

Curves and Surfaces

Parametric Representations
Cubic Polynomial Forms
Hermite Curves
Bezier Curves and Surfaces
[Angel 10.1-10.6]

February 11, 2003
Frank Pfenning
Carnegie Mellon University
<http://www.cs.cmu.edu/~fp/courses/graphics/>

Goals

- How do we draw surfaces?
 - Approximate with polygons
 - Draw polygons
- How do we specify a surface?
 - Explicit, implicit, parametric
- How do we approximate a surface?
 - Interpolation (use only points)
 - Hermite (use points and tangents)
 - Bezier (use points, and more points for tangents)
- Next lecture: splines, realization in OpenGL

02/11/2003

15-462 Graphics I

2

Explicit Representation

- Curve in 2D: $y = f(x)$
- Curve in 3D: $y = f(x), z = g(x)$
- Surface in 3D: $z = f(x, y)$
- Problems:
 - How about a vertical line $x = c$ as $y = f(x)$?
 - Circle $y = \pm (r^2 - x^2)^{1/2}$ two or zero values for x
- Too dependent on coordinate system
- Rarely used in computer graphics

02/11/2003

15-462 Graphics I

3

Implicit Representation

- Curve in 2D: $f(x, y) = 0$
 - Line: $ax + by + c = 0$
 - Circle: $x^2 + y^2 - r^2 = 0$
- Surface in 3D: $f(x, y, z) = 0$
 - Plane: $ax + by + cz + d = 0$
 - Sphere: $x^2 + y^2 + z^2 - r^2 = 0$
- $f(x, y, z)$ can describe 3D object:
 - Inside: $f(x, y, z) < 0$
 - Surface: $f(x, y, z) = 0$
 - Outside: $f(x, y, z) > 0$

02/11/2003

15-462 Graphics I

4

Algebraic Surfaces

- Special case of implicit representation
- $f(x, y, z)$ is polynomial in x, y, z
- Quadrics: degree of polynomial ≤ 2
- Render more efficiently than arbitrary surfaces
- Implicit form often used in computer graphics
- How do we represent curves implicitly?

02/11/2003

15-462 Graphics I

5

Parametric Form for Curves

- Curves: single parameter u (e.g. time)
- $x = x(u), y = y(u), z = z(u)$
- Circle: $x = \cos(u), y = \sin(u), z = 0$
- Tangent described by derivative

$$\mathbf{p}(u) = \begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix} \quad \frac{d\mathbf{p}(u)}{du} = \begin{bmatrix} \frac{dx(u)}{du} \\ \frac{dy(u)}{du} \\ \frac{dz(u)}{du} \end{bmatrix}$$

- Magnitude is "velocity"

02/11/2003

15-462 Graphics I

6

Parametric Form for Surfaces

- Use parameters u and v
- $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$
- Describes surface as both u and v vary
- Partial derivatives describe tangent plane at each point $p(u,v) = [x(u,v) \ y(u,v) \ z(u,v)]^T$

$$\frac{\partial p(u,v)}{\partial u} = \begin{bmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \\ \frac{\partial z(u,v)}{\partial u} \end{bmatrix} \quad \frac{\partial p(u,v)}{\partial v} = \begin{bmatrix} \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial v} \\ \frac{\partial z(u,v)}{\partial v} \end{bmatrix}$$

02/11/2003

15-462 Graphics I

7

Assessment of Parametric Forms

- Parameters often have natural meaning
- Easy to define and calculate
 - Tangent and normal
 - Curves segments (for example, $0 \leq u \leq 1$)
 - Surface patches (for example, $0 \leq u,v \leq 1$)



02/11/2003

15-462 Graphics I

8

Parametric Polynomial Curves

- Restrict $x(u)$, $y(u)$, $z(u)$ to be polynomial in u
- Fix degree n
- Each c_k is a column vector

$$p(u) = \sum_{k=0}^n c_k u^k$$

$$c_k = \begin{bmatrix} c_{xk} \\ c_{yk} \\ c_{zk} \end{bmatrix}$$

02/11/2003

15-462 Graphics I

9

Parametric Polynomial Surfaces

- Restrict $x(u,v)$, $y(u,v)$, $z(u,v)$ to be polynomial of fixed degree n

$$p(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix} = \sum_{i=0}^n \sum_{k=0}^n c_{ik} u^i v^k$$

- Each c_{ik} is a 3-element column vector
- Restrict to simple case where $0 \leq u,v \leq 1$

02/11/2003

15-462 Graphics I

10

Approximating Surfaces

- Use parametric polynomial surfaces
- Important concepts:
 - Join points for segments and patches
 - Control points to interpolate
 - Tangents and smoothness
 - Blending functions to describe interpolation
- First curves, then surfaces



02/11/2003

15-462 Graphics I

11

Outline

- Parametric Representations
- Cubic Polynomial Forms
- Hermite Curves
- Bezier Curves and Surfaces

02/11/2003

15-462 Graphics I

12

Cubic Polynomial Form

- Degree 3 appears to be a useful compromise
- Curves:

$$p(u) = c_0 + c_1u + c_2u^2 + c_3u^3 = \sum_{k=0}^3 c_k u^k$$

- Each c_k is a column vector $[c_{kx} \ c_{ky} \ c_{kz}]^T$
- From control information (points, tangents) derive 12 values c_{kx}, c_{ky}, c_{kz} for $0 \leq k \leq 3$
- These determine cubic polynomial form
- Later: how to render

02/11/2003

15-462 Graphics I

13

Interpolation by Cubic Polynomials

- Simplest case, although rarely used
- Curves:
 - Given 4 control points p_0, p_1, p_2, p_3
 - All should lie on curve: 12 conditions, 12 unknowns
- Space $0 \leq u \leq 1$ evenly
 - $p_0 = p(0), p_1 = p(1/3), p_2 = p(2/3), p_3 = p(1)$

02/11/2003

15-462 Graphics I

14

Equations to Determine c_k

- Plug in values for $u = 0, 1/3, 2/3, 1$

$$p_0 = p(0) = c_0$$

$$p_1 = p(1/3) = c_0 + \frac{1}{3}c_1 + (\frac{1}{3})^2c_2 + (\frac{1}{3})^3c_3$$

$$p_2 = p(2/3) = c_0 + \frac{2}{3}c_1 + (\frac{2}{3})^2c_2 + (\frac{2}{3})^3c_3$$

$$p_3 = p(1) = c_0 + c_1 + c_2 + c_3$$

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & (\frac{1}{3})^2 & (\frac{1}{3})^3 \\ 1 & \frac{2}{3} & (\frac{2}{3})^2 & (\frac{2}{3})^3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Note: p_k and c_k are vectors!

02/11/2003

15-462 Graphics I

15

Interpolating Geometry Matrix

- Invert A to obtain interpolating geometry matrix

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5.5 & 9 & -4.5 & 1 \\ 9 & -22.5 & 18 & 4.5 \\ -4.5 & 13.5 & -13.5 & 4.5 \end{bmatrix} \quad c = A^{-1}P$$

02/11/2003

15-462 Graphics I

16

Joining Interpolating Segments

- Do not solve degree n for n points
- Divide into overlap sequences of 4 points
- p_0, p_1, p_2, p_3 then p_3, p_4, p_5, p_6 , etc.



- At join points
 - Will be continuous (C^0 continuity)
 - Derivatives will usually not match (no C^1 continuity)

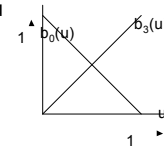
02/11/2003

15-462 Graphics I

17

Blending Functions

- Make explicit, how control points contribute
- Simplest example: straight line with control points p_0 and p_3
- $p(u) = (1-u)p_0 + up_3$
- $b_0(u) = 1-u, b_3(u) = u$



02/11/2003

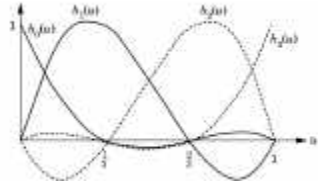
15-462 Graphics I

18

Blending Polynomials for Interpolation

- Each blending polynomial is a cubic
- Solve (see [Angel, p. 427]):

$$p(u) = b_0(u)p_0 + b_1(u)p_1 + b_2(u)p_2 + b_3(u)p_3$$



02/11/2003

15-462 Graphics I

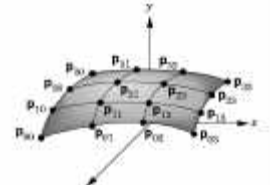
19

Cubic Interpolation Patch

- Bicubic surface patch with 4×4 control points

$$p(u, v) = \sum_{i=0}^3 \sum_{k=0}^3 u^i v^k c_{ik}$$

Note: each c_{ik} is
3 column vector
(48 unknowns)



[Angel, Ch. 10.4.2]

02/11/2003

15-462 Graphics I

20

Outline

- Parametric Representations
- Cubic Polynomial Forms
- Hermite Curves
- Bezier Curves and Surfaces

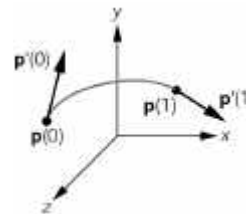
02/11/2003

15-462 Graphics I

21

Hermite Curves

- Another cubic polynomial curve
- Specify two endpoints and their tangents



02/11/2003

15-462 Graphics I

22

Deriving the Hermite Form

- As before

$$p(0) = p_0 = c_0$$

$$p(1) = p_3 = c_0 + c_1 + c_2 + c_3$$

- Calculate derivative

$$p'(u) = \begin{bmatrix} \frac{dx}{du} \\ \frac{dy}{du} \\ \frac{dz}{du} \end{bmatrix} = c_1 + 2uc_2 + 3u^2c_3$$

- Yields

$$p'_0 = p'(0) = c_1$$

$$p'_3 = p'(1) = c_1 + 2c_2 + 3c_3$$

02/11/2003

15-462 Graphics I

23

Summary of Hermite Equations

- Write in matrix form
- Remember p_k and p'_k and c_k are vectors!

$$\begin{bmatrix} p_0 \\ p_3 \\ p'_0 \\ p'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- Let $q = [p_0 \ p_3 \ p'_0 \ p'_3]^T$ and invert to find Hermite geometry matrix M_H satisfying

$$c = M_H q$$

02/11/2003

15-462 Graphics I

24

Blending Functions

- Explicit Hermite geometry matrix

$$M_H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix}$$

- Blending functions for $u = [1 \ u \ u^2 \ u^3]^T$

$$b(u) = M_H^T u = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}$$

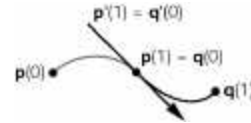
02/11/2003

15-462 Graphics I

25

Join Points for Hermite Curves

- Match points and tangents (derivates)



- Much smoother than point interpolation
- How to obtain the tangents?
- Skip Hermite surface patch
- More widely used: Bezier curves and surfaces

02/11/2003

15-462 Graphics I

26

Parametric Continuity

- Matching endpoints (C^0 parametric continuity)

$$p(1) = \begin{bmatrix} p_x(1) \\ p_y(1) \\ p_z(1) \end{bmatrix} = \begin{bmatrix} q_x(0) \\ q_y(0) \\ q_z(0) \end{bmatrix} = q(0)$$

- Matching derivatives (C^1 parametric continuity)

$$p'(1) = \begin{bmatrix} p'_x(1) \\ p'_y(1) \\ p'_z(1) \end{bmatrix} = \begin{bmatrix} q'_x(0) \\ q'_y(0) \\ q'_z(0) \end{bmatrix} = q'(0)$$

02/11/2003

15-462 Graphics I

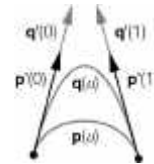
27

Geometric Continuity

- For matching tangents, less is required

$$p'(1) = \begin{bmatrix} p'_x(1) \\ p'_y(1) \\ p'_z(1) \end{bmatrix} = k \begin{bmatrix} q'_x(0) \\ q'_y(0) \\ q'_z(0) \end{bmatrix} = kq'(0)$$

- G^1 geometric continuity
- Extends to higher derivatives



02/11/2003

15-462 Graphics I

28

Outline

- Parametric Representations
- Cubic Polynomial Forms
- Hermite Curves
- Bezier Curves and Surfaces

02/11/2003

15-462 Graphics I

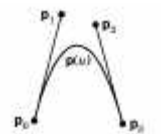
29

Bezier Curves

- Widely used in computer graphics
- Approximate tangents by using control points

$$p'(0) = 3(p_1 - p_0)$$

$$p'(1) = 3(p_3 - p_2)$$



02/11/2003

15-462 Graphics I

30

Equations for Bezier Curves

- Set up equations for cubic parametric curve
- Recall:

$$\begin{cases} p(u) = c_0 + c_1u + c_2u^2 + c_3u^3 \\ p'(u) = c_1 + 2c_2u + 3c_3u^2 \end{cases}$$

- Solve for c_k

$$\begin{aligned} p_0 &= p(0) = c_0 \\ p_3 &= p(1) = c_0 + c_1 + c_2 + c_3 \\ p'(0) &= 3p_1 - 3p_0 = c_1 \\ p'(1) &= 3p_3 - 3p_2 = c_1 + 2c_2 + 3c_3 \end{aligned}$$

02/11/2003

15-462 Graphics I

31

Bezier Geometry Matrix

- Calculate Bezier geometry matrix M_B

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = M_B \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \text{ so } M_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

- Have C^0 continuity, not C^1 continuity
- Have C^1 continuity with additional condition

02/11/2003

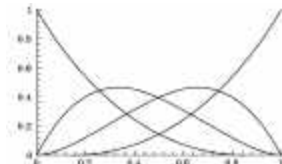
15-462 Graphics I

32

Blending Polynomials

- Determine contribution of each control point

$$b(u) = M_B^T u = \begin{bmatrix} (1-u)^3 \\ 3u(1-u)^2 \\ 3u^2(1-u) \\ u^3 \end{bmatrix}$$



Smooth blending polynomials

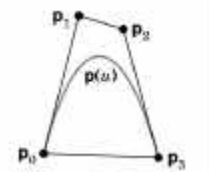
02/11/2003

15-462 Graphics I

33

Convex Hull Property

- Bezier curve contained entirely in convex hull of control points
- Determined choice of tangent coefficient (?)



02/11/2003

15-462 Graphics I

34

Bezier Surface Patches

- Specify Bezier patch with 4×4 control points



- Bezier curves along the boundary

$$\begin{aligned} p(0,0) &= p_{00} \\ \frac{\partial p}{\partial u}(0,0) &= 3(p_{10} - p_{00}) \\ \frac{\partial p}{\partial v}(0,0) &= 3(p_{01} - p_{00}) \end{aligned}$$

02/11/2003

15-462 Graphics I

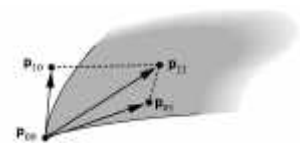
35

Twist

- Inner points determine twist at corner

$$\frac{\partial^2 p}{\partial u \partial v}(0,0) = 9(p_{00} - p_{01} + p_{10} - p_{11})$$

- Flat means $p_{00}, p_{10}, p_{01}, p_{11}$ in one plane
- $(\partial^2 p / \partial u \partial v)(0,0) = 0$



02/11/2003

15-462 Graphics I

36

Summary

- Parametric Representations
- Cubic Polynomial Forms
- Hermite Curves
- Bezier Curves and Surfaces

02/11/2003

15-462 Graphics I

37

Preview

- B-Splines: more continuity (C^2)
- Non-uniform B-splines ("heavier" points)
- Non-uniform rational B-splines (NURBS)
 - Rational functions instead of polynomials
 - Based on homogeneous coordinates
- Rendering and recursive subdivision
- Curves and surfaces in OpenGL

02/11/2003

15-462 Graphics I

38

Announcements

- Handing back Assignment 2 Thursday
- Model solution coming soon
- Assignment 3 due a week from Thursday
- Movie from Assignment 1!
- Thursday: Texture Mapping [Ian Graham]
- Next Tuesday: Splines

02/11/2003

15-462 Graphics I

39