15-462 Computer Graphics I
Lecture 11

Splines

Cubic B-Splines
Nonuniform Rational B-Splines
Rendering by Subdivision
Curves and Surfaces in OpenGL
[Angel, Ch 10.7-10.14]

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http://www.cs.cmu.edu/~fp/courses/graphics/

Review

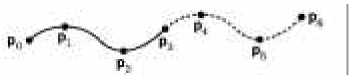
- Cubic polynomial form for curve

$$p(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3 = \sum_{k=0}^{3} c_k u^k$$

- Each c_k is a column vector $[c_{kx}\ \ c_{ky}\ \ c_{kz}]^T$
- Solve for c_k given control points
- Interpolation: 4 points
- Hermite curves: 2 endpoints, 2 tangents
- Bezier curves: 2 endpoints, 2 tangent points

Splines

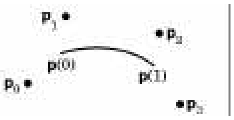
- Approximating more control points



- C^0 continuity: points match
- C^1 continuity: tangents (derivatives) match
- C^2 continuity: curvature matches
- With Bezier segments or patches: C^0

B-Splines

- Use 4 points, but approximate only middle two



- Draw curve with overlapping segments
  0-1-2-3, 1-2-3-4, 2-3-4-5, 3-4-5-6, etc.
- Curve may miss all control points
- Smoother at joint points

Cubic B-Splines

- Need m+2 control points for m cubic segments
- Computationally 3 times more expensive than simple interpolation
- C^2 continuous at each interior point
- Derive as follows:
  - Consider two overlapping segments
  - Enforce C^0 and C^1 continuity
  - Employ symmetry
  - C^2 continuity follows

Deriving B-Splines

- Consider points
  - $p_{i-2}, p_{i-1}, p_i, p_{i+1}$
  - $p(0)$ approx p_{i-1}, $p(1)$ approx p_i
  - $p_{i-3}, p_{i-2}, p_{i-1}, p_i$
  - $q(0)$ approx p_{i-2}, $q(1)$ approx p_{i-1}
- Condition 1: $p(0) = q(1)$
  - Symmetry: $p(0) = q(1) = 1/6(p_{i-2} + 4\, p_{i-1} + p_i)$
- Condition 2: $p'(0) = q'(1)$
  - Geometry: $p'(0) = q'(1) = 1/2\ ((p_i - p_{i-1}) + (p_{i-1} - p_{i-2}))$
    $= 1/2\ (p_i - p_{i-2})$

B-Spline Geometry Matrix

- Conditions at u = 0
  - $p(0) = c_0 = 1/6\ (p_{i-2} + 4p_{i-1} + p_i)$
  - $p'(0) = c_1 = 1/2\ (p_i - p_{i-2})$
- Conditions at u = 1
  - $p(1) = c_0 + c_1 + c_2 + c_3 = 1/6\ (p_{i-1} + 4p_i + p_{i+1})$
  - $p'(1) = c_1 + 2c_2 + 3c_3 = 1/2\ (p_{i+1} - p_{i-1})$

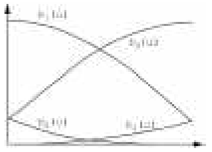
$$\begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix} = \mathbf{M}_S \begin{bmatrix} \mathbf{p}_{i-2} \\ \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \end{bmatrix}, \mathbf{M}_S = \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Blending Functions

- Calculate cubic blending polynomials

$$\mathbf{b}(u) = \mathbf{M}_S^T \mathbf{u} = \frac{1}{6} \begin{bmatrix} (1-u)^3 \\ 4 - 6u^2 + 3u^3 \\ 1 + 3u + 3u^2 - 3u^3 \\ u^3 \end{bmatrix}$$

- Note symmetries



Convex Hull

- For $0 \leq u \leq 1$, have $0 \leq b_k(u) \leq 1$
- Recall:

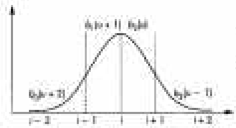
$p(u) = b_{i-2}(u)p_{i-2} + b_{i-1}(u)p_{i-1} + b_i(u)p_i + b_{i+1}(u)p_{i+1}$

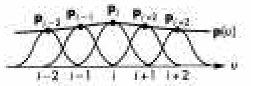
- So each point p(u) lies in convex hull of p_k

Spline Basis Functions

- Total contribution $B_i(u)p_i$ of p_i is given by

$$\mathbf{B}_i(u) = \begin{cases} 0 & u < i-2 \\ b_0(u+2) & i-2 \leq u < i-1 \\ b_1(u+1) & i-1 \leq u \leq i \\ b_2(u) & i \leq u < i+1 \\ b_3(u-1) & i+1 \leq u < i+2 \\ 0 & i-2 \leq u \end{cases}$$



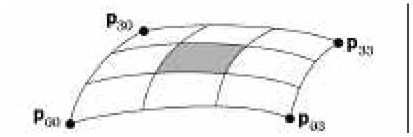


Spline Surface

- As for Bezier patches, use 16 control points
- Start with blending functions

$$\mathbf{p}(u,v) = \sum_{i=0}^{3} \sum_{k=0}^{3} b_i(u) b_k(v) \mathbf{p}_{ik}$$

- Need 9 times as many splines as for Bezier



Assessment: Cubic B-Splines

- More expensive than Bezier curves or patches
- Smoother at join points
- Local control
  - How far away does a point change propagate?
- Contained in convex hull of control points
- Preserved under affine transformation
- How to deal with endpoints?
  - Closed curves (uniform periodic B-splines)
  - Non-uniform B-Splines (multiplicities of knots)

General B-Splines

- Generalize from cubic to arbitrary order
- Generalize to different basis functions
- Read: [Angel, Ch 10.8]
- Knot sequence $u_{min} = u_0 \leq \ldots \leq u_n = u_{max}$
- Repeated points have higher “gravity”
- Multiplicity 4 means point must be interpolated
- {0, 0, 0, 0, 1, 2, ..., n-1, n, n, n, n} solves boundary problem

Nonuniform Rational B-Splines (NURBS)

- Exploit homogeneous coordinates

$$\mathbf{p}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \simeq w_i \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix} = \mathbf{q}_i$$

- Use perspective division to renormalize

$$\mathbf{p}(u) = \frac{\sum_{i=0}^{n} \mathbf{B}_i(u) w_i \mathbf{p}_i}{\sum_{i=0}^{n} \mathbf{B}_i(u) w_i}$$

- Each component of p(u) is rational function of u
- Points not necessarily uniform (NURBS)

NURBS Assessment

- Convex-hull and continuity props. of B-splines
- Preserved under perspective transformations
  - Curve with transformed points = transformed curve
- Widely used (including OpenGL)

Outline

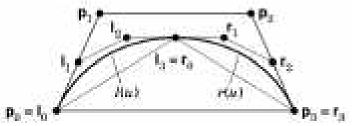
- Cubic B-Splines
- Nonuniform Rational B-Splines (NURBS)
- Rendering by Subdivision
- Curves and Surfaces in OpenGL

Rendering by Subdivision

- Divide the curve into smaller subpieces
- Stop when “flat” or at fixed depth
- How do we calculate the sub-curves?
  - Bezier curves and surfaces: easy (next)
  - Other curves: convert to Bezier!

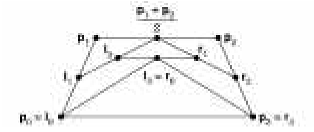
Subdividing Bezier Curves

- Given Bezier curve by p_0, p_1, p_2, p_3
- Find l_0, l_1, l_2, l_3 and r_0, r_1, r_2, r_3
- Subcurves should stay the same!



Construction of Bezier Subdivision

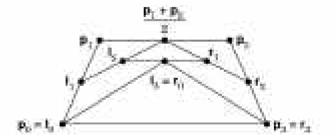
- Use algebraic reasoning



- $l(0) = l_0 = p_0$
- $l(1) = l_3 = p(1/2) = 1/8(p_0 + 3p_1 + 3p_2 + p_3)$
- $l'(0) = 3(l_1 - l_0) = p'(0) = 3/2 (p_1 - p_0)$
- $l'(1) = 3(l_3 - l_2) = p'(1/2) = 3/8(-p_0 - p_1 + p_2 + p_3)$
- Note parameter substitution $v = 2u$ so $dv = 2du$

Geometric Bezier Subdivision

- Can also calculate geometrically



- $l_1 = ½(p_0 + p_1)$, $r_2 = ½ (p_2 + p_3)$
- $l_2 = ½ (l_1 + ½ (p_1 + p_2))$, $r_1 = ½ (r_2 + ½(p_1 + p_2))$
- $l_3 = r_0 = ½ (l_2 + r_1)$, $l_0 = p_0$, $r_3 = p_3$

Recall: Bezier Curves

- Recall $\mathbf{u}^T = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix}$
- Express $\mathbf{p}(u) = \mathbf{c}_0 + \mathbf{c}_1 u + \mathbf{c}_2 u^2 + \mathbf{c}_3 u^3$

$$= \mathbf{u}^T \begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix} = \mathbf{u}^T \mathbf{M}_\mathbf{B} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$\mathbf{M}_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & -1 \end{bmatrix}$$

Subdividing Other Curves

- Calculations more complex
- Trick: transform control points to obtain identical curve as Bezier curve!
- Then subdivide the resulting Bezier curve
- Bezier: $p(u) = u^T M_b p$
- Other curve: $p(u) = u^T M q$, M geometry matrix
- Solve: $q = M^{-1} M_b p$ with $p = M_b^{-1} M q$

Example Conversion

- From cubic B-splines to Bezier:

$$\mathbf{M}_B^{-1}\mathbf{M}_S = \frac{1}{6}\begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix}$$

- Calculate Bezier points p from q
- Subdivide as Bezier curve

Subdivision of Bezier Surfaces

- Slightly more complicated
- Need to calculate interior point
- Cracks may show with uneven subdivision
- See [Angel, Ch 10.9.4]

Outline

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- Curves and Surfaces in OpenGL

Curves and Surface in OpenGL

- Central mechanism is evaluator
- Defined by array of control points
- Evaluate coordinates at u (or u and v) to generate vertex
- Define Bezier curve: *type* = GL_MAP_VERTEX_3
  glMap1f(*type*, u_0, u_1, stride, order, point_array)
- Enable evaluator
  glEnable(*type*)
- Evaluate Bezier curve
  glEvalCoord1f(u)

Example: Drawing a Bezier Curve

- 4 control points

```
GLfloat ctrlpoints[4][3] = {
  {-4.0, -4.0, 0.0}, { -2.0, 4.0, 0.0},
  {2.0, -4.0, 0.0}, {4.0, 4.0, 0.0}};
```

- Initialize

```
void init()
{ ...
   glMap1f(GL_MAP1_VERTEX_3, 0.0, 1.0, 3, 4,
              &ctrlpoints[0][0]);
   glEnable(GL_MAP1_VERTEX_3);
}
```

Evaluating Coordinates

- Use a fixed number of points, num_points

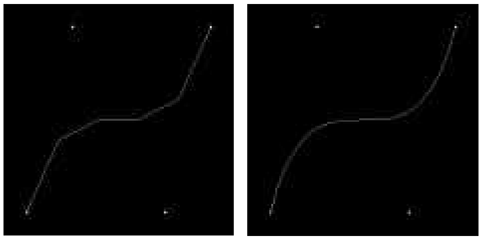
```
void display()
{ ...
  glBegin(GL_LINE_STRIP);
     for (i = 0; i <= num_points; i++)
        glEvalCoord1f((GLfloat)i/(GLfloat)num_points);
   glEnd();
   ...
}
```

Drawing the Control Points

- To illustrate Bezier curve

```
void display()
{ ...
   glPointSize(5.0);
   glColor3f(1.0, 1.0, 0.0);
   glBegin(GL_POINTS);
     for (i = 0; i < 4; i++)
        glVertex3fv(&ctrlpoints[i][0]);
   glEnd();
   glFlush();
}
```

Resulting Images



n = 5 n = 20

Bezier Surfaces

- Create evaluator in two parameters u and v

glMap2f(GL_MAP2_VERTEX_3,
u_0, u_1, ustride, uorder,
v_0, v_1, vstride, vorder, point_array);

- Enable, also automatic calculation of normal

```
glEnable(GL_MAP2_VERTEX_3);
glEnable(GL_AUTO_NORMAL);
```

- Evaluate at parameters u and v

```
glEvalCoord2f(u, v);
```

Grids

- Convenience for uniform evaluators
- Define grid (nu = number of u division)

glMapGrid2f(nu, u_0, u_1, nv, v_0, v_1);

- Evaluate grid

glEvalMesh2(*mode*, i_0, i_1, k_0, k_1);

- *mode* = GL_POINT, GL_LINE, or GL_FILL
- i and k define subrange

Example: Bezier Surface Patch

- Use 16 control points

```
GLfloat ctrlpoints[4][4][3] = {...};
```

- Initialize 2-dimensional evaluator

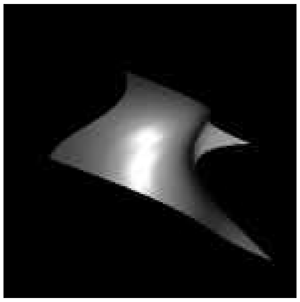
```
void init(void)
{  ...
   glMap2f(GL_MAP2_VERTEX_3, 0, 1, 3, 4,
              0, 1, 12, 4, &ctrlpoints[0][0][0]);
   glEnable(GL_MAP2_VERTEX_3);
   glEnable(GL_AUTO_NORMAL);
   glMapGrid2f(20, 0.0, 1.0, 20, 0.0, 1.0);
}
```

Evaluating the Grid

- Use full range

```
void display(void)
{ ...
   glPushMatrix();
   glRotatef(85.0, 1.0, 1.0, 1.0);
   glEvalMesh2(GL_FILL, 0, 20, 0, 20);
   glPopMatrix();
   glFlush();
}
```

Resulting Image



NURBS Functions

- Higher-level interface
- Implemented in GLU using evaluators
- Create a NURBS renderer

```
theNurb = gluNewNurbsRenderer();
```

- Set NURBS properties

```
gluNurbsProperty(theNurb, GLU_DISPLAY_MODE, GLU_FILL);
gluNurbsCallback(theNurb, GLU_ERROR,  nurbsError);
```

Displaying NURBS Surfaces

- Specify knot arrays for splines

```
GLfloat knots[8] = {0.0, 0.0, 0.0, 0.0, 1.0, 1.0, 1.0, 1.0};
gluBeginSurface(theNurb);
  gluNurbsSurface(theNurb,
              8, knots, 8, knots,
              4 * 3, 3, &ctlpoints[0][0][0],
              4, 4, GL_MAP2_VERTEX_3);
 gluEndSurface(theNurb);
```

- For more see [Red Book, Ch. 12]

Summary

- Cubic B-Splines
- Nonuniform Rational B-Splines (NURBS)
- Rendering by Subdivision
- Curves and Surfaces in OpenGL

Reminders

- Assignment 3 due Thursday
- Assignment 4 out Thursday
- Midterm will cover curves and surfaces
- Thursday: Pixel Shading (Nvidia guest lecture)