \vee E^{x,y}$$

$$\text{no } \bot I \qquad \frac{\bot \mathrel{@} w}{C \mathrel{@} w''} \bot E$$

Figure 1: Kripke logic, propositional connectives

In anticipation of the modal connectives which move between worlds, we have given the $\vee E$ and $\bot E$ rules with a general conclusion $C \mathrel{@} w''$. This will cause us some grief in the operational interpretation because the action of disjunction is somehow not local, despite our goals of locality. On the other hand, it is not simple to just restrict these rules to the case $w'' = w$ and preserve cut elimination (see Exercise 2).

Next are the modal connectives. We use the judgment $w \leq w'$ to express that w' is *reachable* or *accessible* from w. We may also say that w' lies in the future of w. For the moment we remain agnostic which laws govern this relation.

Necessity. In order for $\Box A$ to be true at world w, we have to show that A is true at all reachable worlds. We model this by introducing a new world parameter α and verifying that A is true at α, knowing only that α is reachable from w.

$$\dfrac{\begin{array}{c}\overline{w \leq \alpha}\\ \vdots \\ A \mathbin{@} \alpha\end{array}}{\Box A \mathbin{@} w}\ \Box I^{\alpha}$$

We write w, w', w'' for arbitrary worlds, and α, β, γ for world parameters that are bound in inference rules. In the rule for $\Box I$, the deduction in the premise is parametric in α and hypothetical in $w \leq \alpha$. Note that $w \leq \alpha$ is a *judgment*, not a proposition. It is therefore not localized at any world, and it can not be mentioned in a proposition.

The elimination rule expresses that if $\Box A$ is true at some world w, then A is true at any reachable world w'.

$$\dfrac{\Box A \mathbin{@} w \quad w \leq w'}{A \mathbin{@} w'}\ \Box E$$

The local reduction exploits the parametricity in the introduction rule in a new familiar manner.

$$\dfrac{\dfrac{\begin{array}{c}\overline{w \leq \alpha}\\ \mathcal{D}\\ A \mathbin{@} \alpha\end{array}}{\Box A \mathbin{@} w}\ \Box I^{\alpha} \quad \begin{array}{c}\mathcal{E}\\ w \leq w'\end{array}}{A \mathbin{@} w'}\ \Box E \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{E}\\ w \leq w'\\ {[w'/\alpha]\mathcal{D}}\\ A \mathbin{@} w'\end{array}$$

The local expansion straightforwardly introduces the modality.

$$\begin{array}{c}\mathcal{D}\\ \Box A \mathbin{@} w\end{array} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Box A \mathbin{@} w\end{array} \quad \overline{w \leq \alpha}}{A \mathbin{@} \alpha}\ \Box E}{\Box A \mathbin{@} w}\ \Box I^{\alpha}$$

Possibility. In order for $\Diamond A$ to be true at a world w, we have to show that A is true at some reachable world w'.

$$\dfrac{w \leq w' \quad A \mathbin{@} w'}{\Diamond A \mathbin{@} w}\ \Diamond I$$

In the elimination rule, we assume that A is true at some reachable world α about which we know nothing else.

$$\dfrac{\Diamond A \mathbin{@} w \qquad \begin{array}{c} \overline{w \leq \alpha} \quad \overline{A \mathbin{@} \alpha}^{\,x} \\ \vdots \\ C \mathbin{@} w'' \end{array}}{C \mathbin{@} w''}\ \Diamond E^{\alpha,x}$$

The right premise here is parametric in α and hypothetical in $w \leq \alpha$ and $A \mathbin{@} \alpha$, labeled by x. This implies that $w'' \neq \alpha$, because otherwise α would escape its scope to the conclusion.

Like $\vee E$ and $\perp E$, this rule is not local in the sense that w may be different from w''.

Local reduction again exploits parametricity in an expected way.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \mathbin{@} w'\end{array} \quad \begin{array}{c}\mathcal{E}\\ w \leq w'\end{array}}{\Diamond A \mathbin{@} w}\ \Diamond I \qquad \begin{array}{c} \overline{w \leq \alpha} \quad \overline{A \mathbin{@} \alpha}^{\,x} \\ \mathcal{F} \\ C \mathbin{@} w'' \end{array}}{C \mathbin{@} w''}\ \Diamond E^{\alpha,x} \quad \Longrightarrow_R \quad \begin{array}{c} \begin{array}{c}\mathcal{E}\\ w \leq w'\end{array} \quad \overline{\begin{array}{c}\mathcal{D}\\ A \mathbin{@} w'\end{array}}^{\,x} \\ {[w'/\alpha]\mathcal{F}} \\ C \mathbin{@} w'' \end{array}$$

The local expansion mirrors the local expansion of disjunction.

$$\begin{array}{c}\mathcal{D}\\ \Diamond A \mathbin{@} w\end{array} \quad \Longrightarrow_E \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Diamond A \mathbin{@} w\end{array} \qquad \dfrac{\overline{w \leq \alpha} \quad \overline{A \mathbin{@} \alpha}^{\,x}}{\Diamond A \mathbin{@} w}\ \Diamond I}{\Diamond A \mathbin{@} w}\ \Diamond E^{\alpha,x}$$

The rules for the modalities are summarized in Figure 3. We obtain various particular modal logics by specifying specific laws for the accessibility relation $\leq$. We name the logics by prefixing the corresponding classical modal logic with "I".

Propertie of $\leq$	Modal logic
none	IK
reflexive	IT
transitive	I4
reflexive and transitive	IS4
symmetric	I5
reflexive, transitive, and symmetric	IS5

$$\dfrac{\begin{array}{c}\overline{w \leq \alpha}\\ \vdots \\ A \mathbin{@} \alpha\end{array}}{\Box A \mathbin{@} w}\ \Box I^{\alpha} \qquad \dfrac{\Box A \mathbin{@} w \quad w \leq w'}{A \mathbin{@} w'}\ \Box E$$

$$\dfrac{w \leq w' \quad A \mathbin{@} w'}{\Diamond A \mathbin{@} w}\ \Diamond I \qquad \dfrac{\Diamond A \mathbin{@} w \quad \begin{array}{c}\overline{w \leq \alpha} \quad \overline{A \mathbin{@} \alpha}^{\,x}\\ \vdots \\ C \mathbin{@} w''\end{array}}{C \mathbin{@} w''}\ \Diamond E^{\alpha,x}$$

Figure 2: Kripke logic, modal connectives

We can see the effects of these by writing out the proofs of a few sample theorems. The first, the familiar axiom $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$. We prove this at an arbitrary world h (for "here").

$$\dfrac{\dfrac{\overline{\Diamond A \mathbin{@} h}^{\,y} \quad \dfrac{\overline{h \leq \alpha} \quad \dfrac{\dfrac{\overline{\Box(A \supset B) \mathbin{@} h}^{\,x} \quad \overline{h \leq \alpha}}{A \supset B \mathbin{@} \alpha}\ \Box E \quad \overline{A \mathbin{@} \alpha}^{\,z}}{B \mathbin{@} \alpha}\ {\supset}E}{\Diamond B \mathbin{@} h}\ \Diamond I}{\Diamond B \mathbin{@} h}\ \Diamond E^{\alpha,z}}{\Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \mathbin{@} h}\ {\supset}I^{x}, {\supset}I^{y}$$

As expected, this proof does not require any reasoning about the accessibility relation: we directly use the assumption $h \leq \alpha$ twice. We would expect $\Box A \supset A$ to require reflexivity, and indeed this is the case.

$$\dfrac{\dfrac{\overline{\Box A \mathbin{@} h}^{\,x} \quad \overline{h \leq h}^{\,\mathsf{refl}}}{A \mathbin{@} h}\ \Box E}{\Box A \supset A \mathbin{@} h}\ {\supset}I^{x}$$

On the other hand, something like $\Diamond\Diamond A \supset \Diamond A$ should require transitivity.

$$
\dfrac{\dfrac{\dfrac{}{\Diamond\Diamond A \mathbin{@} h}\,x \quad \dfrac{\dfrac{}{\Diamond A \mathbin{@} \alpha}\,y \quad \dfrac{\dfrac{\dfrac{}{h \leq \alpha} \quad \dfrac{}{\alpha \leq \beta}}{h \leq \beta}\,\mathsf{trans} \quad \dfrac{}{A \mathbin{@} \beta}\,z}{\Diamond A \mathbin{@} h}\,\Diamond I}{\Diamond A \mathbin{@} h}\,\Diamond E^{\beta,z}}{\Diamond A \mathbin{@} h}\,\Diamond E^{\alpha,y}}{\Diamond\Diamond A \supset \Diamond A \mathbin{@} h}\,{\supset}I^{x}
$$

Perhaps surprisingly, the axioms laid out in Lecture 9 are not sufficient to prove all the true propositions in the corresponding intuitionistic modal logic under the Kripke semantics. We return to this very important point in the next lecture.

3 Proof Term Assignment

A natural attempt to give an operational interpretation for this form of intuitionistic modal logic is as a language for distributed computation, where accessibility is related to connectivity between loci of computation, e.g., hosts in a network. This interpretation will require some restrictions; instead of assuming them a priori will develop the language and read off the restriction from its definition.

We start with a proof term assignment. The primary intuition, to be refined in the next section, is that if we type $M : A \mathbin{@} w$ then M is located at world w. $\Box$ and $\Diamond$ are then concerned with movement between the worlds. We present a localized version of the inference rules annotated with proof terms. For the non-modal connectives, we use the same proof terms as before. For the modal connectives, we have two forms of hypotheses: $w \leq \alpha$ and $A \mathbin{@} w$. When we write $w \leq \alpha$, α is considered a new parameter declared at that occurrence. This avoids us having to introduce a new kind of hypothesis such as α *world*. As usual, we assume all world parameters and also all ordinary variables declared in the context to be distinct.

We have chosen not to have an explicit representation for the proofs of accessibility, since in all the modal logics we consider the reachability relation is easily decidable by saturating the assumptions under the appropriate rules (reflexivity for declared worlds, transitivity, and symmetry, as available). Instead, we show the accessibility judgment itself that was used. We write the term in such a way that the world in which a terms is checked

$$\frac{x{:}A@w \in \Gamma}{\Gamma \vdash x : A @ w}\ \mathsf{hyp}$$

$$\frac{\Gamma, w \leq \alpha \vdash M : A @ \alpha}{\Gamma \vdash \Lambda w \leq \alpha.\, M : \Box A @ w}\ \Box I$$

$$\frac{\Gamma \vdash M : \Box A @ w \quad \Gamma \vdash w \leq w'}{\Gamma \vdash M\,[w \leq w'] : A @ w'}\ \Box E$$

$$\frac{\Gamma \vdash w \leq w' \quad \Gamma \vdash M : A @ w'}{\Gamma \vdash \langle w \leq w' \rangle M : \Diamond A @ w}\ \Diamond I$$

$$\frac{\Gamma \vdash M : \Diamond A @ w \quad \Gamma, w \leq \alpha, x{:}A @ \alpha \vdash N : C @ w''}{\Gamma \vdash \textbf{let}\,\langle w \leq \alpha \rangle\, x = M \textbf{ in } N : C @ w''}\ \Diamond E$$

Figure 3: Kripke logic, proof terms for modal connectives

is close to that term:

Term	Location
$\Lambda w \leq \alpha.\, M$	$M @ \alpha$
$M\,[w \leq w']$	$M @ w$
$\langle w \leq w' \rangle M$	$M @ w'$
$\textbf{let}\,\langle w \leq \alpha \rangle\, x = M \textbf{ in } N$	$M @ w, x @ \alpha$

We rewrite the earlier examples in the form of proof terms.

$$\begin{array}{rcl} K^{\Diamond} & : & \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) @ h \\ & = & \lambda x@h.\, \lambda y@h.\, \textbf{let}\,\langle h \leq \alpha \rangle\, z = y \textbf{ in } \langle h \leq \alpha \rangle((x\,[h \leq \alpha])\, z) \end{array}$$

$$\begin{array}{rcl} T^{\Box} & : & \Box A \supset A \\ & = & \lambda x@h.\, x\,[h \leq h] \end{array}$$

$$\begin{array}{rcl} 4^{\Diamond} & : & \Diamond\Diamond A \supset \Diamond A \\ & = & \lambda x@h.\, \textbf{let}\,\langle h \leq \alpha \rangle\, y = x \textbf{ in let}\,\langle \alpha \leq \beta \rangle\, z = y \textbf{ in } \langle h \leq \beta \rangle z \end{array}$$

To check accessibility we can read the term left-to-right, collection the hypotheses from $\Lambda w \leq \alpha$ and **let** $\langle w \leq \alpha \rangle$ and verifying if the conditions $[w \leq w']$ and $\langle w \leq w' \rangle$ hold, under the reachability relation under consideration.

We also rewrite the earlier local reduction on the proof terms. For the sake of clarity (as far as locations are concerned), we have built some redundancy into the terms. On well-typed terms, the two occurrences of worlds labeled w in the reductions below must match.

$$
\begin{array}{lll}
(\Lambda w \leq \alpha.\, M)\,[w \leq w'] & \Longrightarrow_R & [w'/\alpha]M \\
\textbf{let}\,\langle w \leq \alpha\rangle\, x = \langle w \leq w'\rangle M \;\textbf{in}\; N & \Longrightarrow_R & [M/x][w'/\alpha]N
\end{array}
$$

4 Distributed Computation

Here is the basic intended interpretation of values of modal type. A value $V : \Box A$ should be a *mobile computation* of type A which can be moved to a reachable host according to the accessibility relation. Conversely, $V : \Diamond A$ should be the address of a *remote value* of type A at some reachable host. While references to remote values may be mobile, the remote values themselves are not.

We present the computational interpretation as a *substructural operational semantics* [Pfe04, PS09]. In this style of presentation the state of the computation is represented by an ordered context of ephemeral propositions. The general invariant is that if we have a typed term $\Gamma \vdash M : A$ then we are in a situation where we are either computing the value of M or returning the value of M, written as $\mathsf{eval}\, M$ or $\mathsf{ret}\, V$. The remaining computation called the *continuation* is represented by a stack of frames $\mathsf{cont}\, F$, each of which carries out the computation described by F when a value is returned to it. In the particular kind of call-by-value operational semantics we are interested in here, variables are bound by to values $!\mathsf{bind}\, x\, V$. Such bindings are not ordered and persistent, in the sense that once made they remain in effect throughout the rest of the computation. The operational semantics itself is described by a set of rules for transforming the state. The left-hand side is matched against part of the ordered context, which is then replaced by the right-hand side.

For the current application, we generalize the predicates of evaluation, return, continuation, and binding to be located at particular worlds. Some of this information is redundant, but may help to clarify the location of the computational object we are considering. We have the following correspondence:

Static	Dynamic
$M : A \,@\, w$	$\mathsf{eval}\, M\, w$
$V : A \,@\, w$	$\mathsf{ret}\, V\, w$
$x : A \,@\, w$	$!\mathsf{bind}\, x\, V\, w$

In addition we have the predicate $\mathsf{cont}\,F\,w$ which means that continuation frame F is waiting for a value at world w. We will introduce appropriate notions of values and frames as we present the semantics.

As a warm-up, we consider call-by-value evaluation in substructural style. We use $A \bullet B$ for adjacent propositions in the current state, and $A \twoheadrightarrow B$ for the transition rule that replaces A by B, where $\bullet$ binds more tightly than $\twoheadrightarrow$. We begin with the rules for functions $A \supset B$.

$$
\begin{array}{lcl}
\mathsf{eval}\,(M_1\,M_2)\,w & \twoheadrightarrow & \mathsf{eval}\,M_1\,w \bullet \mathsf{cont}\,(\_\,M_2)\,w \\
\mathsf{eval}\,(\lambda x.\,M)\,w & \twoheadrightarrow & \mathsf{ret}\,(\lambda x.\,M)\,w \\
\mathsf{ret}\,V_1\,w \bullet \mathsf{cont}\,(\_\,M_2)\,w & \twoheadrightarrow & \mathsf{eval}\,M_2\,w \bullet \mathsf{cont}\,(V_1\,\_)\,w \\
\mathsf{ret}\,V_2\,w \bullet \mathsf{cont}\,((\lambda x.\,M)\,\_)\,w & \twoheadrightarrow & \exists x.\,!\mathsf{bind}\,x\,V_2\,w \bullet \mathsf{eval}\,M\,w \\
\mathsf{eval}\,x\,w & \twoheadrightarrow & \mathsf{cont}\,x\,w \\
!\mathsf{bind}\,x\,V_2 \bullet \mathsf{cont}\,x\,w & \twoheadrightarrow & \mathsf{ret}\,V_2\,w
\end{array}
$$

In the rule where a function is applied to an argument, we create a new parameter, called x, using the existential on the right-hand side. In the presentation we chose a new name identical to the name of the variable bound in $\lambda x.\,M$, but using α-conversion we could write the right-hand side equivalently as $\exists y.\,!\mathsf{bind}\,y\,V_2\,w \bullet \mathsf{eval}\,([y/x]M)\,w$. We have values $\lambda x.\,M$ and continuation frames $(\_\,M_2)$ and $(V_1\,\_)$. Note that all computation takes place at the same world w — computation is entirely local.

We show the rules for mobile computations $\Box A$.

$$
\begin{array}{lcl}
\mathsf{eval}\,(M\,[w \leq w'])\,w' & \twoheadrightarrow & \mathsf{eval}\,M\,w \bullet \mathsf{cont}\,(\_\,[w \leq w'])\,w' \\
\mathsf{eval}\,(\Lambda w \leq \alpha.\,M')\,w & \twoheadrightarrow & \mathsf{ret}\,(\Lambda w \leq \alpha.\,M')\,w \\
\multicolumn{3}{l}{\mathsf{ret}\,(\Lambda w \leq \alpha.\,M')\,w \bullet \mathsf{cont}\,(\_\,[w \leq w'])\,w' \twoheadrightarrow \mathsf{eval}\,([w'/\alpha]M')\,w'}
\end{array}
$$

We see that the source expression M either moves from w' to w (in the reverse direction of the accessibility relation), or is already located at w as the result of compilation and static distribution of the program. The value computed by M, $\Lambda w \leq \alpha \dot{M}'$ flows from w to w' (following accessibility) and is then evaluated at w'.

The rules for remote values $\Diamond A$ are more complex. Recall that a value of type $\Diamond A$ should be a reference to remote value of type A, whose exact location is not known in advance and not accessible from within the program.

$$
\begin{array}{lcl}
\mathsf{eval}\,(\mathbf{let}\,\langle w \leq \alpha\rangle\,x = M\ \mathbf{in}\ N)\,w'' & \twoheadrightarrow & \mathsf{eval}\,M\,w \bullet \mathsf{cont}\,(\mathbf{let}\,\langle w \leq \alpha\rangle\,x = \_\ \mathbf{in}\ N)\,w'' \\
\mathsf{eval}\,(\langle w \leq w'\rangle M')\,w & \twoheadrightarrow & \mathsf{eval}\,M'\,w' \bullet \mathsf{cont}\,(\langle w \leq w'\rangle\,\_) \\
\mathsf{ret}\,V'\,w' \bullet \mathsf{cont}\,(\langle w \leq w'\rangle\,\_) & \twoheadrightarrow & \exists x.\,!\mathsf{bind}\,x\,V'\,w' \bullet \mathsf{ret}\,(\langle w \leq w'\rangle\,x)\,w \\
\multicolumn{3}{l}{\mathsf{ret}\,(\langle w \leq w'\rangle\,x)\,w \bullet \mathsf{cont}\,(\mathbf{let}\,\langle w \leq \alpha\rangle\,x = \_\ \mathbf{in}\ N)\,w'' \twoheadrightarrow \mathsf{eval}\,([w'/\alpha]N)\,w''}
\end{array}
$$

In the last rule, again, we are used some implicit α-conversion, by naming the **let**-bound variable x the same as the reference x to the remote value V. Note that any reference to x in N will automatically take place in the right world w', since x was typed parametrically at α, and w' is substituted for α.

If we assume that all source expressions are already at the right world, we see that in the third rule the new value $\langle w \leq w' \rangle x$ is moved from w' to w, in the opposite direction of accessibility. What is disturbing, however, is that in the last rule the same value is moved from w to w'', even though w and w'' are not required to be connected. This is finally a manifestation of the non-locality of the $\Diamond E$ rule in the sense that w and w'' in that rule do not need to be connected.

So we see that computations (values of type $\Box A$) flow from w to w' where $w \leq w'$ and that references to remote values (of type $\Diamond A$) flow to w from w' where $w \leq w'$. It appears that there is also a non-local interaction, where a continuation at w'' is waiting for a value at the unconnected world w, and the value is needed before computation in the body proceeds.

This tension can be resolved in two ways. One is to restrict one's attention to the modal logic IS5. In IS5, the accessibility relation is reflexive, transitive, and symmetric. This means, if we start from a closed program at a world h, all worlds are inter-accessible. In that case, the non-local actual alluded to above is implementable. This can be done most elegantly if we localize all the rules and introduce separate rules that uniformly move values of type $\Box A$ or $\Diamond A$. This is the approach taken by Murphy et al. [MCHP04, Mur08] and closely related to the solution by Jia and Walker [JW04]. More about this solution in the next lecture.

An alternative is to keep the accessibility relation general, but restrict the rules in other ways to eliminate the need for non-local actions. We will discuss this in the lecture after next using a technique called *tethering*.

Exercises

Exercise 1 *Annotate the rules of natural deduction to isolate verifications and uses among all proofs.*

Exercise 2 *If we restrict* $\vee E$, $\bot E$ *and* $\Diamond E$ *to act entirely locally (that is,* $w'' = w$*) and keep the remaining rules as they are, then some true propositions no longer have verifications. Give examples demonstrating this.*

Exercise 3 *Show the local expansions on proof terms.*

Exercise 4 *Both proof terms and substructural operational semantics contain redundant worlds. Assume that proof terms remain the same and explore which information in the propositions defining the substructural operational semantics can be erased without leading to ambiguity as to which computation takes place where.*

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