Midterm Exam

15-816 Linear Logic
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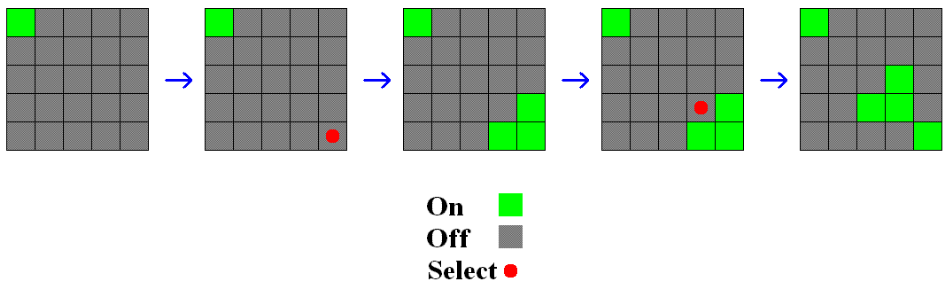
Instructions

- This exam is closed-book, closed-notes.
- You have 80 minutes to complete the exam.
- There are 4 problems.

	Lights Out	Substructural Logics	Focusing	Quotations	
	Prob 1	Prob 2	Prob 3	Prob 4	Total
Score	**40**	**40**	**40**	**30**	**150**
Max	40	40	40	30	150

1 Lights Out (40 pts)

Lights Out is an electronic game, released by Tiger Toys in 1995. The game consists of a 5 by 5 grid of lights. When the game starts, a random number or a stored pattern of these lights is switched on. Pressing any of the lights will toggle it and the four adjacent lights. The goal of the puzzle is to switch all the lights off, preferably in as few button presses as possible.



(For those in black and white: in the second position, the lower right-hand corner is selected; in the third position one square left and up from the lower right-hand corner is selected).

We represent it a position as follows

$\mathsf{sq}(x, y, \mathsf{off})$	square (x, y) is off
$\mathsf{sq}(x, y, \mathsf{on})$	square (x, y) is on

where x and y are integers in unary representation and the lower left-hand corner is $(0, 0)$. For example, the fact that the upper left-hand corner is on in the first position is represented as $\mathsf{sq}(0, \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))), \mathsf{on})$; the fact the the lower right-hand corner is off as $\mathsf{sq}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))), 0, \mathsf{off})$.

We say a state Δ is *well-formed* if there is exactly one ephemeral fact $\mathsf{sq}(x, y, \mathsf{off})$ or $\mathsf{sq}(x, y, \mathsf{on})$ for every square $0 \leq x, y \leq 4$

Task 1 (25 pts). Write a linear logic specification such that for any well-formed state Δ, we have $\Delta, \mathsf{init} \longrightarrow^* \mathsf{allout}$ if and only if all lights are out. It does not matter what happens if the state is not well-formed. You may write inference rules, or logical propositions for forward chaining, at your discretion.

Using inference rules:

$$\frac{\mathsf{sq}(x, y, \mathsf{off})}{\mathsf{allout}} \qquad \frac{\mathsf{allout} \quad \mathsf{allout}}{\mathsf{allout}} \qquad \frac{\mathsf{init}}{\cdot}$$

In propositional form:

$$\forall x.\, \forall y.\, \mathsf{sq}(x, y, \mathsf{off}) \multimap \mathsf{allout}$$
$$\mathsf{allout} \otimes \mathsf{allout} \multimap \mathsf{allout}$$
$$\mathsf{init} \multimap \mathbf{1}$$

These work correctly as a forward-chaining logic program.

Task 2 (15 pts). Under the same assumptions as in Task 1, extend (or rewrite) your specification so that also $\Delta, \mathsf{init} \longrightarrow^* \mathsf{someon}$ if and only if at least one of the lights is on. You may write inference rules, or logical propositions for forward chaining, at your discretion.

We add:

$$\frac{\mathsf{sq}(x, y, \mathsf{on})}{\mathsf{someon}} \qquad \frac{\mathsf{someon} \quad \mathsf{someon}}{\mathsf{someon}} \qquad \frac{\mathsf{someon} \quad \mathsf{allout}}{\mathsf{someon}}$$

In propositional form

$$\forall x. \forall y. \mathsf{sq}(x, y, \mathsf{on}) \multimap \mathsf{someon}$$
$$\mathsf{someon} \otimes \mathsf{someon} \multimap \mathsf{someon}$$
$$\mathsf{someon} \otimes \mathsf{allout} \multimap \mathsf{someon}$$

2 Substructural Logics (40 pts)

In linear logic, linear resources (also called ephemeral resources) must be used exactly once in a proof. We can encode other forms of resources. For example, a resource "twice A" that must be used exactly twice could be defined as $A \otimes A$. In this problem we explore similar definitions for other forms of resources.

Task 1 (10 pts). *Affine resources* can be used **at most once**, but need not be used. Give a definition of an affine resource A as a linear resource $@A$.

$$@A = (A \mathbin{\&} \mathbf{1})$$

Task 2 (5 pts). To *refute* a sequent means to prove that it does not hold in general.
Prove or refute $@A \vdash A$, under your definition.

$$\dfrac{\dfrac{}{A \vdash A}\ \mathsf{id}_A}{A \mathbin{\&} \mathbf{1} \vdash A}\ \&L_1$$

Task 3 (5 pts). Prove or refute $A \vdash @A$, under your definition.

Assume $p \vdash p \mathbin{\&} \mathbf{1}$ for an atom p. By the completeness of the cut-free sequent calculus, we must have $p \Rightarrow p \mathbin{\&} \mathbf{1}$. Only $\&R$ could infer this sequent:

$$\dfrac{p \Rightarrow p \quad p \Rightarrow \mathbf{1}}{p \Rightarrow p \mathbin{\&} \mathbf{1}}\ \&R$$

No rule can infer the second premise, which is a contradiction.

Task 4 (10 pts). *Strict resources* must be used **at least once**, but may be used multiple times. Give a definition of a *strict resource* A as a linear resource $\#A$.

$$\#A = (!A \otimes A)$$

Task 5 (5 pts). Prove or refute $\#A \vdash !A$, under your definition.

Assume $!p \otimes p \vdash !p$. By the completeness of the cut-free sequent calculus, we must have $!p \otimes p \Rightarrow !p$. The last two inferences in its proof must be as follows:

$$\dfrac{\dfrac{p \; ; \; p \Rightarrow !p}{\cdot \; ; \; !p, p \Rightarrow !p}\; !L}{\cdot \; ; \; !p \otimes p \Rightarrow !p}\; \otimes L$$

From this point on, only the copy rule can be applied, never closing the proof, which is a contradiction.

Task 6 (5 pts). Prove or refute $!A \vdash \#A$, under your definition.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \; ; \; A \vdash A}\; \mathsf{id}_A}{A \; ; \; \cdot \vdash A}\; \mathsf{copy}}{A \; ; \; \cdot \vdash !A}\; !R \qquad \dfrac{\dfrac{}{A \; ; \; A \vdash A}\; \mathsf{id}_A}{A \; ; \; \cdot \vdash A}\; \mathsf{copy}}{A \; ; \; \cdot \vdash !A \otimes A}\; \otimes R}{\cdot \; ; \; !A \vdash !A \otimes A}\; !L$$

3 Focusing (40 pts)

Task 1 (5 pts). Explain in your own words the difference between *chaining* (also called *weak focusing*) and *focusing*.

In chaining, we can apply inversion to positive antecedents or negative succedents at any time. In focusing, inversion can be applied only when no formula is in focus, and focusing only when no inversion applies.

Task 2 (5 pts). What is the formal difference between the inference systems for chaining and focusing as introduced in this course? Be as precise as possible.

The three focusing rules for persistent resources (focus!), ephemeral resources (focus$L$), and the succedent (focus$R$), require that the sequent is *stable*. This means that all antecedents must be negative propositions or positive atoms, and that the succedent must be positive proposition or a negative atom. In chaining, there is no such restriction.

Task 3 (15 pts). Define $A \multimap\!\!\circ\, B$ as $(A \multimap B) \,\&\, (B \multimap A)$. Give the rules for $A \circ\!\!\multimap B$ in the focusing calculus. They should be pure (not reference any other connective or constant). They should also be sound and complete with respect to the linear sequent calculus, but you do not need to prove that.

$$\frac{\Gamma \,;\, \Delta \rightarrow [A] \quad \Gamma \,;\, \Delta', [B] \rightarrow C}{\Gamma \,;\, \Delta, \Delta', [A \circ\!\!\multimap B] \rightarrow C} \circ\!\!\multimap L_1 \qquad \frac{\Gamma \,;\, \Delta \rightarrow [B] \quad \Gamma \,;\, \Delta', [A] \rightarrow C}{\Gamma \,;\, \Delta, \Delta', [A \circ\!\!\multimap B] \rightarrow C} \circ\!\!\multimap L_2$$

$$\frac{\Gamma \,;\, \Delta, A \rightarrow B \quad \Gamma \,;\, \Delta, B \rightarrow A}{\Gamma \,;\, \Delta \rightarrow A \circ\!\!\multimap B} \circ\!\!\multimap R$$

Task 4 (15 pts). Recall that we defined $A \supset B$ as $(!A) \multimap B$. Give rules for $A \supset B$ in the focusing calculus. They should be pure (not reference any other connective or constant). They should also be sound and complete with respect to the linear sequent calculus, but you do not need to prove that.

$$\frac{\Gamma, A \,;\, \Delta \supset B}{\Gamma \,;\, \Delta \rightarrow A \supset B} \supset R \qquad \frac{\Gamma \,;\, \cdot \rightarrow A \quad \Gamma \,;\, \Delta, [B] \rightarrow C}{\Gamma \,;\, \Delta, [A \supset B] \rightarrow C} \supset L$$

In the $\supset L$ rule, A is not in focus in the first premise. This is for the same reason that $!R$ does not retain focus: as was explored in an exercise, the resulting calculus would be incomplete. Consider, for example, $a \otimes b \,;\, (a \otimes b) \supset c \rightarrow c$.

4 Quotations (30 pts)

Task 1 (10 pts).

> *A bird in hand is worth two in the bush.* – English proverb

Express this quotation in linear logic, using the following vocabulary:

Constants	$\mathsf{hand}, \mathsf{bush}$	
Predicates	$\mathsf{bird}(x)$	x is a bird
	$\mathsf{in}(x, y)$	x is in y

$$!((\exists z.\, !\mathsf{bird}(z) \otimes \mathsf{in}(z, \mathsf{hand})) \multimapboth (\exists x.\, !\mathsf{bird}(x) \otimes \mathsf{in}(x, \mathsf{bush})) \otimes (\exists y.\, !\mathsf{bird}(y) \otimes \mathsf{in}(y, \mathsf{bush})))$$

Task 2 (20 pts).

> *A banker is a fellow who lends you his umbrella when the sun is shining, but wants it back the minute it begins to rain.* – Mark Twain

Express this quotation in linear logic, using the following vocabulary:

Constants	$\mathsf{umbrella}$	
Predicates	$\mathsf{banker}(x)$	person x is a banker
	$\mathsf{has}(x, y)$	person x has object y
	sun	the sun is shining
	rain	it is raining

$$\begin{aligned} &!(\forall x.\, !\mathsf{banker}(x) \\ &\quad \multimap (\forall y.\, \mathsf{sun} \otimes \mathsf{has}(x, \mathsf{umbrella}) \multimap \mathsf{sun} \otimes \mathsf{has}(y, \mathsf{umbrella}) \\ &\qquad\quad \otimes (\mathsf{rain} \multimap \mathsf{rain} \otimes \mathsf{has}(x, \mathsf{umbrella}))))) \end{aligned}$$