Higher-Order Functions

15-150

Lecture 10: October 1, 2024

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Recap

Recap

- Polymorphic types (type families) and instantiation.

Recap

- Polymorphic types (type families) and instantiation.
  - parameterized data types

Recap

Polymorphic types (type families) and instantiation.

parameterized data types

```
datatype 'a tree =
  Empty | Node of 'a tree * 'a * 'a tree
```

Recap

- Polymorphic types (type families) and instantiation.
  - parameterized data types

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datatype 'a tree =
  Empty | Node of 'a tree * 'a * 'a tree
```

- Typing rules and most general type.

Recap

- Polymorphic types (type families) and instantiation.
  - parameterized data types

```
datatype 'a tree =
  Empty | Node of 'a tree * 'a * 'a tree
```

- Typing rules and most general type.
- Type guarantee:

Recap

- Polymorphic types (type families) and instantiation.
  - parameterized data types

```
datatype 'a tree =
  Empty | Node of 'a tree * 'a * 'a tree
```

- Typing rules and most general type.
- Type guarantee:

$$\text{If } e : t \text{ and } e \hookrightarrow v \text{ then } v : t.$$

Today's topic

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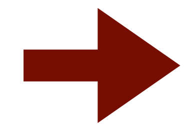
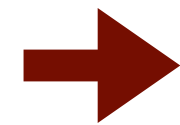
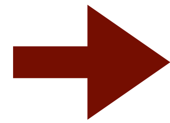


Function currying.

Today’s topic



Function currying.



Higher-order functions:

Today's topic

- Function currying.
- Higher-order functions:
  - Functions whose argument and/or return types are functions.

Today's topic

- Function currying.
- Higher-order functions:
  - Functions whose argument and/or return types are functions.
- More polymorphism.

Currying

Currying

Consider:

```
fun add (x,y) = x + y
```

Currying

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```
fun add (x,y) = x + y
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add:                          *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int          *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Currying

Consider:

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fun add (x,y) = x + y   (* add: int * int -> int *)
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Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Recall, the function definition introduces a binding in the environment:

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Recall, the function definition introduces a binding in the environment:

```
[ env
  fn (x,y) => x + y  /add]
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Let's consider another function:

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Let's consider another function:

```
fun plus x = fn y => x + y
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
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Let’s consider another function:

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fun plus x = fn y => x + y
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Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Let’s consider another function:

```
fun plus x = fn y => x + y
                          (* plus:                   *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Let’s consider another function:

```
fun plus x = fn y => x + y
                           (* plus: int ->           *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Let’s consider another function:

```
fun plus x = fn y => x + y
                          (* plus: int -> int ->     *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)
```

Let's consider another function:

```
fun plus x = fn y => x + y
                            (* plus: int -> int -> int *)
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fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)
```

Notice, arrows right-associate!

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)
```

Notice, arrows right-associate!

t_1 -> t_2 -> t_3

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)
```

Notice, arrows right-associate!

t_1 -> t_2 -> t_3 means

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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```

Notice, arrows right-associate!

t_1 -> t_2 -> t_3 means t_1 -> (t_2 -> t_3)

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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Notice, arrows right-associate!

t_1 -> t_2 -> t_3 means t_1 -> (t_2 -> t_3)

Correspondingly, function application left-associates!

Currying

Consider:

```
fun add (x,y) = x + y    (* add: int * int -> int *)

fun plus x = fn y => x + y
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Notice, arrows right-associate!

t_1 -> t_2 -> t_3 means t_1 -> (t_2 -> t_3)

Correspondingly, function application left-associates!

```
f x y
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                                  (* plus: int -> int -> int *)
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Notice, arrows right-associate!

t_1 -> t_2 -> t_3 means t_1 -> (t_2 -> t_3)

Correspondingly, function application left-associates!

`f x y`   means

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                                (* plus: int -> int -> int *)
```

Notice, arrows right-associate!

t_1 -> t_2 -> t_3 means t_1 -> (t_2 -> t_3)

Correspondingly, function application left-associates!

`f x y` means `(f x) y`

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                             (* plus: int -> int -> int *)
```

Binding for plus:

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)
```

Binding for plus:

[env fn x => fn y => x + y /plus]

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)
```

Currying

Consider:

```
fun add (x,y) = x + y    (* add: int * int -> int *)

fun plus x = fn y => x + y
                         (* plus: int -> int -> int *)
```

Function `plus` is the curried form of `add`.

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                              (* plus: int -> int -> int *)
```

Function `plus` is the curried form of `add`.

Currying: changing "`*`" to "`->`".

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)
```

- Function `plus` is the curried form of `add`.
- Currying: changing "*" to "->".
  - Named after Haskell Curry.

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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```

- Function `plus` is the curried form of `add`.
- Currying: changing "`*`" to "`->`".
  - Named after Haskell Curry.
  - Useful for staging (next lecture!).

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)
```

Let's evaluate:

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                              (* plus: int -> int -> int *)
```

Let’s evaluate:

```
[…,(fn (x,y) => x+y)/add] add (3,4)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                            (* plus: int -> int -> int *)
```

Let’s evaluate:

```
[…,(fn (x,y) => x+y)/add] add (3,4)
==> […] (fn (x,y) => x+y) (3,4)
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)
```

Let’s evaluate:

```
[…,(fn (x,y) => x+y)/add] add (3,4)
==> […] (fn (x,y) => x+y) (3,4)
==> […, 3/x, 4/y] x+y
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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```

Let's evaluate:

```
[…,(fn (x,y) => x+y)/add] add (3,4)
==> […] (fn (x,y) => x+y) (3,4)
==> […, 3/x, 4/y] x+y
==> […, 3/x, 4/y] 3+y
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Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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Let’s evaluate:

```
[…,(fn (x,y) => x+y)/add] add (3,4)
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==> […, 3/x, 4/y] x+y
==> […, 3/x, 4/y] 3+y
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```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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```

Let's evaluate:

```
[…,(fn (x,y) => x+y)/add] add (3,4)
==> […] (fn (x,y) => x+y) (3,4)
==> […, 3/x, 4/y] x+y
==> […, 3/x, 4/y] 3+y
==> […, 3/x, 4/y] 3+4
==> […] 7
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

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Currying

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fun add (x,y) = x + y   (* add: int * int -> int *)

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Let's define yet another function:

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                          (* plus: int -> int -> int *)
```

Let's define yet another function:

```
val incr3 = plus 3
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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fun add (x,y) = x + y   (* add: int * int -> int *)

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```

Let's define yet another function:

```
val incr3 = plus 3      (* incr3:              *)
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Currying

Consider:

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fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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Let's define yet another function:

```
val incr3 = plus 3      (* incr3: int ->     *)
```

Currying

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fun add (x,y) = x + y    (* add: int * int -> int *)

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fun add (x,y) = x + y   (* add: int * int -> int *)

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```

Currying

Consider:

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fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)

val incr3 = plus 3      (* incr3: int -> int *)
```

Binding for incr3:

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)

val incr3 = plus 3      (* incr3: int -> int *)
```

Binding for incr3:

```
[ env
  [3/x]
  fn y => x + y   / incr3 ]
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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val incr3 = plus 3      (* incr3: int -> int *)
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Consider:

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fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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val incr3 = plus 3      (* incr3: int -> int *)
```

Let's evaluate:

Currying

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fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)

val incr3 = plus 3      (* incr3: int -> int *)
```

Let's evaluate:

```
[…,(fn x => fn y => x+y)/plus] plus 3
```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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Let's evaluate:

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[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
```

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Consider:

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fun add (x,y) = x + y   (* add: int * int -> int *)

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Let's evaluate:

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[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
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```

Currying

Consider:

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fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
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val incr3 = plus 3      (* incr3: int -> int *)
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Let's evaluate:

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[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
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Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)

val incr3 = plus 3      (* incr3: int -> int *)
```

Let's evaluate:

```
[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
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Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                        (* plus: int -> int -> int *)

val incr3 = plus 3      (* incr3: int -> int *)
```

Let's evaluate:

```
[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
==> […, 3/x] fn y => x+y
```

It's a value!

lambda

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                          (* plus: int -> int -> int *)

val incr3 = plus 3        (* incr3: int -> int *)
```

Let's evaluate:

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[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
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Consider:

```
fun add (x,y) = x + y    (* add: int * int -> int *)

fun plus x = fn y => x + y
                          (* plus: int -> int -> int *)

val incr3 = plus 3        (* incr3: int -> int *)
```

Let's evaluate:

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[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
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```

Currying

Consider:

```
fun add (x,y) = x + y   (* add: int * int -> int *)

fun plus x = fn y => x + y
                            (* plus: int -> int -> int *)

val incr3 = plus 3          (* incr3: int -> int *)
```

Let's evaluate:

```
[…,(fn x => fn y => x+y)/plus] plus 3
==> […] (fn x => fn y => x+y) 3
==> […, 3/x] fn y => x+y
```

closure

Currying

Consider:

```
fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)

val incr3 = plus 3         (* incr3: int -> int *)
```

Currying

Consider:

```
fun plus x = fn y => x + y
                              (* plus: int -> int -> int *)

val incr3 = plus 3            (* incr3: int -> int *)
```

Let's evaluate:

```
[…, 3/x, (fn y => x+y)/incr3] incr3 4
```

Currying

Consider:

```
fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)

val incr3 = plus 3         (* incr3: int -> int *)
```

Let's evaluate:

```
[…, 3/x, (fn y => x+y)/incr3] incr3 4
==> […, 3/x] (fn y => x+y) 4
```

Currying

Consider:

```
fun plus x = fn y => x + y
                              (* plus: int -> int -> int *)

val incr3 = plus 3            (* incr3: int -> int *)
```

Let's evaluate:

```
[…, 3/x, (fn y => x+y)/incr3] incr3 4
==> […, 3/x] (fn y => x+y) 4
==> […, 3/x, 4/y] x+y
```

Currying

Consider:

```
fun plus x = fn y => x + y
                           (* plus: int -> int -> int *)

val incr3 = plus 3         (* incr3: int -> int *)
```

Let's evaluate:

```
[…, 3/x, (fn y => x+y)/incr3] incr3 4
==> […, 3/x] (fn y => x+y) 4
==> […, 3/x, 4/y] x+y
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```

Currying

Consider:

```
fun plus x = fn y => x + y
                              (* plus: int -> int -> int *)

val incr3 = plus 3            (* incr3: int -> int *)
```

Let’s evaluate:

```
[…, 3/x, (fn y => x+y)/incr3] incr3 4
==> […, 3/x] (fn y => x+y) 4
==> […, 3/x, 4/y] x+y
==> […, 3/x, 4/y] 3+y
==> […, 3/x, 4/y] 3+4
```

Currying

Consider:

```
fun plus x = fn y => x + y
                              (* plus: int -> int -> int *)

val incr3 = plus 3            (* incr3: int -> int *)
```

Let’s evaluate:

```
[…, 3/x, (fn y => x+y)/incr3] incr3 4
==> […, 3/x] (fn y => x+y) 4
==> […, 3/x, 4/y] x+y
==> […, 3/x, 4/y] 3+y
==> […, 3/x, 4/y] 3+4
==> […] 7
```

Currying: syntactic sugar

Currying: syntactic sugar

A maybe more convenient notation

Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
```

Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
```

which is syntactic sugar for

```
fun plus x = fn y => x + y
```

Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
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which is syntactic sugar for

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fun plus x = fn y => x + y
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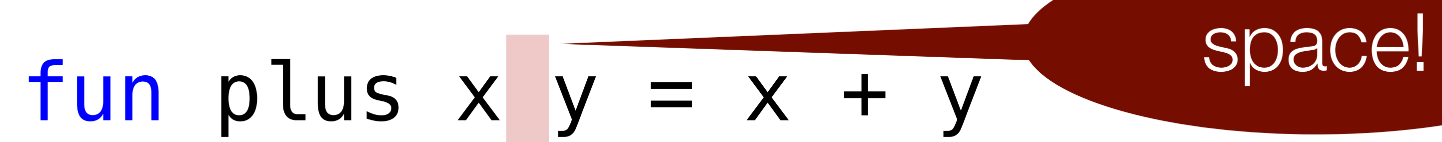
Currying: syntactic sugar

A maybe more convenient notation




```
fun plus x y = x + y
```

which is syntactic sugar for

```
fun plus x = fn y => x + y
```

Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
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which is syntactic sugar for

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Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
```

which is syntactic sugar for

```
fun plus x = fn y => x + y
```

which is itself syntactic sugar for

```
val rec plus = fn x => fn y => x + y
```

Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
```

which is syntactic sugar for

```
fun plus x = fn y => x + y
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which is itself syntactic sugar for

```
val rec plus = fn x => fn y => x + y
```

Currying: syntactic sugar

A maybe more convenient notation

```
fun plus x y = x + y
```

which is syntactic sugar for

```
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which is itself syntactic sugar for

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val rec plus = fn x => fn y => x + y
```

Currying: some more examples

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```
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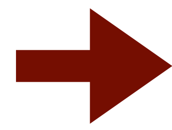
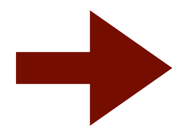
Higher-order functions

Higher-order functions



Higher-order functions are functions whose argument type and/or return type are functions.

Higher-order functions



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Filtering elements in a list, given a predicate:

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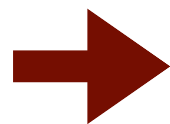

`filter` is higher-order

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- `filter` is predefined as `List.filter`.

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                          (* int -> bool *)        type?

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Filter in action:

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val keepevens = filter (fn n => n mod 2 = 0)

val [2,~6, 10] = keepevens [1,2,~5,~6,11,10,13]
```

Higher-order function: composition

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Function composition, abstractly:

Higher-order function: composition

Function composition, abstractly: $(f \circ g)\ (x) = f\ (g\ (x))$

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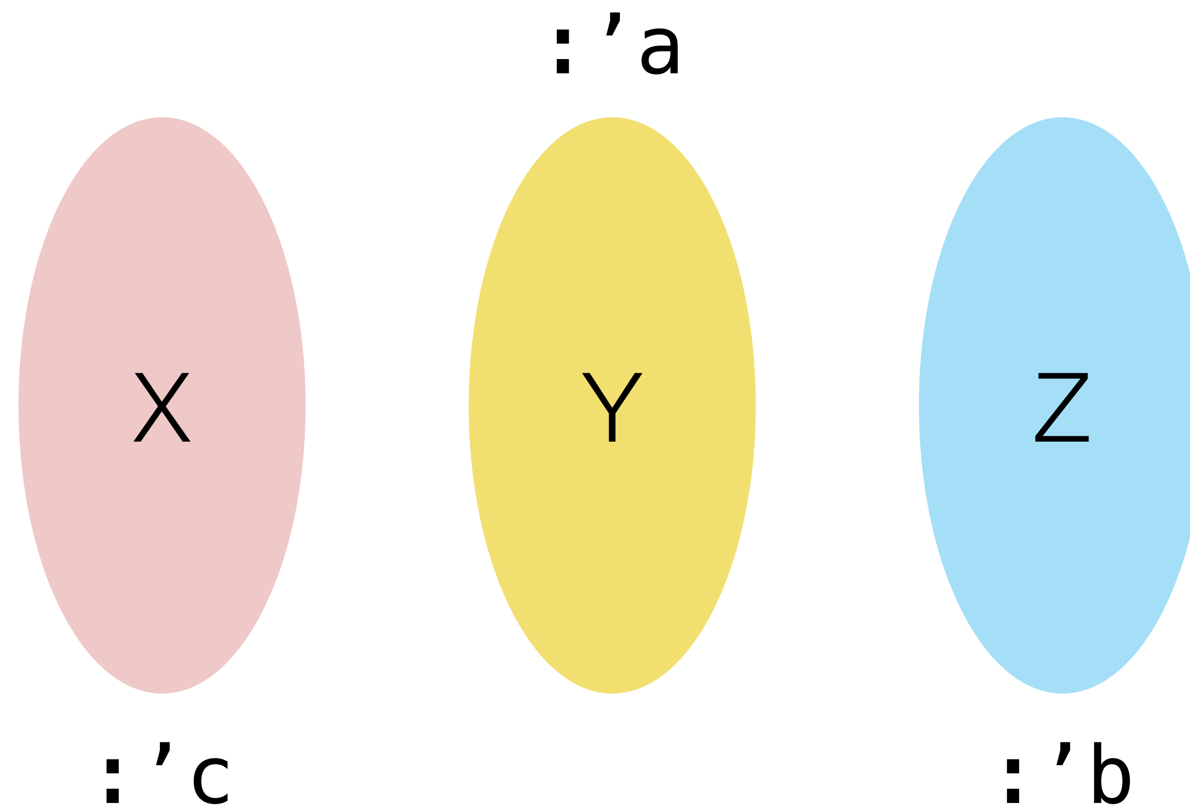

Higher-order function: composition

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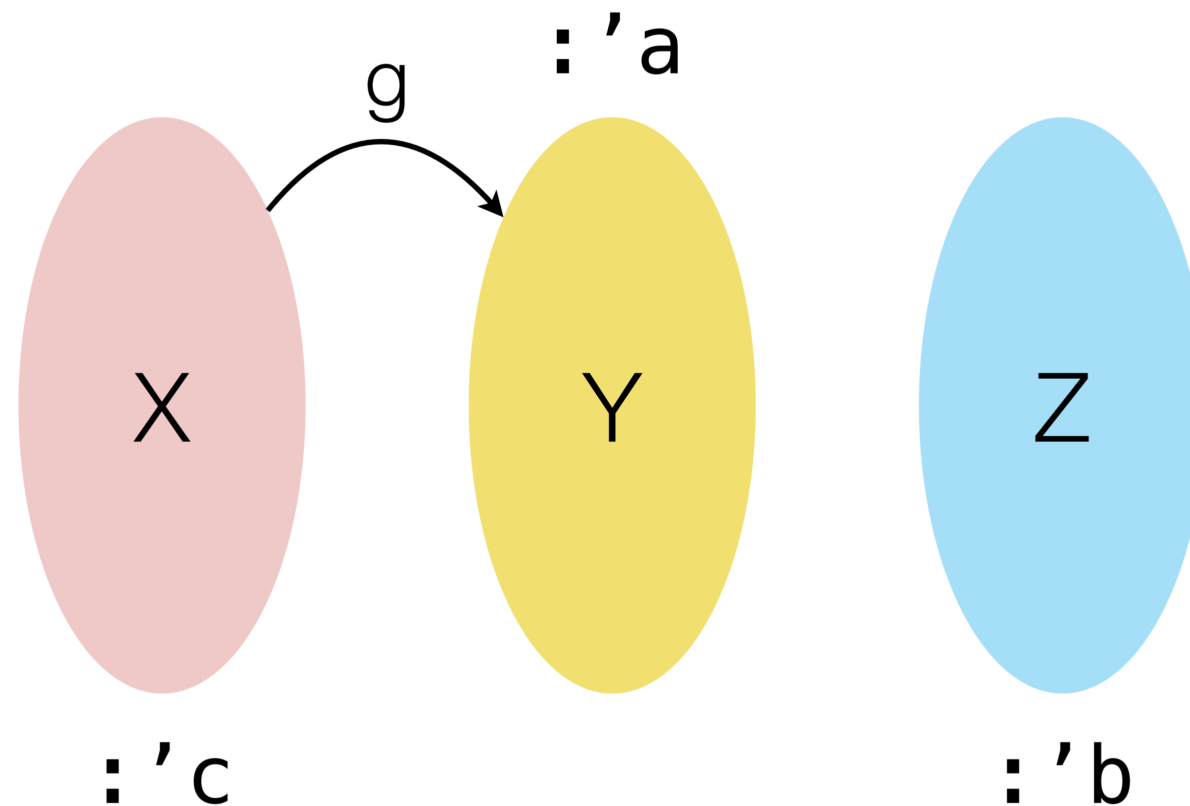

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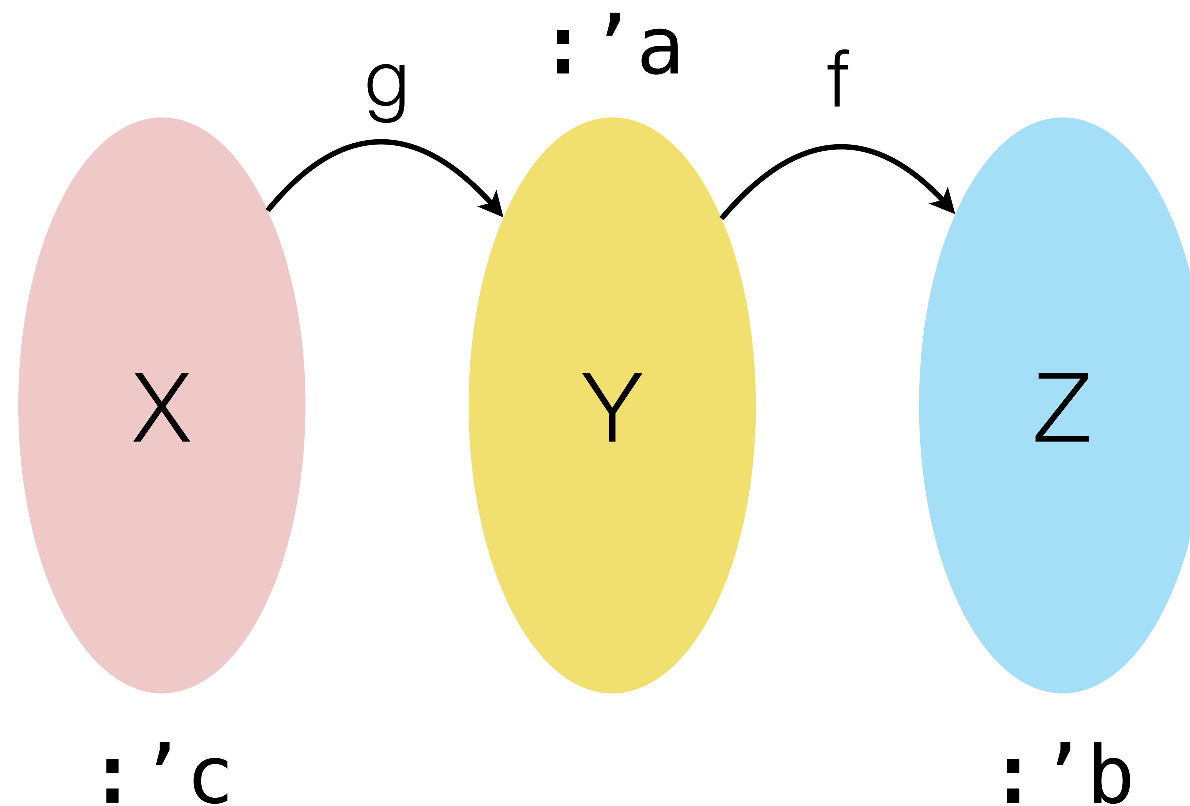

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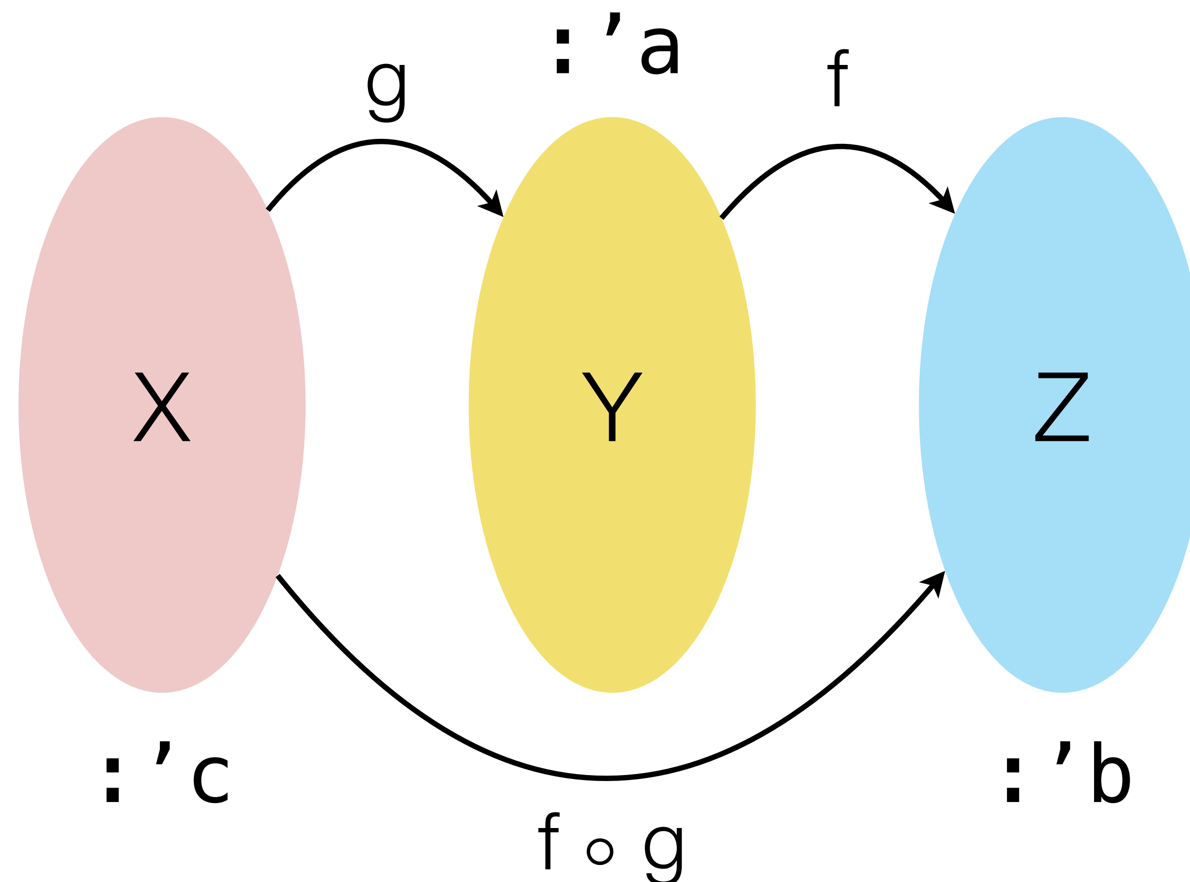

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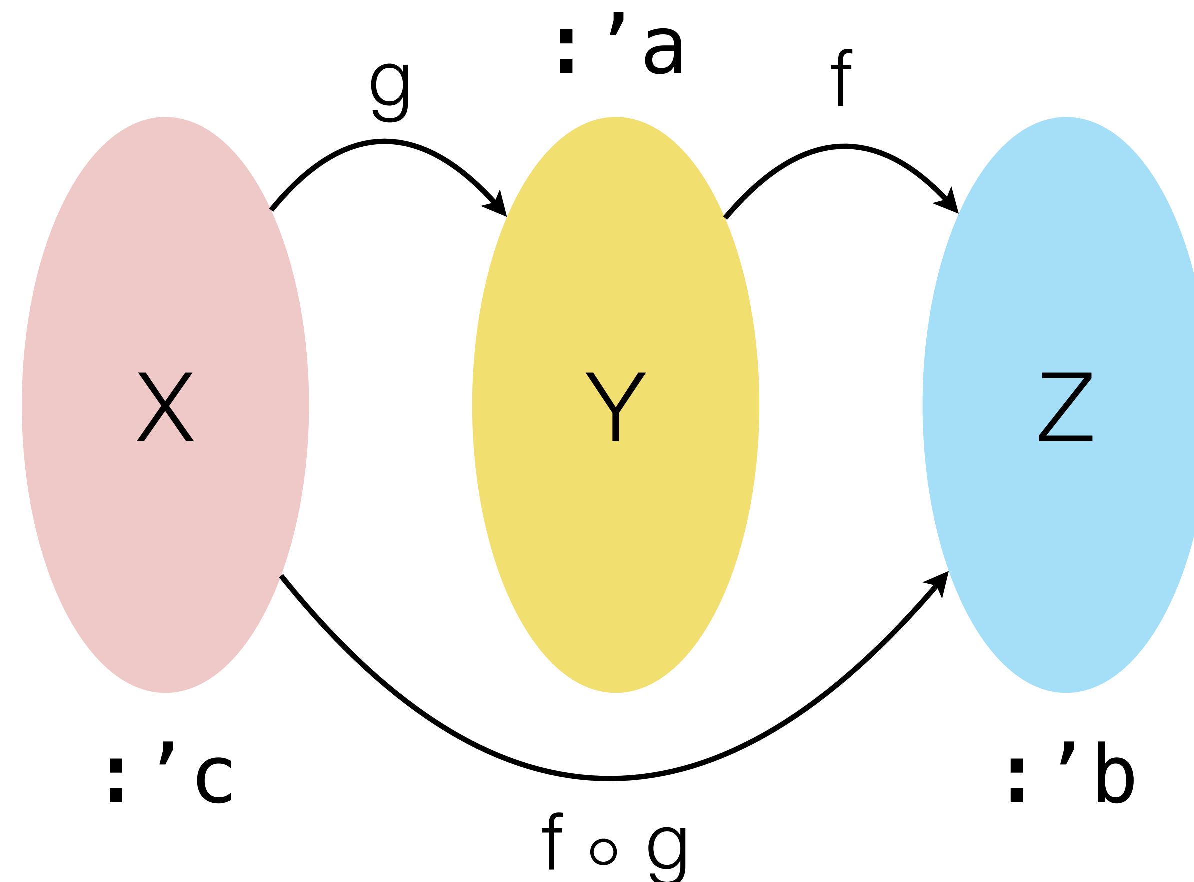

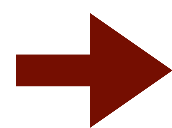

∘ is a higher-order function, i.e., a combinator, expecting two functions as arguments and returning a function.

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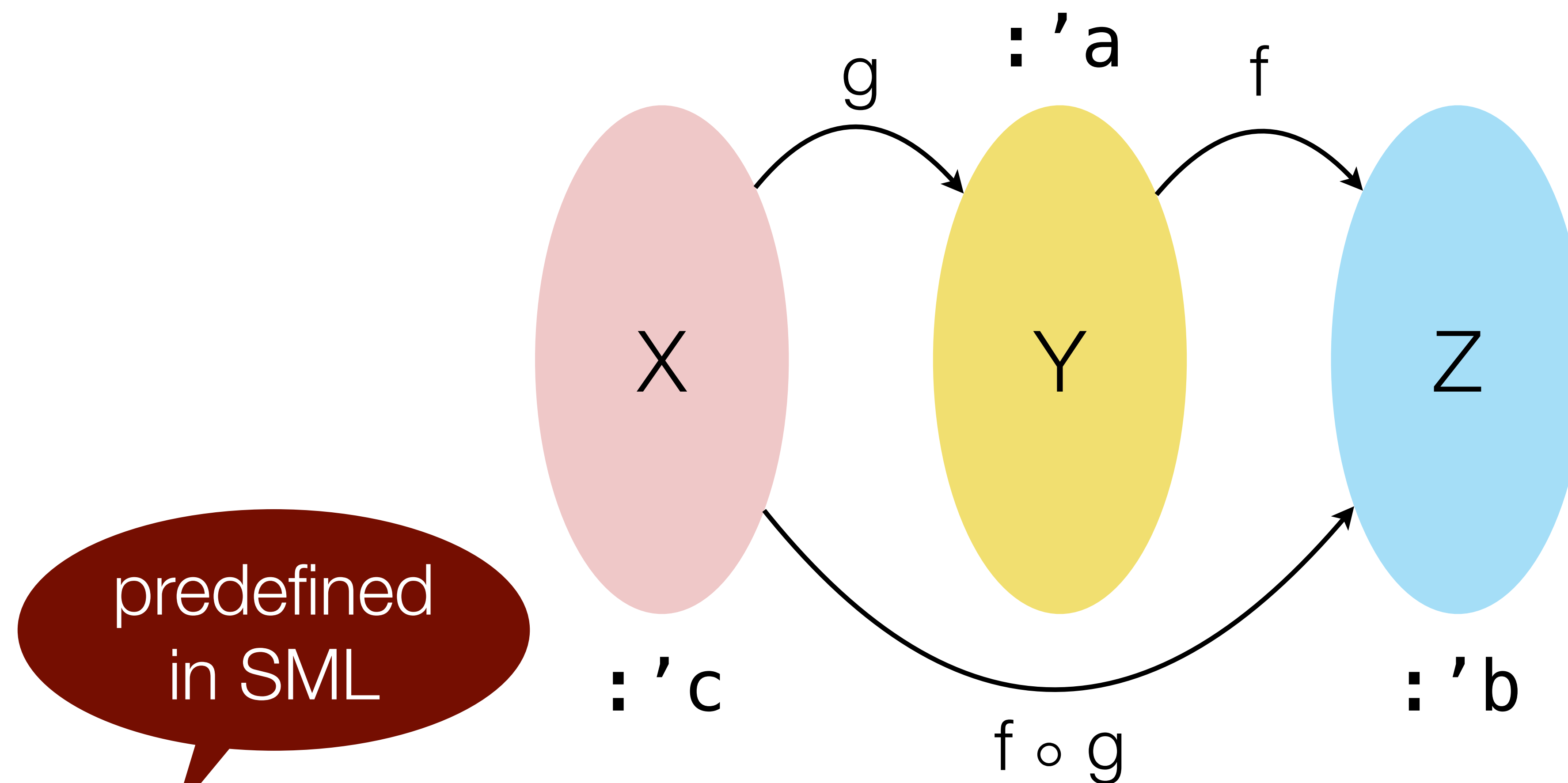

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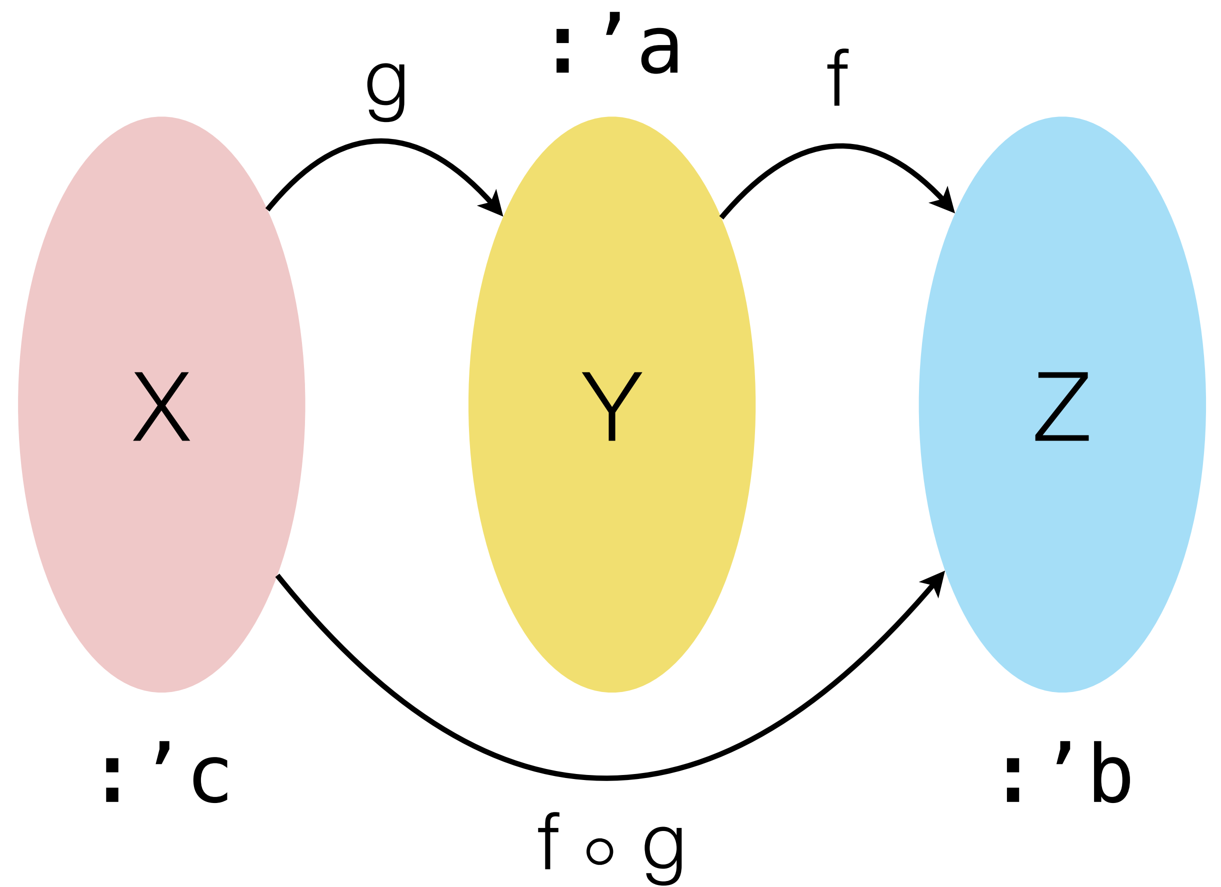

Higher-order function: composition

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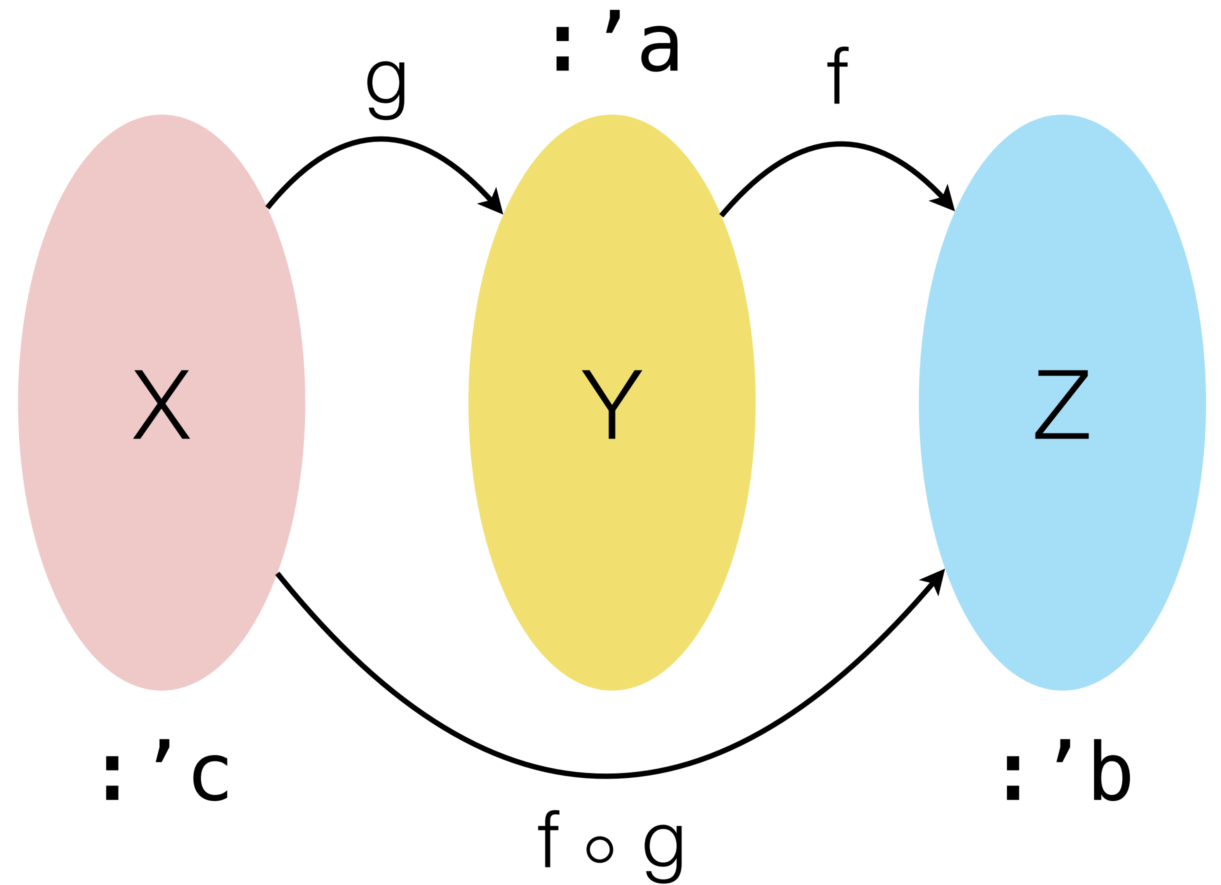
infix o
fun f o g = fn x => f(g(x))
```



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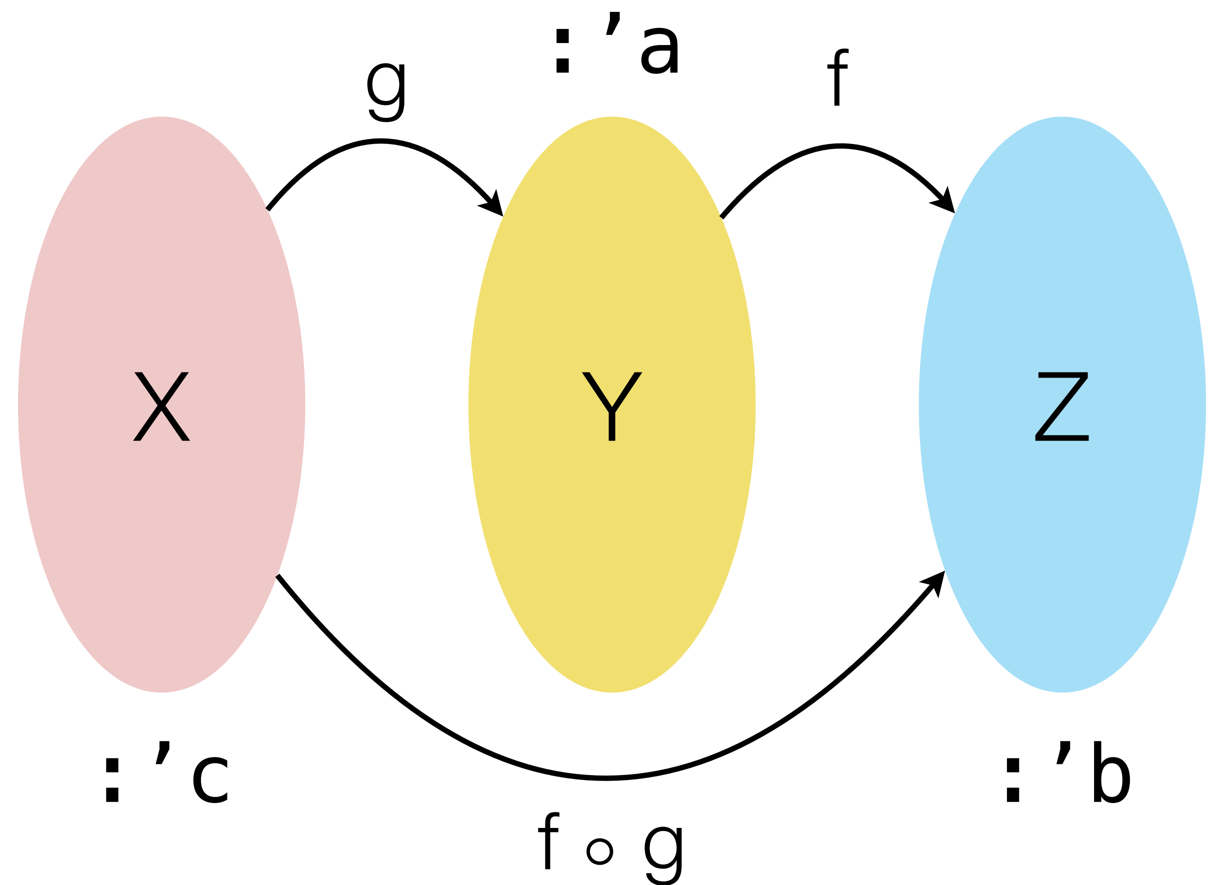
What is its type?



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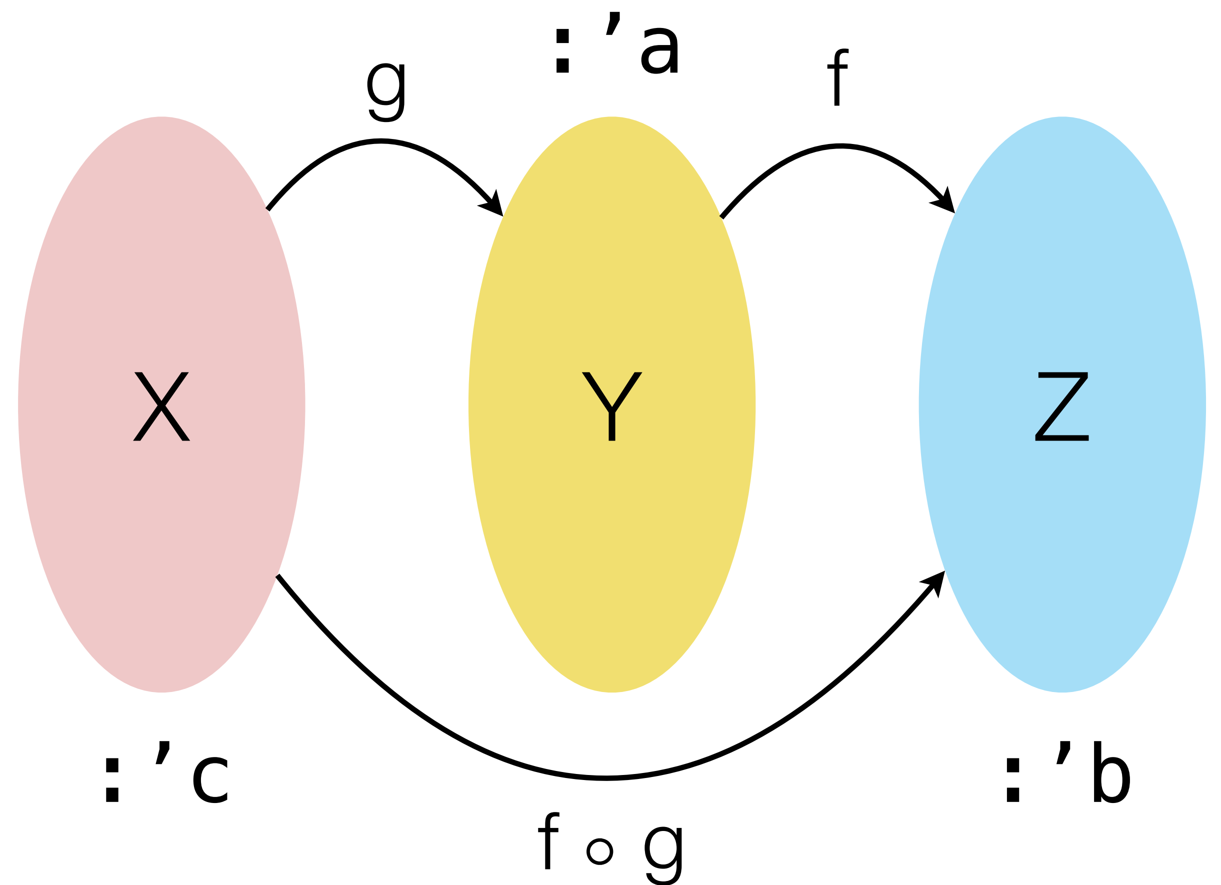


```
(* (op o):                                          *)
```

Higher-order function: composition

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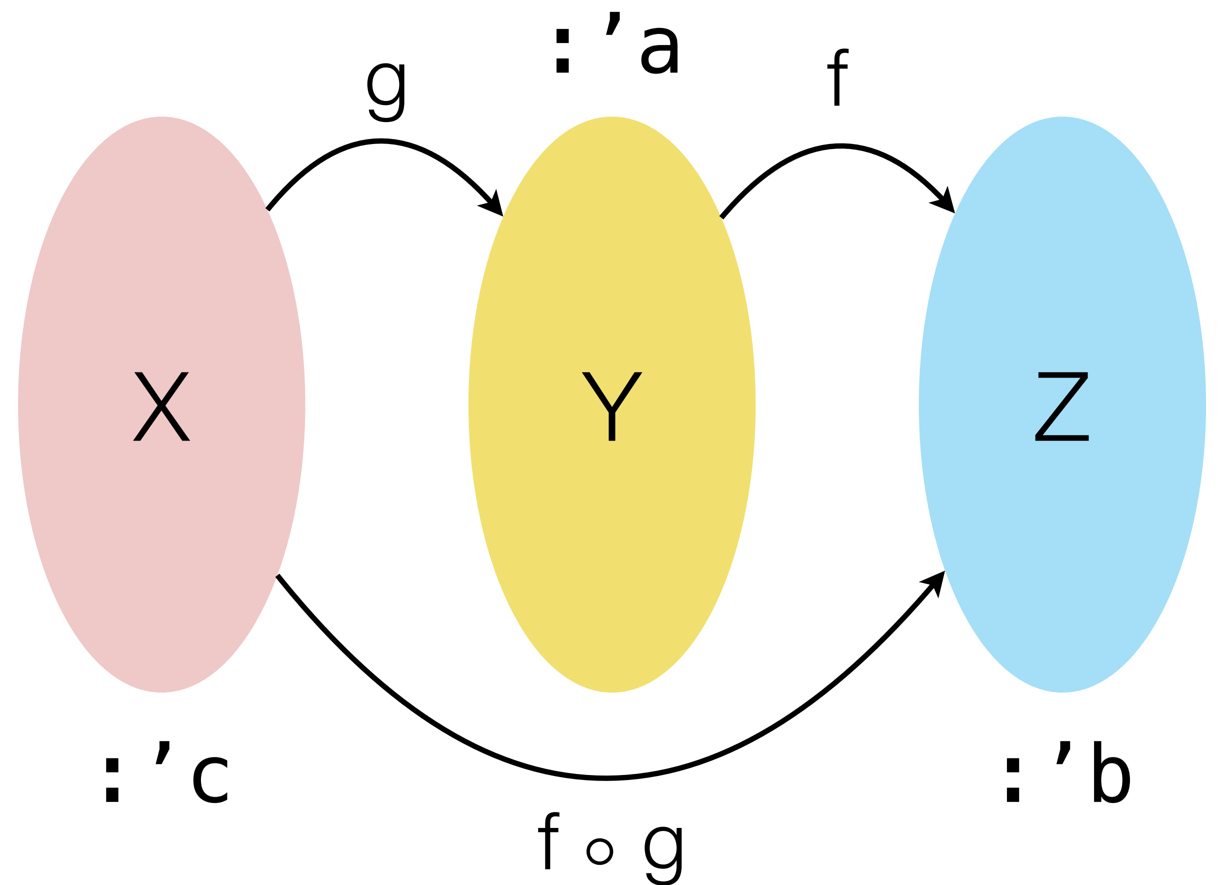


```
(* (op o): ('a -> 'b) *                                    *)
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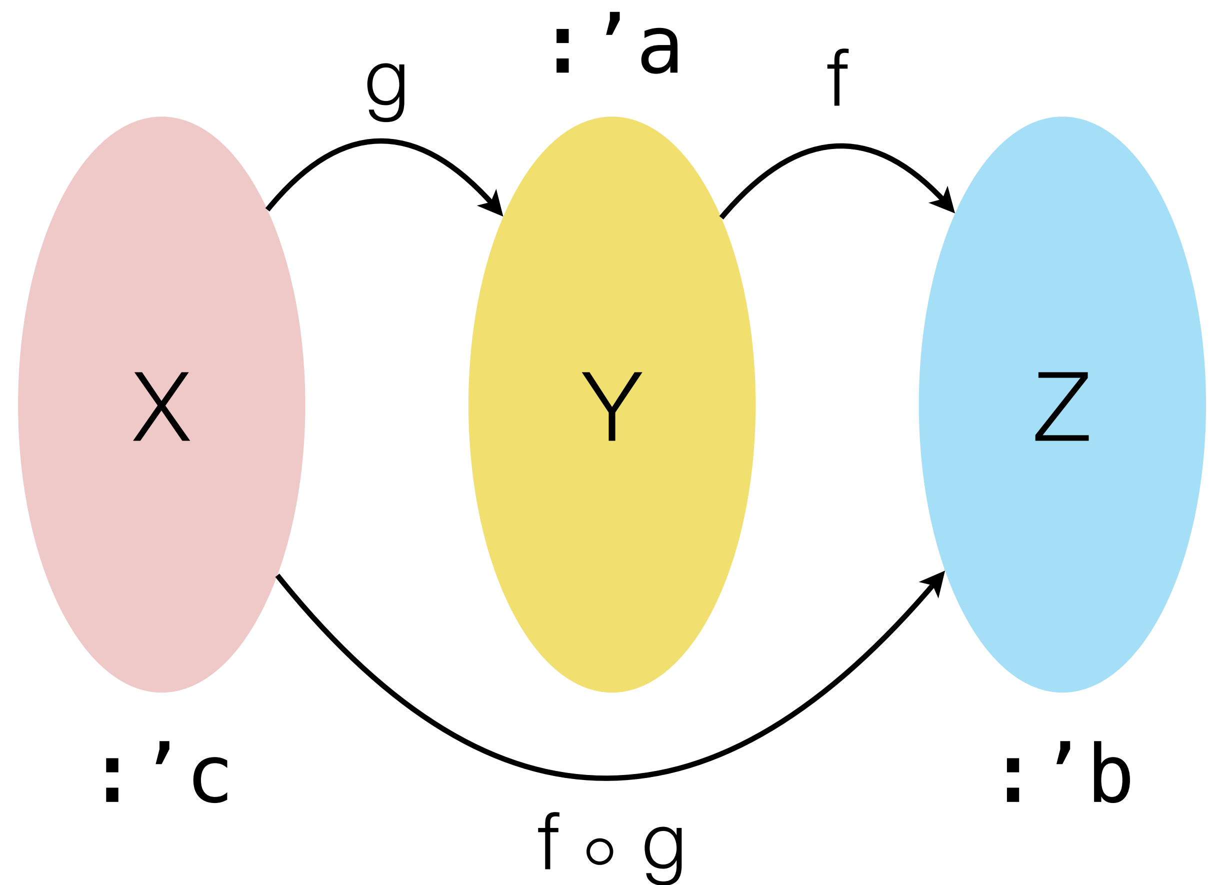


```
(* (op o): ('a -> 'b) * ('c -> 'a)                *)
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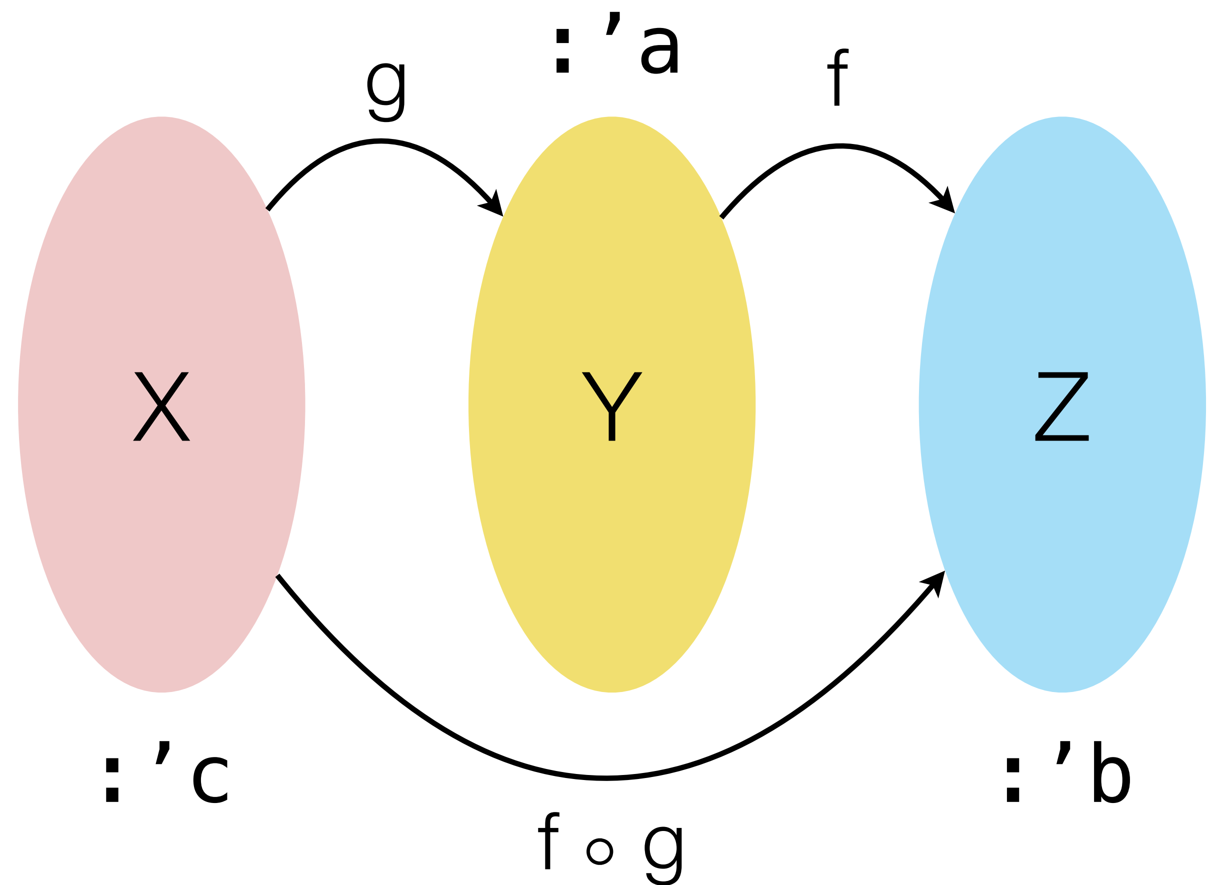


```
(* (op o): ('a -> 'b) * ('c -> 'a) -> 'c -> 'b *)
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Higher-order function: composition

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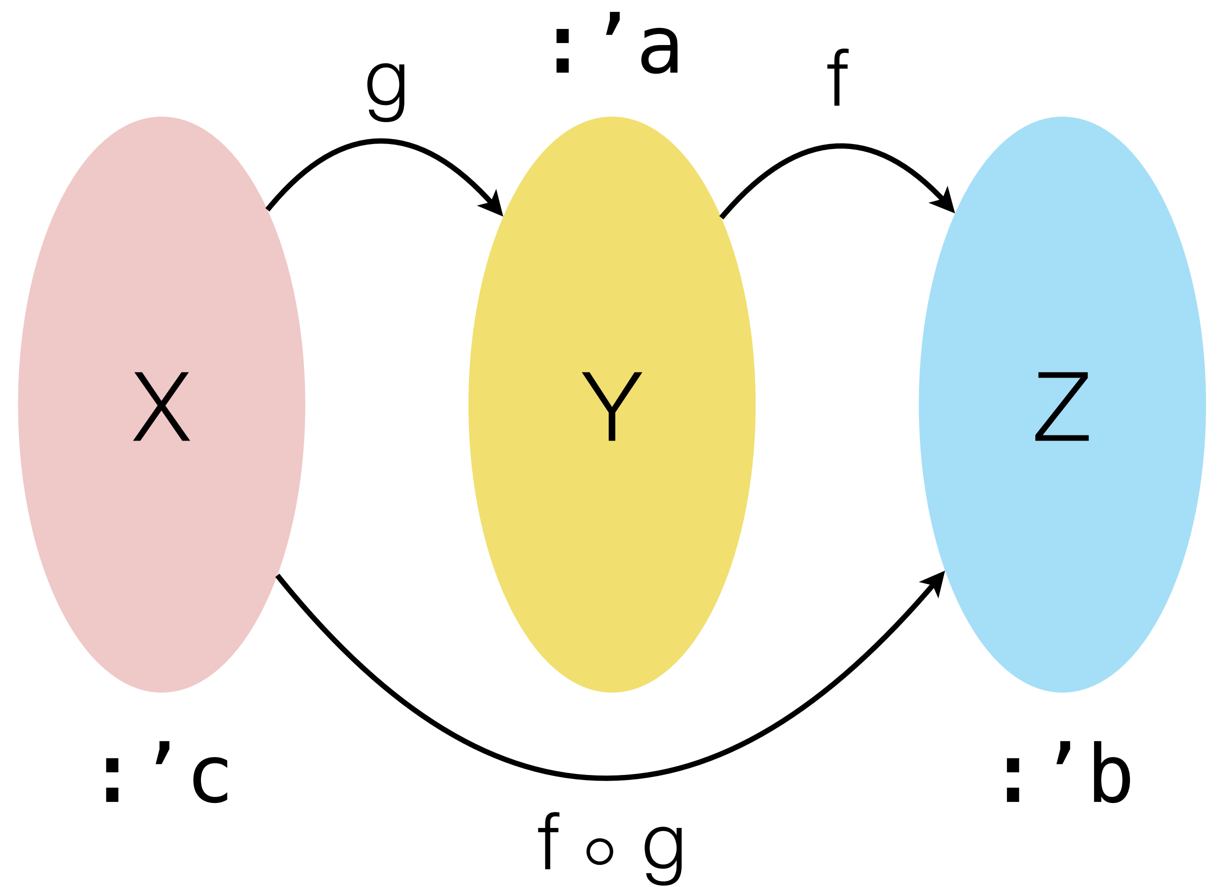

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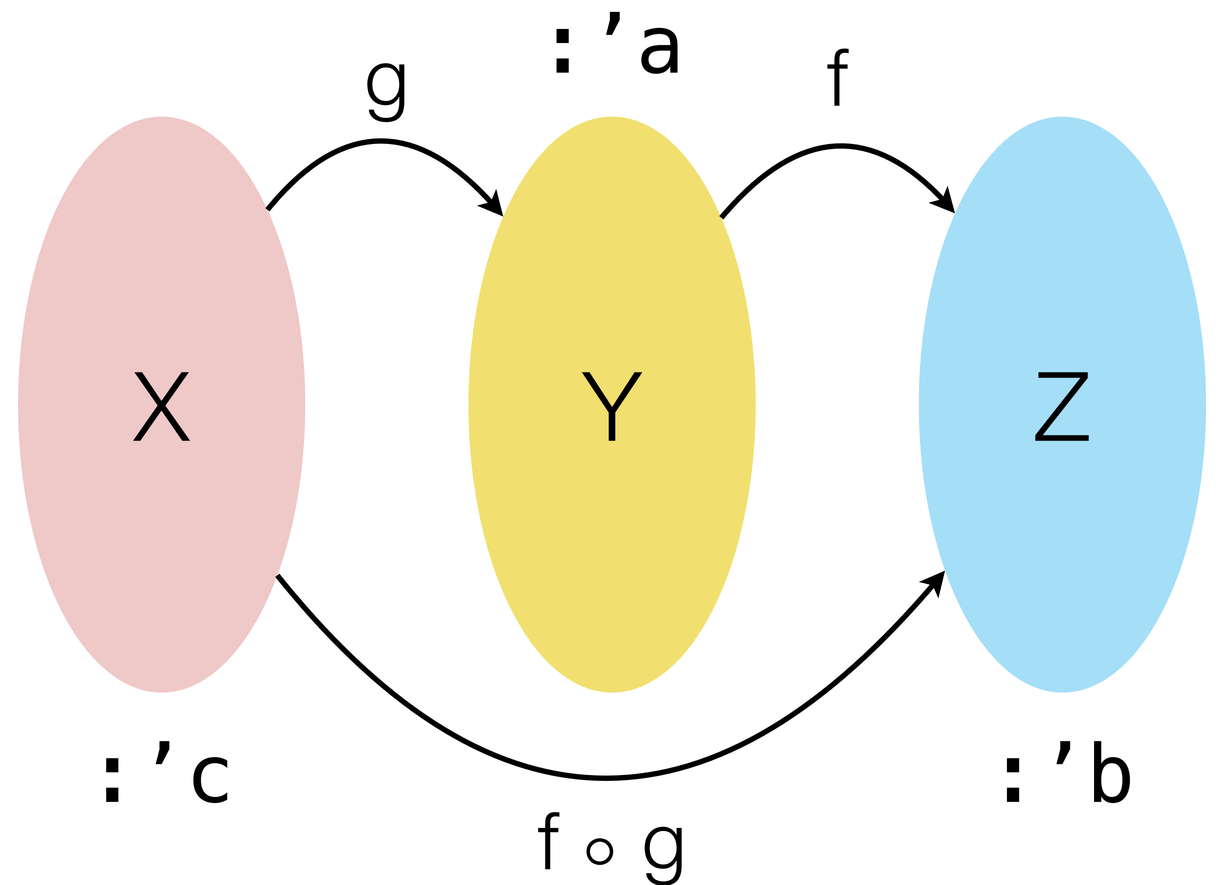

Higher-order function: composition

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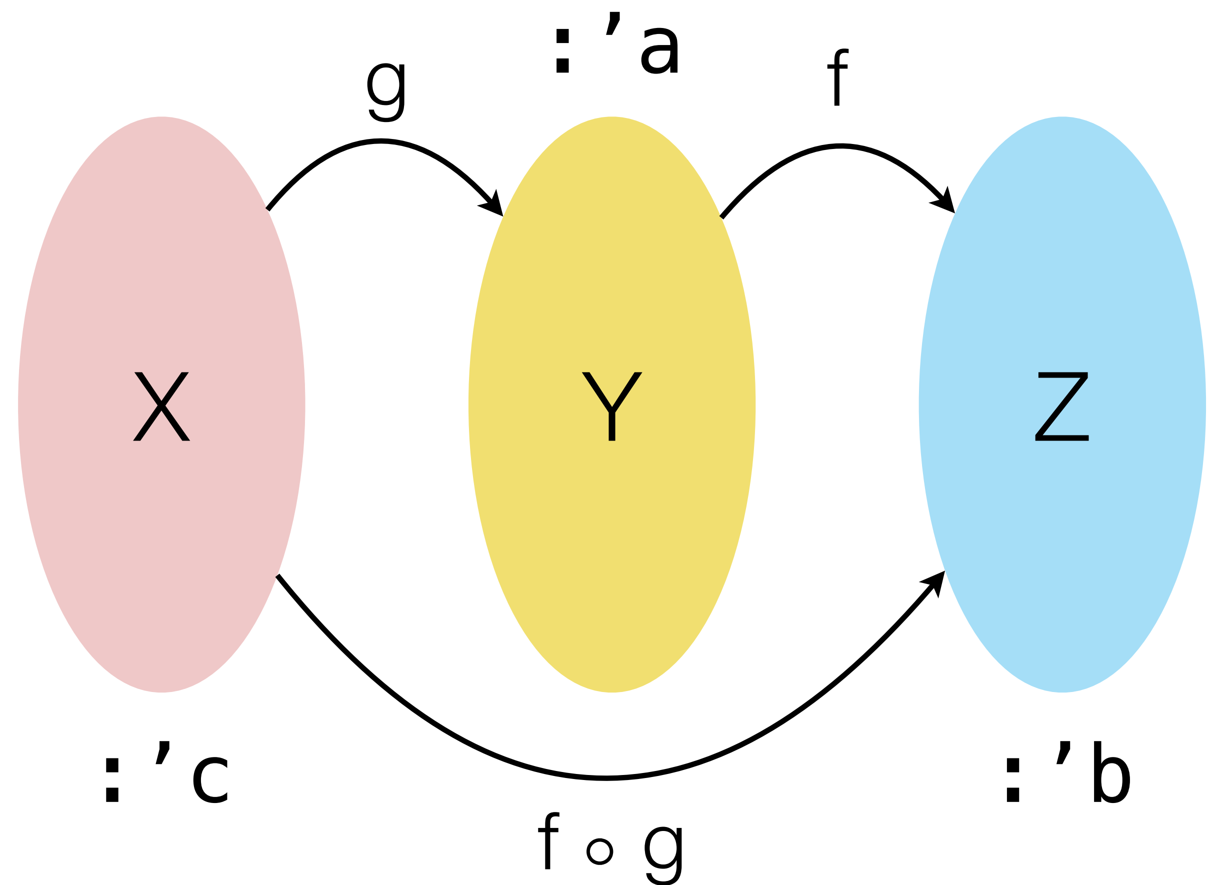
Examples:

```
fun incr x = x + 1
fun double x = 2 * x
```



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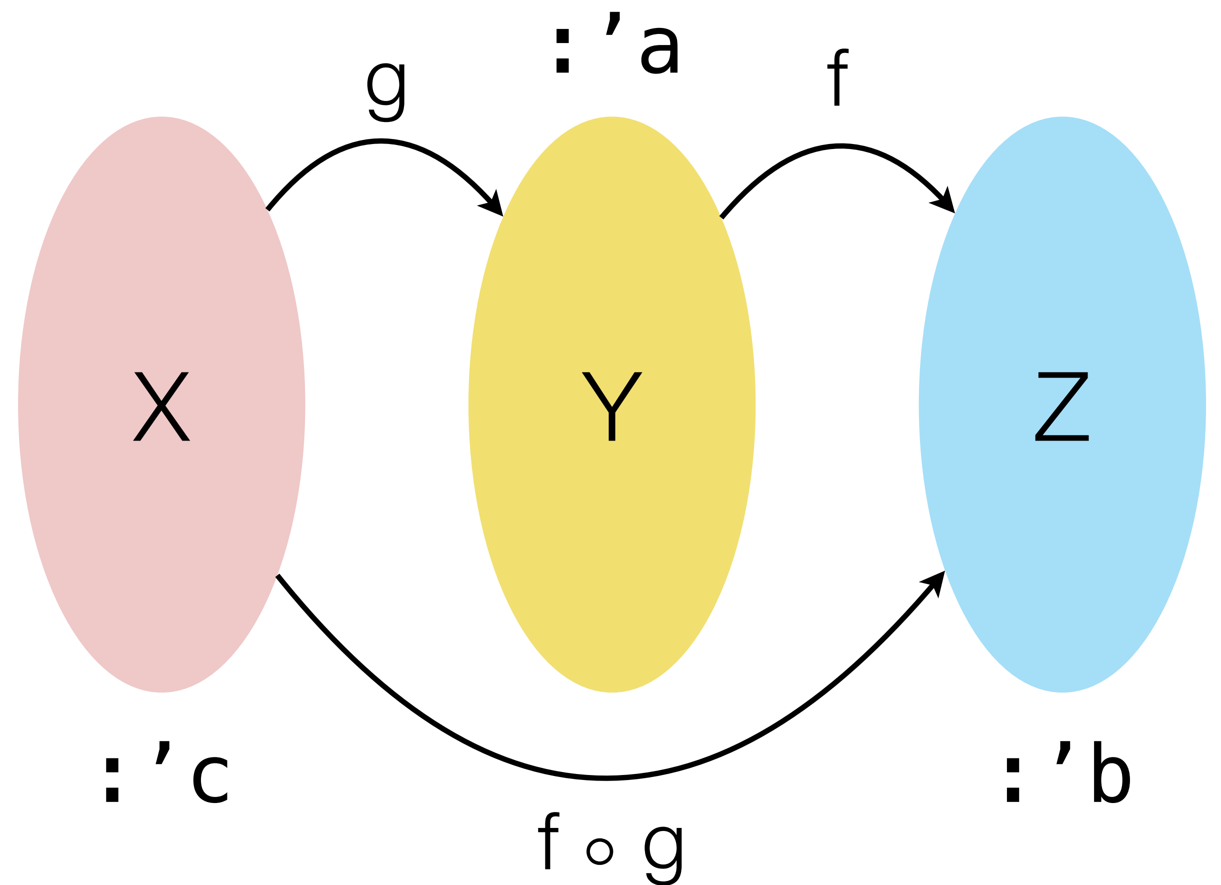

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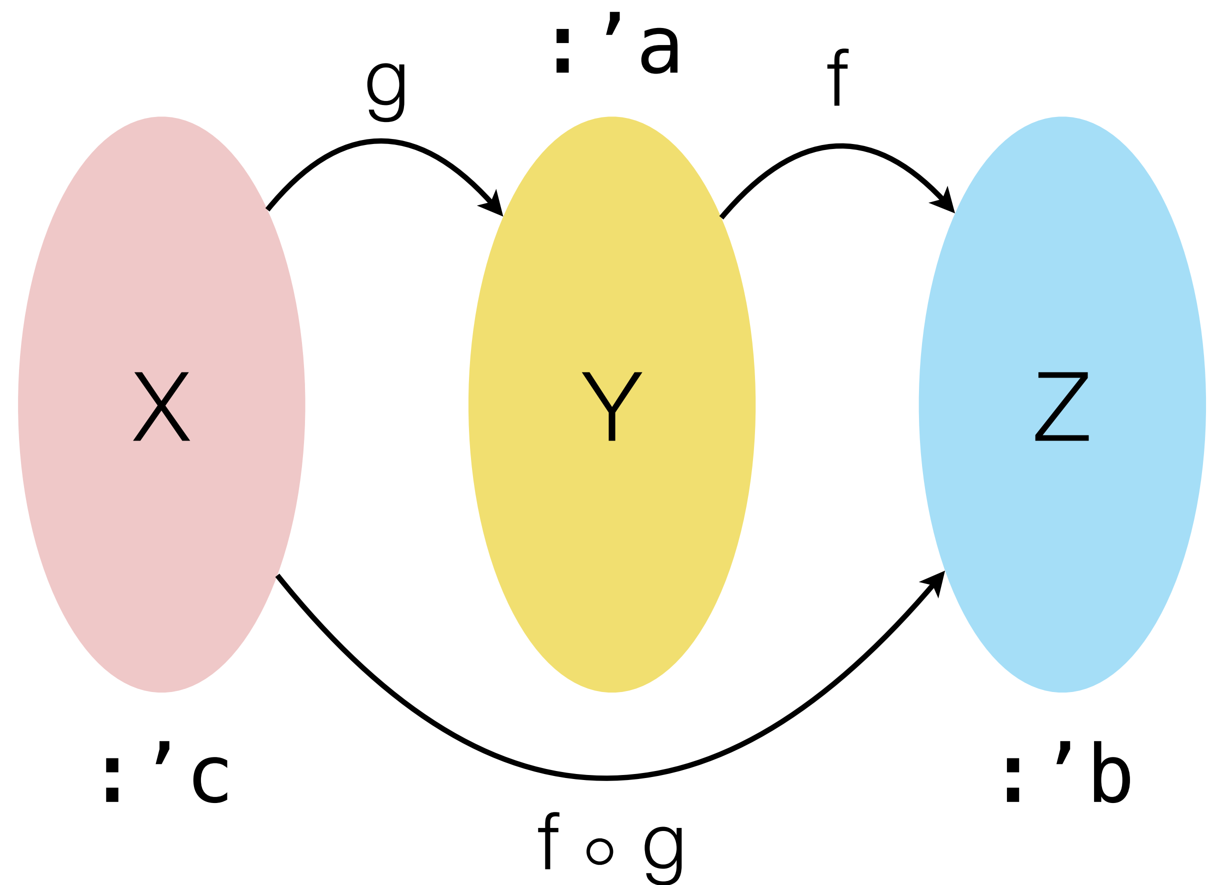

Examples:

```
fun incr x = x + 1
fun double x = 2 * x
```

Then we have:

```
double o incr  ≅  fn x => 2x + 2
incr o double  ≅  fn x => 2x + 1
```

Higher-order function: map

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Transforming elements in a list, given a transformation function:

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```
(* map: ('a -> 'b) -> 'a list -> 'b list
   REQUIRES: true
   ENSURES: For all n≥0, map f [x1,…,xn] ≅ [f x1,…,f xn]
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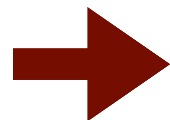

`map` is higher-order

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takes a function `f: ‘a -> ‘b` as an argument

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- `map` is higher-order
  - takes a function `f: 'a -> 'b` as an argument
- `map` is predefined in SML

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Example:

```
map double [1,2,3] ==> [2,4,6]
```

Higher-order function: fold

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Combining elements in a list, given a binary operation and base value:

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(* fold: ('a * 'b -> 'b) -> 'b -> 'a list -> 'b *)
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combining function:

‘a: type of list elements

‘b: type of base value and of combined value

Higher-order function: fold

Combining elements in a list, given a binary operation and base value:

```
(* fold: ('a * 'b -> 'b) -> 'b -> 'a list -> 'b *)
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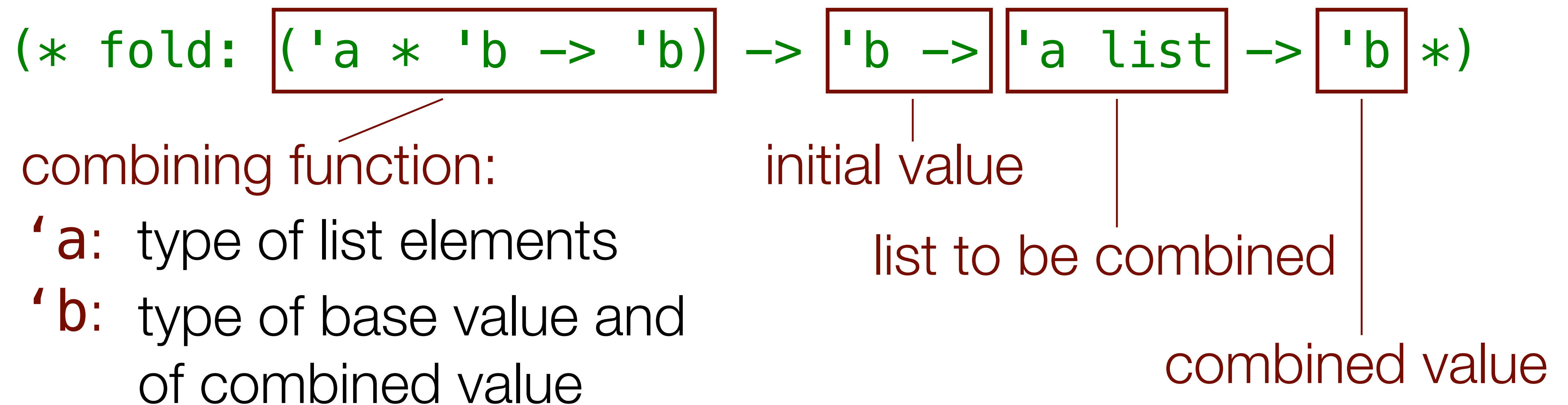

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Two implementations:

```
foldl f z [x1,…,xn]  ≅  f(xn,…f(x3,f(x2,f(x1,z))))
```

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Two implementations:

foldl f z $[x_1, \ldots, x_n] \cong f(x_n, \ldots f(x_3, f(x_2, f(x_1, z))))$

foldr f z $[x_1, \ldots, x_n] \cong f(x_1, \ldots f(x_{n-2}, f(x_{n-1}, f(x_n, z))))$

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Examples:

```
foldl (op -) 0 [1,2,3,4] ==>
```

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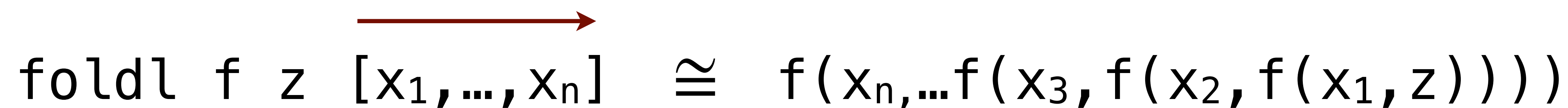

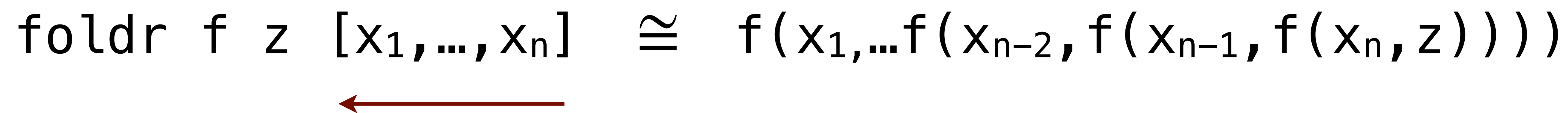

Examples:

```
foldl (op -) 0 [1,2,3,4] ==> 2
```

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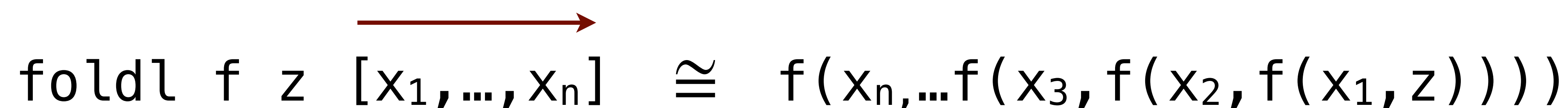

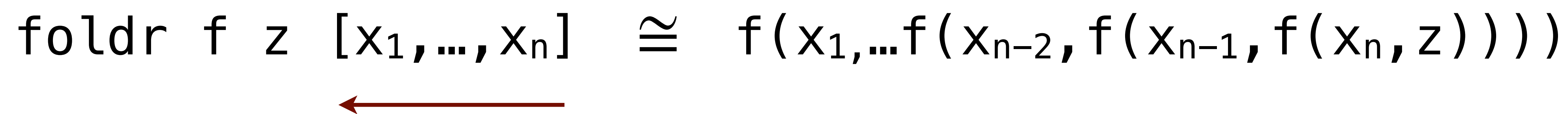

Examples:

```
foldl (op -) 0 [1,2,3,4] ==> 2
foldr (op -) 0 [1,2,3,4] ==>
```

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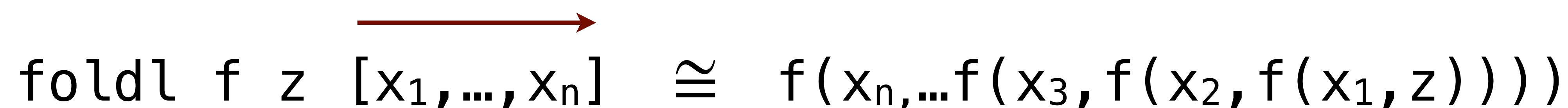

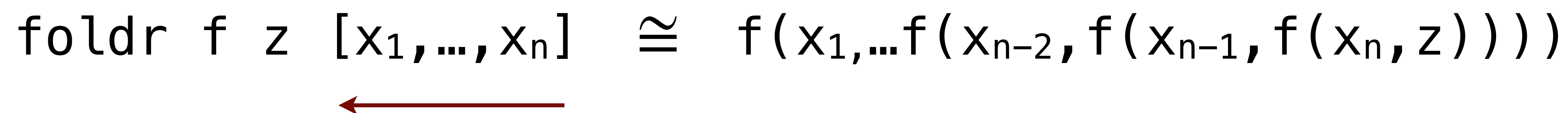

Examples:

```
foldl (op -) 0 [1,2,3,4] ==> 2
foldr (op -) 0 [1,2,3,4] ==> ~2
```

Higher-order function: fold

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Let’s implement foldl and foldr:

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```
fun foldl f z [] =
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```

```
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  | foldr f z (x::xs) = f(x, foldr f z xs)
```

Homework:

```
foldl (op ::) [] [1,2,3,4] ==> ?
foldr (op ::) [] [1,2,3,4] ==> ?
```