Lecture Notes on Completeness and Canonical Models

15-816: Modal Logic
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Lecture 20
April 6, 2010

1 Introduction to This Lecture

In this lecture, we study completeness of (Hilbert-style) proof systems for propositional modal logics. The device of canonical models gives a rich and systematic framework for understanding completeness questions and other advanced properties. Also see [HC96, Sch03].

2 Normal Modal Logics

In this lecture we consider a logic as the set of its tautologies. The following definition captures the closure properties that the we expect from this set of tautologies:

Definition 1 (Normal modal logic) *A set L of formulas is called a* normal modal logic *if:*

1. *L contains all propositional tautologies*
2. *$\Box(p \to q) \to (\Box p \to \Box q) \in L$ for all propositional letters p, q*
3. *$A \in L, (A \to B) \in L$ implies $B \in L$ (closed under modus ponens)*
4. *$A \in L$ implies $\Box A \in L$ (Gödel)*
5. *$A \in L$ implies $A' \in L$ for all instances A' of A (closed under instantiation). An instance results by substituting any number of propositional letters by arbitrary propositional modal formulas.*

Definition 2 (Normal modal logic proof system) *A proof system* $\mathbf{S}$ *of modal logic is called a* normal modal logic proof system*, if*

1. $\mathbf{S}$ *can derive all propositional tautologies*
2. $\Box(p \to q) \to (\Box p \to \Box q)$ *is an axiom of* $\mathbf{S}$
3. *Modus ponens and Gödel generalization are proof rules of* $\mathbf{S}$*.*

The set $\{A : \vdash_{\mathbf{S}} A\}$ of all formulas provable in a normal modal logic proof systems is a normal modal logic. The proof systems for $\mathbf{K}$, $\mathbf{T}$ and $\mathbf{S4}$ that we have seen before are normal.

Other properties that we have seen before can also be shown easily to hold in normal modal logics.

Lemma 3 *Let L be a normal modal logic. Then for any formulas A, B, C:*

1. $\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B) \in L$
2. $(A \to B) \in L$ *implies* $(\Box A \to \Box B) \in L$
3. $(A \leftrightarrow B) \in L$ *implies* $(C \leftrightarrow D) \in L$ *where D results from C by replacing subformula A by B*

3 Consistency

Definition 4 (Consistency) *Let L be a normal modal logic. A set S of formulas of propositional modal logic is called L-*consistent *iff there are no formulas $A_1, \ldots, A_n \in S$ with*

$$(A_1 \wedge \cdots \wedge A_n \to \mathit{false}) \in L$$

*Otherwise S is called L-*inconsistent*. A consistent set S of propositional modal formulas is called $maximally consistent$ iff, for every formula A either $A \in S$ or $\neg A \in S$.*

We assume normal modal logics L to be consistent.

Lemma 5 *Let L be a normal modal logic and S maximally L-consistent, then*

1. *For every formula A exactly one of the following cases holds, either $A \in S$ or $\neg A \in S$.*
2. *$A \in S, (A \to B) \in S$ then $B \in S$ (closed under modus ponens).*

3. $(A \wedge B) \in S$ *iff* $A \in S$ *and* $B \in S$
4. $(A \vee B) \in S$ *iff* $A \in S$ *or* $B \in S$
5. $L \subseteq S$

Proof: 1. One of A or $\neg A$ must be in S, which is maximally consistent. If both were in S then S would be inconsistent, because the propositional tautology $(A \wedge \neg A \rightarrow \mathit{false}) \in L$.

2. Let $A \in S, (A \rightarrow B) \in S$ but $B \notin S$. By maximal consistency, $\neg B \in S$. Consider tautology $(A \wedge (A \rightarrow B) \wedge \neg B \rightarrow \mathit{false}) \in L$. This contradicts the consistency of S.

3. Similar to the next case.

4. Let us prove the direction from left to right. Let $(A \vee B) \in S$ and $A \notin S, B \notin S$. Hence, by maximal consistency, $\neg A \in S, \neg B \in S$. Also the tautology $(\neg A \wedge \neg B \wedge (A \vee B) \rightarrow \mathit{false}) \in L$. That contradicts the consistency of S.

   Conversely, let $A \in S, (A \vee B) \notin S$. Then maximal consistency shows $\neg(A \vee B) \in S$. But the tautology $(A \wedge (A \vee B) \rightarrow \mathit{false}) \in L$ contradicts the consistency of F.

5. Let $A \in L$. Then $\{\neg A\}$ is L-inconsistent. Thus $\neg A \notin S$. By maximal consistency, $A \in S$.

$\square$

Lemma 6 *For every consistent set S there is a maximally consistent superset M.*

Proof: Fix an ordering $A_0, A_1, A_2, \dots, A_n, \dots$ of all propositional model formulas ordered. Define an ascending chain of sets of formulas $S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots \subseteq S_n \subseteq \dots$ by:

$$S_0 := S$$

$$S_{n+1} := \begin{cases} S_n \cup \{A_n\} & \text{if this set is consistent} \\ S_n \cup \{\neg A_n\} & \text{otherwise} \end{cases}$$

We prove by induction on n that S_n is consistent. The case $n = 0$ follows from the fact that F was assumed consistent. Suppose S_{n+1} was inconsistent. By construction $S_n \cup \{A_n\}$ and $S_n \cup \{\neg A_n\}$ are both inconsistent then.

Hence there are formulas $B_1, \ldots, B_k, C_1, \ldots, C_l \in S_n$:

$$(B_1 \wedge \cdots \wedge B_k \wedge A_n \to \mathit{false}) \in L$$
$$(C_1 \wedge \cdots \wedge C_l \wedge \neg A_n \to \mathit{false}) \in L$$

Now L contains all propositional tautologies and is closed under modus ponens (Lemma 5), thus the above lines imply

$$(B_1 \wedge \cdots \wedge B_k \wedge C_1 \wedge \cdots \wedge C_l \to \mathit{false}) \in L$$

which contradicts the induction hypothesis that S_n is consistent.

Define $M := \bigcup_{n=0}^{\infty} S_n$. Then

- M is consistent: otherwise there is an F_n in which the inconsistency witness lies, but F_n is consistent.
- M is maximally consistent: because, for each formula A_i, S_i contains either A_i or $\neg A_i$, hence so does the union M.
- $S \subseteq M$

□

Lemma 7 *Let S be a consistent set of formulas and $\neg\Box A \in S$, then $\Box^- S \cup \{\neg A\}$ is consistent where $\Box^- S := \{A : \Box A \in S\}$.*

Proof: Suppose $\Box^- S \cup \{\neg A\}$ is inconsistent then there are $A_1, \ldots, A_n \in \Box^- S$ such that

$$(A_1 \wedge \cdots \wedge A_n \wedge \neg A \to \mathit{false}) \in L$$

Note that we can assume $\neg A$ to occur in this inconsistency witness because $(X \to \mathit{false}) \in L$ implies $(X \wedge \neg A \to \mathit{false}) \in L$. Now propositional reasoning implies

$$(A_1 \wedge \cdots \wedge A_n \to A) \in L$$

Hence the monotonicity property (Lemma 32 of normal modal logics implies

$$(\Box(A_1 \wedge \cdots \wedge A_n) \to \Box A) \in L$$

Now the property of conjunctive distributitivity (Lemma 31) with the substitution property (Lemma 33) of normal modal logics imply

$$(\Box A_1 \wedge \cdots \wedge \Box A_n \to \Box A) \in L$$

Propositional reasoning implies the following witness of the inconsistency of F:

$$(\Box A_1 \land \cdots \land \Box A_n \land \neg\Box A \rightarrow \mathit{false}) \in L$$

□

Beware that the consistency of S does not imply that $\Box^- S$ is consistent. For the trivial Kripke structure with empty accessibility relation and only one world s, $S := \{A \ : \ K, s \models A\}$ is maximally **K**-consistent. Especially $\Box A, \Box\neg A \in S$ for any formula A. But that means that $\Box^- S$ is inconsistent.

4 Canonical Kripke Structure

Let L be a normal propositional modal logic, considered as the set of its tautologies.

Theorem 8 (Canonical Kripke Structure) *For a normal propositional modal logic L, let $K_L = (W_L, \rho_L, v_L)$ be the* canonical Kripke structure *of L, i.e.:*

- *W_L is the set of all maximally L-consistent sets of propositional modal formulas (built from the vocabulary);*
- *$S\rho_L T$ iff $\Box^- S \subseteq T$ where $\Box^- S := \{A \ : \ \Box A \in S\}$;*
- $v_L(S)(q) := \begin{cases} 1 & \text{if } q \in S \\ 0 & \text{if } q \notin S \end{cases}$

Then for any world $S \in W_L$ and any formula A:

$$K_L, S \models A \quad \text{iff} \quad A \in S$$

Proof: The proof is by induction on A.

0. The case where A is a propositional letter is by definition.

1. If A is of the form $A_1 \land A_2$ then by Lemma 5 and by induction hypothesis we have that

$$\begin{aligned} & K_L, S \models A_1 \land A_2 \\ \text{iff } & K_L, S \models A_1 \text{ and } K_L, S \models A_2 \\ \text{iff } & A_1 \in S \text{ and } A_2 \in S \\ \text{iff } & (A_1 \land A_2) \in S \end{aligned}$$

2. If A is of the form $\Box B$ then we reason by cases. First assume $\Box B \in S$. Consider any world $T \in W_L$ with $S \rho_L T$. That is $\Box^- S \subseteq T$, hence $B \in T$. Thus, by induction hypothesis, $K_L, T \models B$, which implies $K_L, S \models \Box B$, because T was arbitrary.

   Now assume $\Box B \notin S$. Thus $\neg\Box B \in S$ by maxi-consistency. Hence by Lemma 7 the set $\Box^- S \cup \{\neg B\}$ is consistent and, by Lemma 6 there is a (maximally consistent extension) world $T \in W_L$ with $T \supseteq \Box^- S \cup \{\neg B\}$. Especially, $S \rho_L T$. By induction hypothesis, $\neg B \in T$ yields $K_L, T \models \neg B$, which implies $K_L, S \models \neg\Box B$.

□

Corollary 9 *Let K_L be the canonical Kripke structure of normal modal logic L, then:*

$$A \in L \quad \textit{iff} \quad K_L \models A$$

Proof: By Lemma 5, L is a subset of every world $S \in W_L$. Thus the direction from left to right follows from Theorem 8.

Conversely let $K_L \models A$, i.e., $K_L, S \models A$ for all $S \in W_L$. Suppose $A \notin L$. But then $L \cup \{\neg A\}$ would be consistent: otherwise there were $A_1, \ldots A_n \in L$ with $(A_1 \wedge \ldots A_n \wedge \neg A \rightarrow \mathit{false}) \in L$ which would imply $A \in L$ for the logic. Since $L \cup \{\neg A\}$ is consistent, there, thus, is a (maximally consistent extension) world $T \in W_L$ with $T \supseteq L \cup \{\neg A\}$. In particular, $\neg A \in T$, such that Theorem 8 implies $K_L, T \models \neg A$, which would contradict $K_L \models A$. □

This implies a kind of completeness, but is surprising in that it connects provability in a system with validity, not in all, but only in one Kripke structure.

Corollary 10 *Let $\vdash_{\mathbf{S}}$ be a provability relation for a normal modal logic proof system and K_L the canonical Kripke structure for the logic $L := \{A : \vdash_{\mathbf{S}} A\}$, then*

$$\vdash_{\mathbf{S}} A \quad \textit{iff} \quad K_L \models A$$

Proof: Consider $L := \{A : \vdash_{\mathbf{S}} A\}$ in the last corollary. □

This corollary is a starting point for proving full completeness.

Proposition 11 (Completeness for K) *For every modal logic formula A*

$$\vdash_{\mathbf{K}} A \quad \textit{iff} \quad \vDash_{\mathbf{K}} A \quad \textit{iff} \quad K \models A \quad \textit{for every Kripke structure } K$$

Proof: If $K \models A$ for every Kripke structure K, then also for the canonical Kripke structure, thus Corollary 10 implies $\vdash_{\mathbf{K}} A$.

The converse direction is soundness that every axiom of $\mathbf{K}$ holds in all Kripke structures and every proof rule of $\mathbf{K}$ preserves validity (see Lecture 7). $\square$

Proposition 12 (Completeness for T) *For every modal logic formula A*

$$\vdash_{\mathbf{K}} A \quad \textit{iff} \quad \vDash_{\mathbf{T}} A \quad \textit{iff} \quad K \models A \quad \textit{for every reflexive Kripke structure } K$$

Proof: The only new part is the need to show that the T-axiom is true in all reflexive Kripke structures (which follows from Lecture 7), and that the canonical Kripke structure for $\mathbf{T}$ is reflexive. Consider a maximal $\mathbf{T}$-consistent set S. We have to show that $\Box^{-}S \subseteq S$. Consider any $\Box A \in S$. By Lemma 5.5 the T-instance $\Box A \to A$ is an element of S, thus $A \in S$ by Lemma 5.2. $\square$

In a similar way, completeness can be shown for the modal logics $\mathbf{S4}$ and $\mathbf{S5}$ [HC96].

Theorem 13 (Strong completeness) *Let $\mathbf{S}$ be the normal modal logic (Hilbert) proof system $\mathbf{K}$ or $\mathbf{T}$ (or $\mathbf{S4}$ or $\mathbf{S5}$) and let Γ be a set of (propositional) modal formulas and A a modal formula. Then the global consequence relation $\vDash^g_{\mathbf{S}}$ of $\mathbf{S}$ and its provability relation $\vdash_{\mathbf{S}}$ coincide:*

$$\Gamma \vdash_{\mathbf{S}} A \quad \textit{iff} \quad \Gamma \vDash^g_{\mathbf{S}} A$$

Proof: The soundness direction is as usual. For the completeness direction, it is easy to see that $L := \{A : \Gamma \vdash_{\mathbf{S}} A\}$ is a normal modal logic. Let K_L be the canonical Kripke structure for L. Assume $\Gamma \vDash^g_{\mathbf{S}} A$. Now the fact that $\Gamma \subseteq L$ implies that $K_L \models \Gamma$. Thus $K_L \models A$. Now Corollary 9 implies that $A \in L$, i.e., $\Gamma \vdash_{\mathbf{S}} A$. $\square$

References

[HC96] G.E. Hughes and M.J. Cresswell. *A New Introduction to Modal Logic*. Routledge, 1996.

[Sch03] Peter H. Schmitt. Nichtklassische Logiken. Vorlesungsskriptum Fakultät für Informatik , Universität Karlsruhe, 2003.