Lecture Notes on
Tethered Semantics

15-816: Modal Logic
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Lecture 17
March 25, 2010

1 Introduction

In this lecture we continue our search for a multiple-world semantics for the intutionistic logic of validity and possibility, dubbed JS4. As we have seen in the previous lecture, the fragment without possibility, disjunction, and falsehood has a sound and complete semantics with IS4. Once those are added, however, the IS4 semantics proves too many propositions. For example, the IS4 axiom (or theorem) $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$ requires a decision *here* about a disjunction which is only known to be true at some reachable world. The objection lodged by JS4 can also be understood if we interpret reachability in a temporal way. In that case, the above proposition would mean that we have to make a decision now based on information that will only be available to us in the future.

One solution, which will also be useful later when we discuss the modal interpretation of so-called *substructural logics*, is to give a mixed semantics which employs both multiple judgments and worlds. We will take great care to make sure all actions in this logic are local. It turns out that this is enough to rule out temporal paradoxes such as the one above and obtain a semantics that is both sound and complete for JS4. This lecture is based on joint work with William Lovas and Jason Reed.

2 Tethering

The basic idea of the tethered sequent calculus is that there is a current world, namely the world associated with the proposition on the right-hand side of the sequent, and that all left rules are restricted to take place in the current world. The name "tether" derives from the intuition that the proposition on the left are tied to the current world on the right.

For the ordinary intuitionistic connectives, tethering is completely straightforward. We have two judgments: A is true at world w which can be either in the antecedent or succedent of the sequent.

$$\frac{P@w \in \Gamma}{\Gamma \Longrightarrow P \;@\; w}\;\mathsf{init}$$

$$\frac{\Gamma, A@w \Longrightarrow B \;@\; w}{\Gamma \Longrightarrow A \supset B \;@\; w}\;{\supset}R$$

$$\frac{\Gamma, A \supset B@w \Longrightarrow A \;@\; w \quad \Gamma, A \supset B@w, B@w \Longrightarrow C \;@\; w}{\Gamma, A \supset B@w \Longrightarrow C \;@\; w}\;{\supset}L$$

$$\frac{\Gamma \Longrightarrow A \;@\; w}{\Gamma \Longrightarrow A \vee B \;@\; w}\;{\vee}R_1 \qquad \frac{\Gamma \Longrightarrow B \;@\; w}{\Gamma \Longrightarrow A \vee B \;@\; w}\;{\vee}R_2$$

$$\frac{\Gamma, A \vee B@w, A@w \Longrightarrow C \;@\; w \quad \Gamma, A \vee B@w, B@w \Longrightarrow C \;@\; w}{\Gamma, A \vee B@w \Longrightarrow C \;@\; w}\;{\vee}L$$

$$\text{no } {\perp}R \qquad\qquad \frac{}{\Gamma, \perp@w \Longrightarrow C \;@\; w}\;{\perp}L$$

Figure 1: Tethered sequent calculus, fragment with $\supset$, $\vee$, $\perp$

When we try to tether the modal operators, however, we find some difficulties. For example, with the rules

$$\frac{\Gamma, w \leq \alpha \Longrightarrow A \;@\; \alpha}{\Gamma \Longrightarrow \Box A \;@\; w}\;\Box R^{\alpha}? \qquad \frac{\Gamma^{\leq} \vdash w \leq w' \quad \Gamma, \Box A@w, A@w' \Longrightarrow C \;@\; w}{\Gamma, \Box A@w \Longrightarrow C \;@\; w}\;\Box L?$$

the identity principle fails: after

$$\frac{\Box A@h \Longrightarrow A \;@\; \alpha}{\Box A@h \Longrightarrow \Box A \;@\; h}\;\Box R^{\alpha}$$

the left rule cannot be applied to the assumption $\Box A@h$ because $h \neq \alpha$.

Perhaps surprisingly, the solution is to eliminate accessibility altogether. This goes back to the time we introduced the modal logic of validity, where we mentioned multiple worlds, but we did not mention accessibility (nor did it seem to come up). However, this does require us to introduce a new jugment which we write $A\,!$ which means that A is true in all worlds. We then have

$$\frac{\Gamma \Longrightarrow A \mathbin{@} \alpha}{\Gamma \Longrightarrow \Box A \mathbin{@} w}\ \Box R^{\alpha} \qquad \frac{\Gamma, \Box A@w, A\,! \Longrightarrow C \mathbin{@} w}{\Gamma, \Box A@w \Longrightarrow C \mathbin{@} w}\ \Box L$$

$$\frac{\Gamma, A\,!, A@w \Longrightarrow C \mathbin{@} w}{\Gamma, A\,! \Longrightarrow C \mathbin{@} w}\ \mathsf{copy}$$

In the $\Box R$ rule, α must be a new world parameter, and would make sense to make this explicit as a hypothesis $\alpha\,\mathit{world}$. We elide these additional assumptions.

The judgmental rule copy is also tethered in the sense that the new hypothesis $A@w$ must be at the current world, even though $A\,!$ is universal.

We make a similar adjustment to the rules for possibility. We have a new judgment $A\ ?\ w$ which means that A is possible at w.

$$\frac{\Gamma \Longrightarrow A\ ?\ w}{\Gamma \Longrightarrow \Diamond A \mathbin{@} w}\ \Diamond R \qquad \frac{\Gamma, \Diamond A@w, A@\alpha \Longrightarrow C\ ?\ \alpha}{\Gamma, \Diamond A@w \Longrightarrow C\ ?\ w}\ \Diamond L^{\alpha}$$

$$\frac{\Gamma \Longrightarrow A \mathbin{@} w}{\Gamma \Longrightarrow A\ ?\ w}\ \mathsf{here}$$

In the $\Diamond L$ rule, we use the assumption $\Diamond A@w$ to show that C is possible at w by moving to a new world α where A is true and showing that C is possible there. So tethering affects both the conclusion and the premise.

Now we have to generalize the previous rules left rules to permit a conclusion $A \mathbin{@} w$ or $A\ ?\ w$. We write "$*$" for a variable which is either "$@$" or "$?$". The rules are summarized in Figure 2.

One interesting aspect of the tethered sequent calculus is that all hypotheses are *persistent*: once made, they will be in the context for the rest of the proof (constructed bottom-up). In contrast, in JS4 the $\Box R$ rule clears that truth context, because truth cannot be used in the proof of validity. This means that this calculus is well-suited for implementation in logical frameworks based on hypothetical judgments such as LF.

In the example below we have elided assumptions that are no longer

$$\frac{P@w \in \Gamma}{\Gamma \Longrightarrow P \mathbin{@} w}\ \mathsf{init}$$

$$\frac{\Gamma, A@w \Longrightarrow B \mathbin{@} w}{\Gamma \Longrightarrow A \supset B \mathbin{@} w}\ {\supset}R$$

$$\frac{\Gamma, A \supset B@w \Longrightarrow A \mathbin{@} w \quad \Gamma, A \supset B@w, B@w \Longrightarrow C * w}{\Gamma, A \supset B@w \Longrightarrow C * w}\ {\supset}L$$

$$\frac{\Gamma \Longrightarrow A \mathbin{@} w}{\Gamma \Longrightarrow A \vee B \mathbin{@} w}\ {\vee}R_1 \qquad \frac{\Gamma \Longrightarrow B \mathbin{@} w}{\Gamma \Longrightarrow A \vee B \mathbin{@} w}\ {\vee}R_2$$

$$\frac{\Gamma, A \vee B@w, A@w \Longrightarrow C * w \quad \Gamma, A \vee B@w, B@w \Longrightarrow C * w}{\Gamma, A \vee B@w \Longrightarrow C * w}\ {\vee}L$$

$$\text{no } {\perp}R \qquad \frac{}{\Gamma, \perp@w \Longrightarrow C * w}\ {\perp}L$$

$$\frac{\Gamma \Longrightarrow A \mathbin{@} \alpha}{\Gamma \Longrightarrow \Box A \mathbin{@} w}\ \Box R^{\alpha} \qquad \frac{\Gamma, \Box A@w, A\,! \Longrightarrow C * w}{\Gamma, \Box A@w \Longrightarrow C * w}\ \Box L$$

$$\frac{\Gamma, A\,!, A@w \Longrightarrow C * w}{\Gamma, A\,! \Longrightarrow C * w}\ \mathsf{copy}$$

$$\frac{\Gamma \Longrightarrow A \mathbin{?} w}{\Gamma \Longrightarrow \Diamond A \mathbin{@} w}\ \Diamond R \qquad \frac{\Gamma, \Diamond A@w, A@\alpha \Longrightarrow C \mathbin{?} \alpha}{\Gamma, \Diamond A@w \Longrightarrow C \mathbin{?} w}\ \Diamond L^{\alpha}$$

$$\frac{\Gamma \Longrightarrow A \mathbin{@} w}{\Gamma \Longrightarrow A \mathbin{?} w}\ \mathsf{here}$$

Figure 2: Tethered sequent calculus, $* \in \{@, ?\}$

needed (despite the fact that all are persistent).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A@\alpha \Longrightarrow A \mathbin{@} \alpha}\ \textsf{init} \quad \overline{B@\alpha \Longrightarrow B \mathbin{@} \alpha}\ \textsf{init}}{A \supset B@\alpha, A@\alpha \Longrightarrow B \mathbin{@} \alpha}\ {\supset}L}{(A \supset B)\,!, A@\alpha \Longrightarrow B \mathbin{@} \alpha}\ \textsf{copy}}{(A \supset B)\,!, A@\alpha \Longrightarrow B \mathbin{?} \alpha}\ \textsf{here}}{(A \supset B)\,!, \Diamond A@h \Longrightarrow B \mathbin{?} h}\ \Diamond L}{(A \supset B)\,!, \Diamond A@h \Longrightarrow \Diamond B \mathbin{@} h}\ \Diamond R}{\Box(A \supset B)@h, \Diamond A@h \Longrightarrow \Diamond B \mathbin{@} h}\ \Box L}{\Longrightarrow \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \mathbin{@} h}\ {\supset}I \times 2}{}$$

We can also try a counterexample to the completeness of JS4 with respect to IS4: after one step

$$\dfrac{\Diamond(A \vee B)@h \Longrightarrow \Diamond A \vee \Diamond B \mathbin{@} h}{\Longrightarrow \Diamond(A \vee B) \supset \Diamond A \vee \Diamond B \mathbin{@} h}$$

we are stuck, because $\Diamond L$ requires the succedent to have the form $C \mathbin{?} h$ which it does not.

In general, there seems to be a really close correspondence between proof in the sequent calculus in JS4 and its tethered semantics. Before we establish that, we should verify the *internal* properties of the tethered system.

3 Cut and Identity for the Tethered Sequent Calculus

Taking simple properties such as weakening and contraction for granted, we can conjecture three cut principles, in analogy with JS4. First, cut of a truth assumption.

$$\dfrac{\Gamma \Longrightarrow A \mathbin{@} w \quad \Gamma, A@w \Longrightarrow C * w''}{\Gamma \Longrightarrow C * w''}\ \text{cut}$$

One might wonder if we the right-hand side should be tethered, that is, $C \mathbin{@} w$ should be used instead of $C \mathbin{@} w''$. Indeed, this works as well, because an assumption $A@w$ can never be used when the conclusion is $C \mathbin{@} w''$ unless $w = w''$.

Second, how do we cut an assumption $A\,!$? From the premise of $\Box R$ we can see that we should have A @ α, for a parameter α. In other words, A should be true in an arbitrary word α.

$$\frac{\Gamma \Longrightarrow A \,@\, \alpha \quad \Gamma, A\,! \Longrightarrow C * w}{\Gamma \Longrightarrow C * w}\ \text{cut!}^{\alpha}$$

The side condition here is that α does not occur in Γ and $\alpha \neq w$.

Finally, to cut the assumption of possibility we just have have to repeat the insights above, considering the premise of $\Diamond L$.

$$\frac{\Gamma \Longrightarrow A \mathrel{?} w \quad \Gamma, A@\alpha \Longrightarrow C \mathrel{?} \alpha}{\Gamma \Longrightarrow C \mathrel{?} w}\ \text{cut?}^{\alpha}$$

Here, the restriction on α means that is does not occur in Γ and $\alpha \neq w$. However, α does occur in the succedent of the premise.

As usual, we do not assume these as rules, but show their admissibility. The proof is very similar to the proof of cut elimination for JS4. This might be expected, given the strong correspondence of the two systems given in the next section.

Theorem 1 (Tethered Cut)

(i) *If* $\Gamma \Longrightarrow A \,@\, w$ *and* $\Gamma, A@w \Longrightarrow C * w$ *then* $\Gamma \Longrightarrow C * w$.

(ii) *If* $\Gamma \Longrightarrow A \,@\, \alpha$ *and* $\Gamma, A\,! \Longrightarrow C * w$ *for* α *not in* Γ, $\alpha \neq w$ *then* $\Gamma \Longrightarrow C * w$.

(iii) *If* $\Gamma \Longrightarrow A \mathrel{?} w$ *and* $\Gamma, A@\alpha \Longrightarrow C \,@\, \alpha$ *for* α *not in* Γ, $\alpha \neq w$ *then* $\Gamma \Longrightarrow C \mathrel{?} w$.

Proof: By a nested induction, first on the structure of the cut formula A, second on the type of cut where $(i) < (ii)$ and $(i) < (iii)$, third simultaneously on the structure of the two given proofs (one of which must get smaller while the other one stays the same). $\Box$

Identity is quite straightforward, as in JS4.

Theorem 2 (Tethered Identity) $\Gamma, A@h \Longrightarrow A@h$ *for any* Γ, *proposition* A *and world* h.

Proof: By induction on the structure of A. $\Box$

4 From JS4 to Tethered Sequent Calculus

In this section we should the soundness of the tethered rules with respect to JS4. The translation is entirely straightforward and arises from the basic intuition underlying the systems.

Theorem 3 (From JS4 to Tethered Sequent Calculus)

(i) If $\Delta; \Gamma \Longrightarrow A; \cdot$ *then* $\Delta\,!, \Gamma@h \Longrightarrow A \mathbin{@} h$ *for any* h.

(ii) If $\Delta; \Gamma \Longrightarrow \cdot; A$ *then* $\Delta\,!, \Gamma@h \Longrightarrow A \mathbin{?} h$.

Proof: By induction on the structure of the given derivation. For any rule in JS4 we can apply a corresponding rule in the tethered sequent calculus, since the truth assumptions in Γ are translated to assumptions at the same world h as the succedent. We show only two cases.

Case:

$$\frac{\Delta; \bullet \vdash A; \cdot}{\Delta; \Gamma \vdash \Box A; \cdot}\ \Box R$$

Then we construct

$$\dfrac{\dfrac{\overset{\text{i.h.}}{\Delta\,! \Longrightarrow A \mathbin{@} \alpha}}{\Delta\,!, \Gamma@h \Longrightarrow A \mathbin{@} \alpha}\ \text{weaken}}{\Delta\,!, \Gamma@h \Longrightarrow \Box A \mathbin{@} h}\ \Box R^{\alpha}$$

Case:

$$\frac{\Delta; \bullet, A \Longrightarrow \cdot; C}{\Delta; \Gamma', \Diamond A \Longrightarrow \cdot; C}\ \Diamond L$$

Then we construct

$$\dfrac{\dfrac{\overset{\text{i.h.}}{\Delta\,!, A@\alpha \Longrightarrow C \mathbin{?} \alpha}}{\Delta\,!, \Gamma'@h, \Diamond A@h, A@\alpha \Longrightarrow C \mathbin{?} \alpha}\ \text{weaken}}{\Delta\,!, \Gamma'@h, \Diamond A@h \Longrightarrow C \mathbin{?} h}\ \Diamond L$$

$\Box$

5 From Tethered Sequent Calculus to JS4

From the proof above we see that the key to the proof in the other direction has to be a strengthening property, which is the inverse of the weakening property we have used.

Theorem 4 (Tethered Strengthening) *If $\Gamma, A \;@\; w \Longrightarrow C * w''$ where $w \neq w''$, then $\Gamma \Longrightarrow C * w''$ with the same proof.*

Proof: By induction over the structure of the given derivation. The assumption $A \;@\; w$ can never be used because all the left rules are tethered. □

Now it is easy to give the translation from the tethered sequent calculus to JS4, showing the completeness of JS4 with respect to the tethered semantics.

Theorem 5 (From Tethered Sequent Calculus to JS4)

(i) If $\Delta\,!, \Gamma @ h \Longrightarrow A \;@\; h$ then $\Delta; \Gamma \Longrightarrow A; \cdot$

(ii) If $\Delta\,!, \Gamma @ h \Longrightarrow A \;?\; h$ then $\Delta; \Gamma \Longrightarrow \cdot; A$.

Proof: By mutual induction on the given derivation. In each case we just replay the rule from the tethered calculus in JS4. We only show one case where strengthening is required. We also use that strengthening does not change the proof, so we can apply the induction hypothesis to the result.

Case:

$$\frac{\Delta\,!, \Gamma' @ h, \Diamond B @ h, B @ \alpha \Longrightarrow A \;?\; \alpha}{\Delta\,!, \Gamma' @ h, \Diamond B @ h \Longrightarrow A \;?\; h}\ \Diamond L^{\alpha}$$

Then we have

$\Delta\,!, B @ \alpha \Longrightarrow A \;?\; \alpha$	By strengthening
$\Delta; \bullet, B \Longrightarrow \cdot; A$	By i.h.(ii)
$\Delta; \Gamma, \Diamond B \Longrightarrow \cdot; A$	By rule $\Diamond L$

□

By examining the proofs of the translations in the two directions (Theorem 3 and 5) we see that the two sequent calculi are in bijective correspondence. This allows us to view the tethered sequent calculus as an implementation of the modal sequent calculus with judgments of truth, validity, and possibility. The distinctive property of this representation is that hypotheses are persistent throughout a derivation: assumptions may become unusable after a certain point, but they will still be in the antecedent. This in turn allows the straightforward encoding of the tethered sequent calculus in logical frameworks based on hypothetical judgments such as LF. An implementation of the tethered sequent calculus as well as the formalized proofs of cut and identity can be found at `http://www.cs.cmu.edu/~fp/courses/15816-s10/lectures/17-tethered/`.

6 Tethered Natural Deduction

We can also give a system of natural deduction which is tethered in a similar way. With an abuse of notation, we write the judgments $A\,!$, $A \mathbin{@} w$ and $A\ ?\ w$ as for the sequent calculus. We present it in Figure 3 in the form of a proof term calculus; the pure natural deduction can be extracted by eliminating the proof terms. We conjecture, but have not yet attempted to prove, that tethered natural deduction can be related to the tethered sequent calculus as for other modal logics, and that strengthening is the crucial property to make this possible. We also conjecture that one can directly give a distributed semantics for the tethered calculus, which is quite different from the one for S5 given previously, and which does not require any "action at a distance". We may return to these questions in a future lecture.

$$\frac{x{:}P@w \in \Gamma}{\Gamma \vdash x : P \mathbin{@} w}\ \mathsf{hyp}$$

$$\frac{\Gamma, x{:}A@w \vdash M : B \mathbin{@} w}{\Gamma \vdash \lambda x.\, M : A \supset B \mathbin{@} w}\ {\supset}I$$

$$\frac{\Gamma \vdash M : A \supset B \mathbin{@} w \quad \Gamma \vdash N : A \mathbin{@} w}{\Gamma \vdash M\, N : B \mathbin{@} w}\ {\supset}E$$

$$\frac{\Gamma \vdash M : A \mathbin{@} w}{\Gamma \vdash \mathbf{inl}\, M : A \vee B \mathbin{@} w}\ {\vee}I_1 \qquad \frac{\Gamma \vdash M : B \mathbin{@} w}{\Gamma \vdash \mathbf{inr}\, M : A \vee B \mathbin{@} w}\ {\vee}I_2$$

$$\frac{\Gamma \vdash M : A \vee B \mathbin{@} w \quad \Gamma, x{:}A@w \vdash N : C * w \quad \Gamma, y{:}B@w \vdash O : C * w}{\Gamma \vdash \mathbf{case}\, M\ \mathbf{of\ inl}\, x \Rightarrow N \mid \mathbf{inr}\, y \Rightarrow O : C * w}\ {\vee}E$$

$$\text{no } {\perp}R \qquad \frac{\Gamma \vdash M : \perp \mathbin{@} w}{\Gamma \vdash \mathbf{abort}\, M : C * w}\ {\perp}E$$

$$\frac{\Gamma, \alpha\ \mathit{world} \vdash M : A \mathbin{@} \alpha}{\Gamma \vdash \mathbf{box}(\alpha.\, M) : \Box A \mathbin{@} w}\ \Box I^{\alpha} \qquad \frac{\Gamma \vdash M : \Box A \mathbin{@} w \quad \Gamma, u{:}A\,! \vdash N : C * w}{\Gamma \vdash \mathbf{let\, box}\, u = M\ \mathbf{in}\ N : C * w}\ \Box E$$

$$\frac{u{:}A\,! \in \Gamma}{\Gamma \vdash u[w] : A \mathbin{@} w}\ \mathsf{vhyp}$$

$$\frac{\Gamma \vdash E : A\ ?\ w}{\Gamma \vdash \mathbf{dia}\, E : \Diamond A \mathbin{@} w}\ \Diamond I \qquad \frac{\Gamma \vdash M : \Diamond A \mathbin{@} w \quad \Gamma, \alpha\ \mathit{world}, x{:}A@\alpha \vdash N : C\ ?\ \alpha}{\Gamma \vdash \mathbf{let}\, \langle \alpha \rangle\, x = M\ \mathbf{in}\ N : C\ ?\ w}\ \Diamond E^{\alpha}$$

$$\frac{\Gamma \vdash M : A \mathbin{@} w}{\Gamma \vdash \langle w \rangle M : A\ ?\ w}\ \mathsf{poss}$$

Figure 3: Tethered natural deduction, $* \in \{@, ?\}$

Exercises

Exercise 1 *Show the principal cases for $\Box$ and $\Diamond$ in the proof of tethered cut (Theorem 1).*

Exercise 2 *Show the cases for $\Box$ and $\Diamond$ in the proof of tethered identity (Theorem 2)*

Exercise 3 *Define verification and uses for tethered natural deduction as presented in Figure 3.*