Tail recursion and more about lists, structural induction and extensional equivalence

15-150

Lecture 4: September 5, 2024

Stephanie Balzer
Carnegie Mellon University

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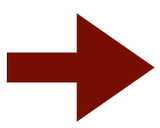

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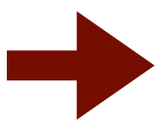

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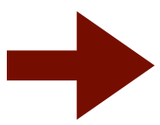

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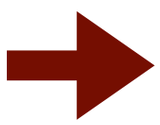

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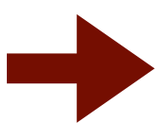

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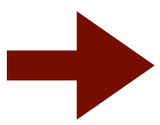

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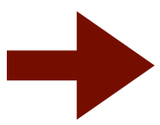

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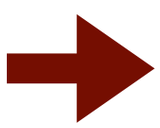

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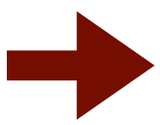

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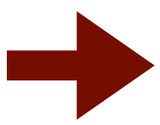

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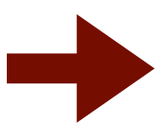

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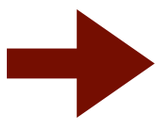

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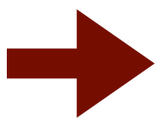

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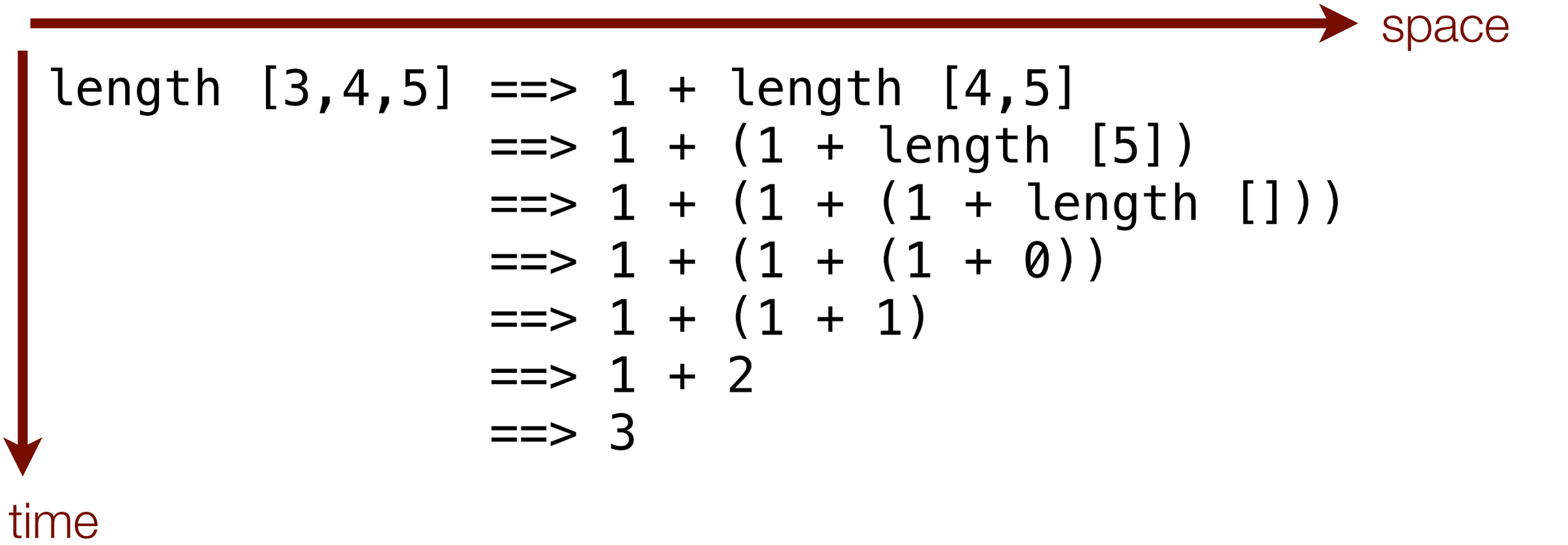

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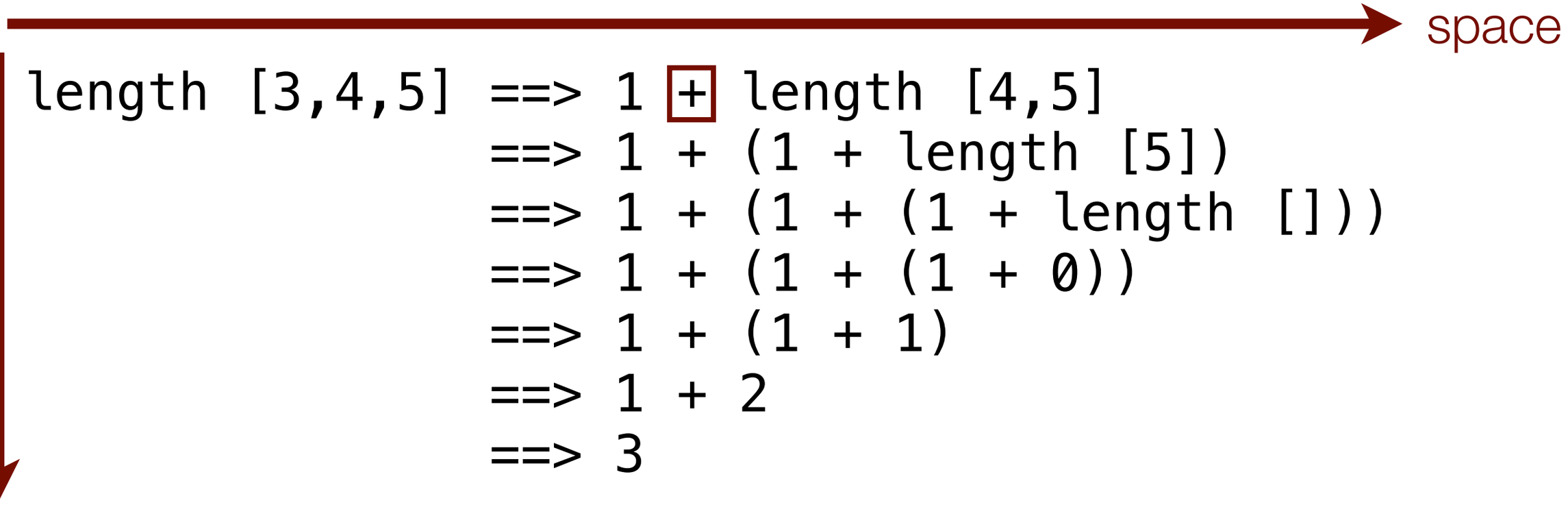

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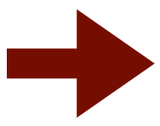

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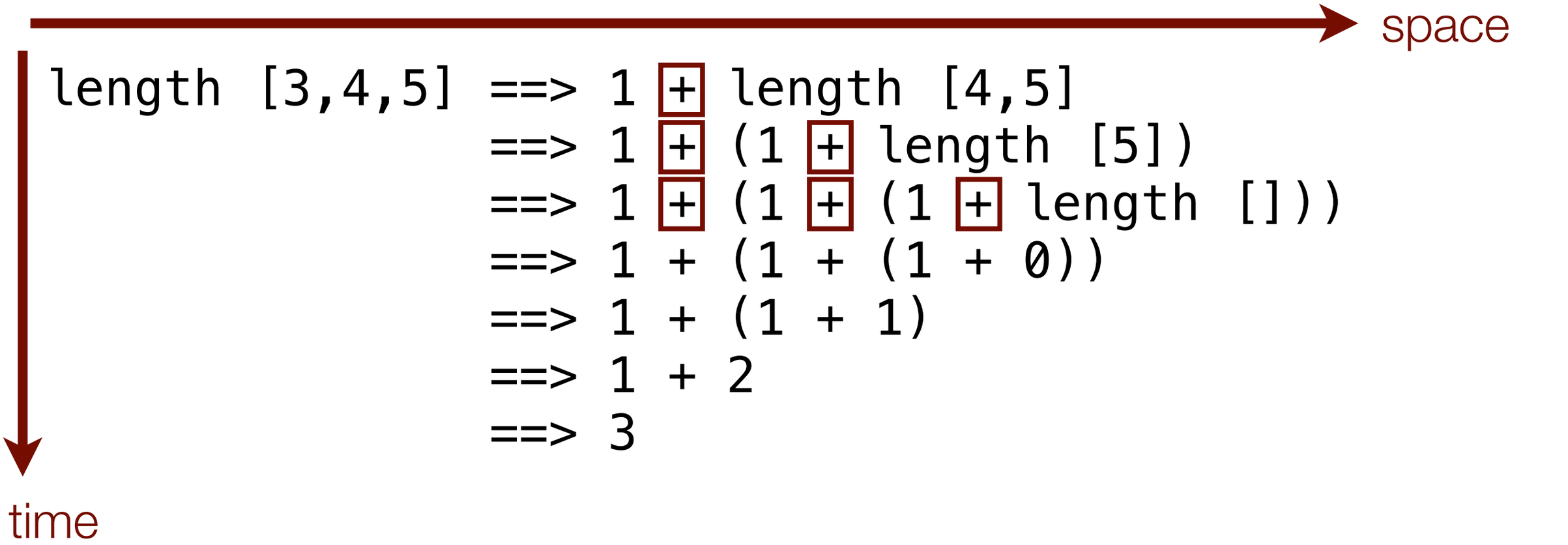

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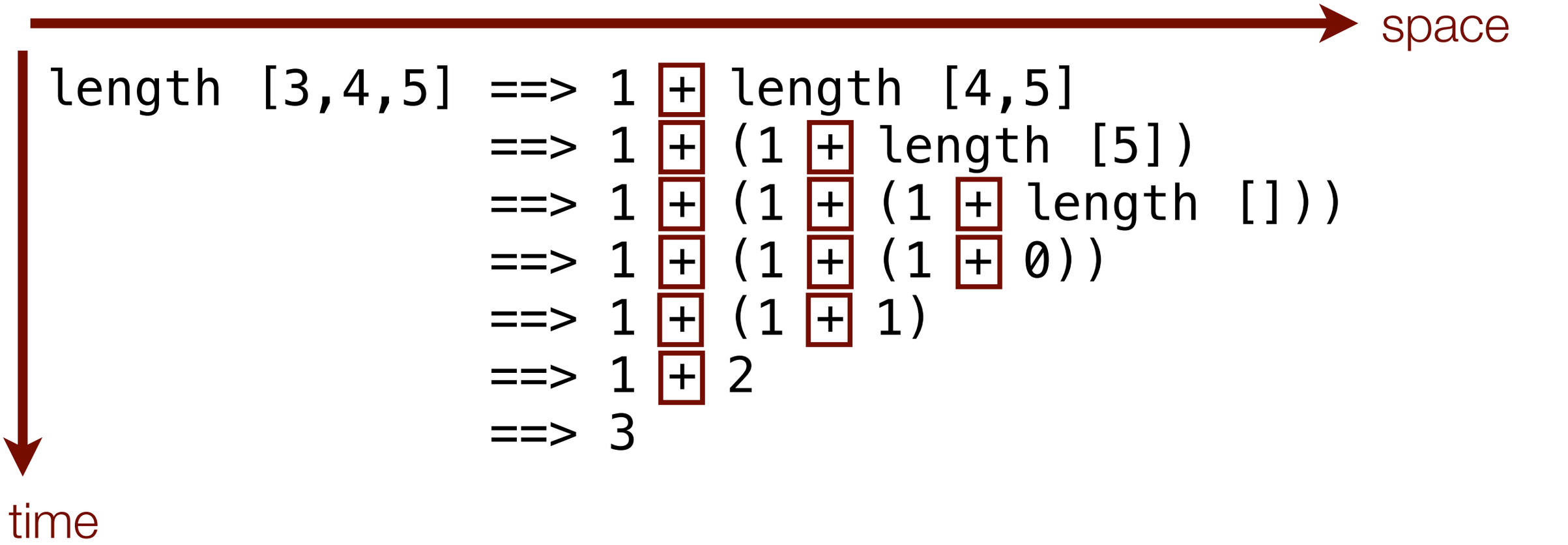

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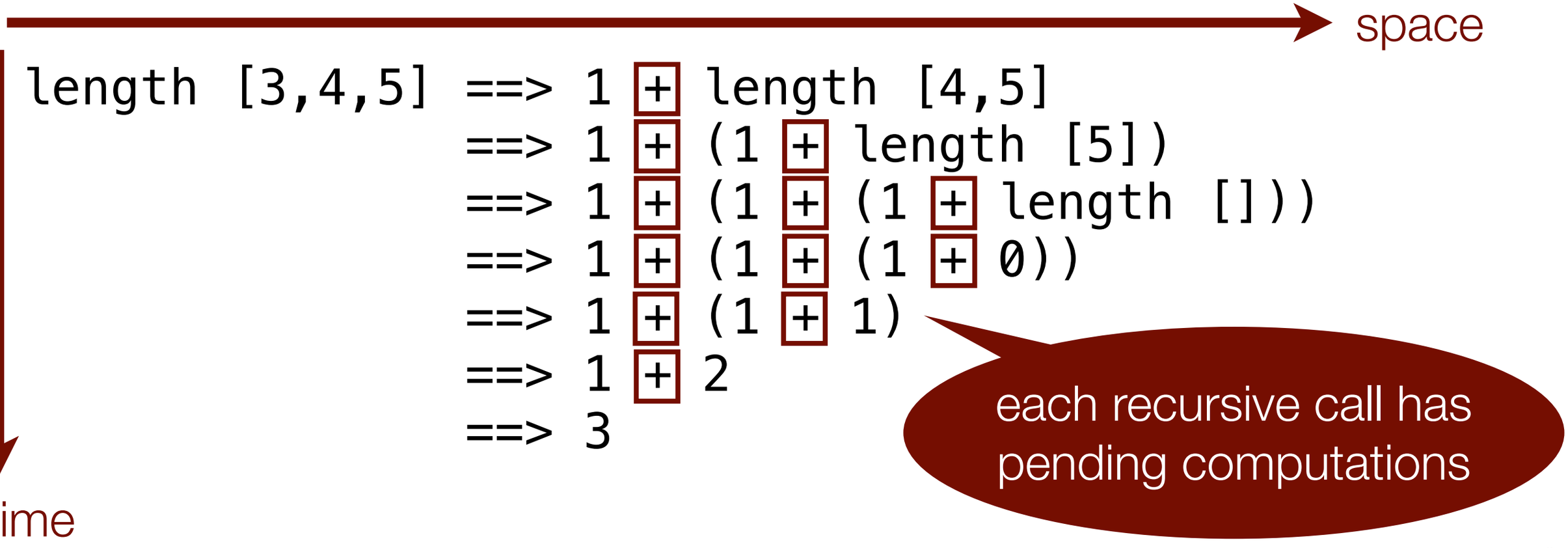

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space-inefficient version

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tail call

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                ==> tlength ([4,5], 1)
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                ==> tlength ([5], 2)
                ==> tlength ([], 2+1)
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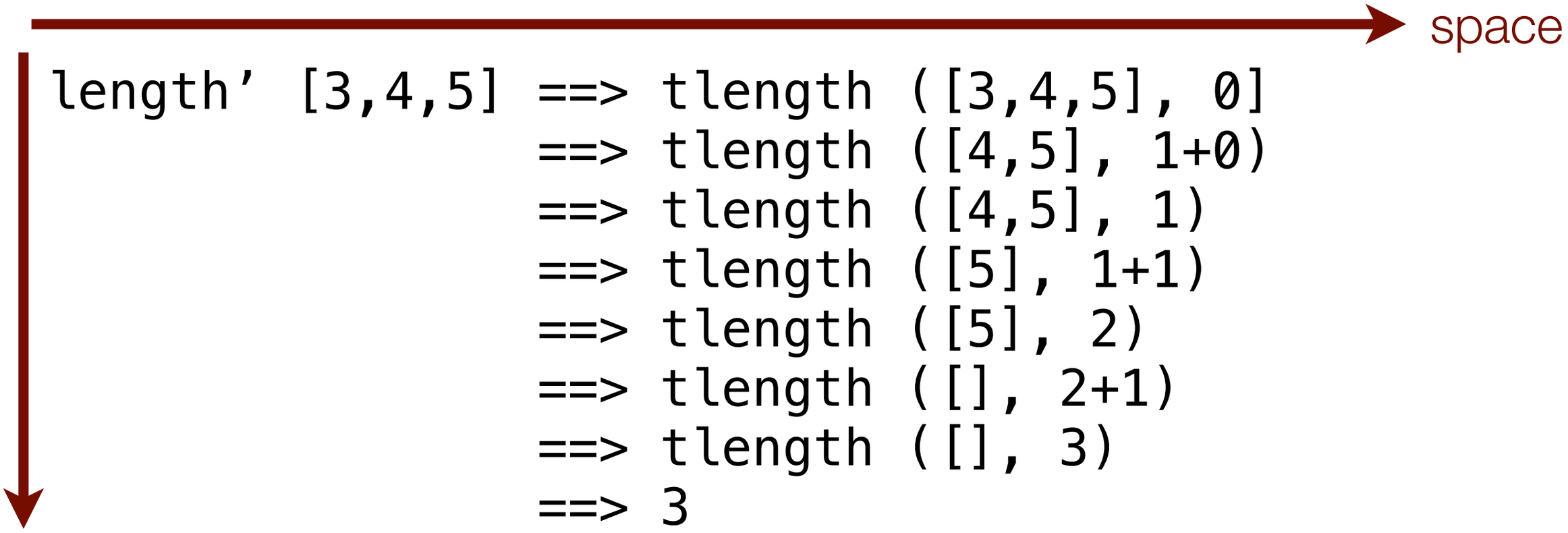


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space

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recursive calls without pending computations

Let’s prove space-efficient version correct!

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Theorem: For all values `L: int list` and `acc: int`, `tlength(L,acc)` ≅ `(length L) + acc`.

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→ Before doing the proof, let's review extensional equivalence!

Extensional equivalence, revisited

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Definition:

- Expression of type `int` are extensionally equivalent,
  - if they evaluate to the same integer, or
  - if they both loop forever, or
  - if they both raise the same exception.
- Functions of type `int -> int` are extensionally equivalent, if they map extensionally equivalent arguments to extensionally equivalent results.

Extensional equivalence, revisited

Definition:

- Expression of type `int` are extensionally equivalent,
  - if they evaluate to the same integer, or
  - if they both loop forever, or
  - if they both raise the same exception.
- Functions of type `int -> int` are extensionally equivalent, if they map extensionally equivalent arguments to extensionally equivalent results.

Facts:

- If $\mathtt{e}_1 \hookrightarrow \mathtt{v}$ and $\mathtt{e}_2 \hookrightarrow \mathtt{v}$, then $\mathtt{e}_1 \cong \mathtt{e}_2$.
- If $\mathtt{e}_1 \Longrightarrow \mathtt{e}$ and $\mathtt{e}_2 \Longrightarrow \mathtt{e}$, then $\mathtt{e}_1 \cong \mathtt{e}_2$.
- If $\mathtt{e}_1 \Longrightarrow \mathtt{e}_2$, then $\mathtt{e}_1 \cong \mathtt{e}_2$.

Extensional equivalence, revisited

Facts:

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However:

From $e_1 \cong e_2$ it does **not** necessarily follow that $e_1 \Longrightarrow e_2$ or $e_2 \Longrightarrow e_1$!

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eg, $1+1+7 \cong 9$

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Moreover:

For f: `int -> int`, it does **not** necessarily follow that `f(1) + f(2)` $\cong$ `f(2) + f(1)`!

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Moreover:

For f: `int -> int`, it does **not** necessarily follow that `f(1) + f(2)` $\cong$ `f(2) + f(1)`!

f may loop or raise an exception (on some inputs)

Let’s prove space-efficient version correct!

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```
fun length ([] : int list) : int = 0
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fun tlength([]: int list, acc: int): int = acc
  | tlength(x::xs, acc) = tlength(xs, 1 + acc)
```

Theorem: For all values `L: int list` and `acc: int`, `tlength(L,acc)` ≅ `(length L) + acc`.

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Proof: By ???

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Proof: By structural induction on `L`.

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Proof: By structural induction on L.

Base case: $L = [\,]$.

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Proof: By structural induction on L.

Base case: L = `[]`.

Need to show: for all values `acc:int`,

`tlength([],acc)` ≅ `length ([]) + acc`.

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$\Longrightarrow$ `acc`

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    tlength ([], acc)
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⟹ `acc` (1st clause of `tlength`)

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Showing:

Evaluating the left side:

`tlength ([], acc)`

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Evaluating the right side:

`length([]) + acc`

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$\Longrightarrow$ `0 + acc`

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→ equivalent because both reduce to the same value.

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Inductive case: L = `x::xs` for some values `x:int` and `xs:int list`.

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Showing:

```
   tlength(x::xs, acc)
≅ tlength(xs, 1 + acc)          (step, 2nd clause of tlength)
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   tlength(x::xs, acc)
≅  tlength(xs, 1 + acc)          (step, 2nd clause of tlength)
≅  length(xs) + (1 + acc)
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   tlength(x::xs, acc)
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≅  length(xs) + (1 + acc)        (IH, assume + total)
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	`tlength(x::xs, acc)`	
≅	`tlength(xs, 1 + acc)`	(step, 2nd clause of `tlength`)
≅	`length(xs) + (1 + acc)`	(IH, assume `+` total)
≅	`(1 + length(xs)) + acc`	(`+` commutative, associative, totality of `length`)

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≅  (1 + length(xs)) + acc        (+ commutative, associative,
                                  totality of length)
≅  length(x::xs) + acc
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Showing:

	`tlength(x::xs, acc)`	
≅	`tlength(xs, 1 + acc)`	(step, 2nd clause of **`tlength`**)
≅	`length(xs) + (1 + acc)`	(IH, assume **+** total)
≅	`(1 + length(xs)) + acc`	(**+** commutative, associative, totality of **`length`**)
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→ Proved by structural induction on lists.

Appending lists

Appending lists

```
(* append: int list * int list -> int list
   REQUIRES: true
   ENSURES: append(L,R) evaluates to a list consisting
            of L followed by R
   NOTE: predefined in SML as the right-associative
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*)

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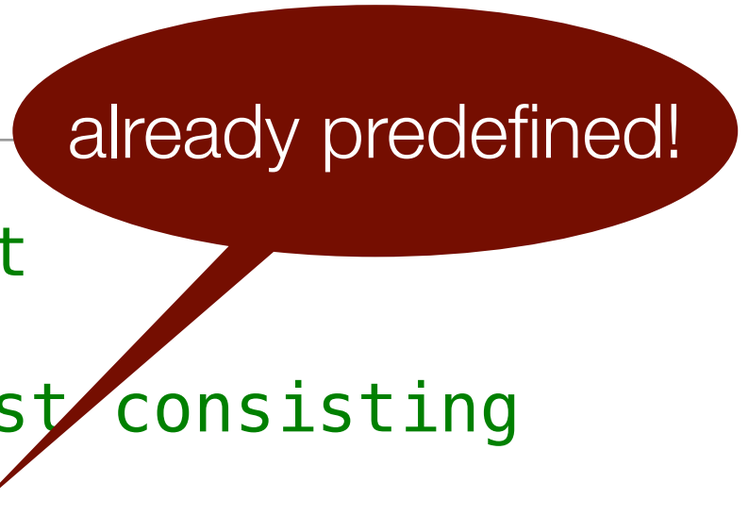
Appending lists

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val [] : int list = append([],[])
val [1,2] = append([], [1,2])
val [1,2,5,6] = append([1,2], [5,6])
```

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What is the time complexity of append?

Appending lists

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(* append: int list * int list -> int list
   REQUIRES: true
   ENSURES: append(L,R) evaluates to a list consisting
            of L followed by R
   NOTE: predefined in SML as the right-associative
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fun append ([] : int list, R : int list) : int list = R
  | append (x::xs, R) = x::append(xs,R)

val [] : int list = append([],[])
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- What is the time complexity of append?
- append(X,Y) has time complexity O(|X|).

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- What is the time complexity of reverse?
- reverse(X) has time complexity $O(|X|^2)$.

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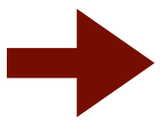

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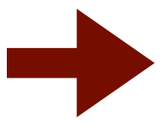

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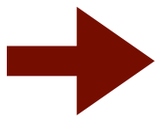

We provide the solution in the notes (rev.pdf). But try first!