Staging (Higher-Order Functions in Action)

15-150

Lecture 11: October 3, 2024

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Can we generalize map and fold?

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- Yes! Let’s work it out.

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- Yes! Let's work it out.
- It may be helpful to visualize map and fold for lists diagrammatically first, to capture the underlying pattern.

The “pattern” underlying map

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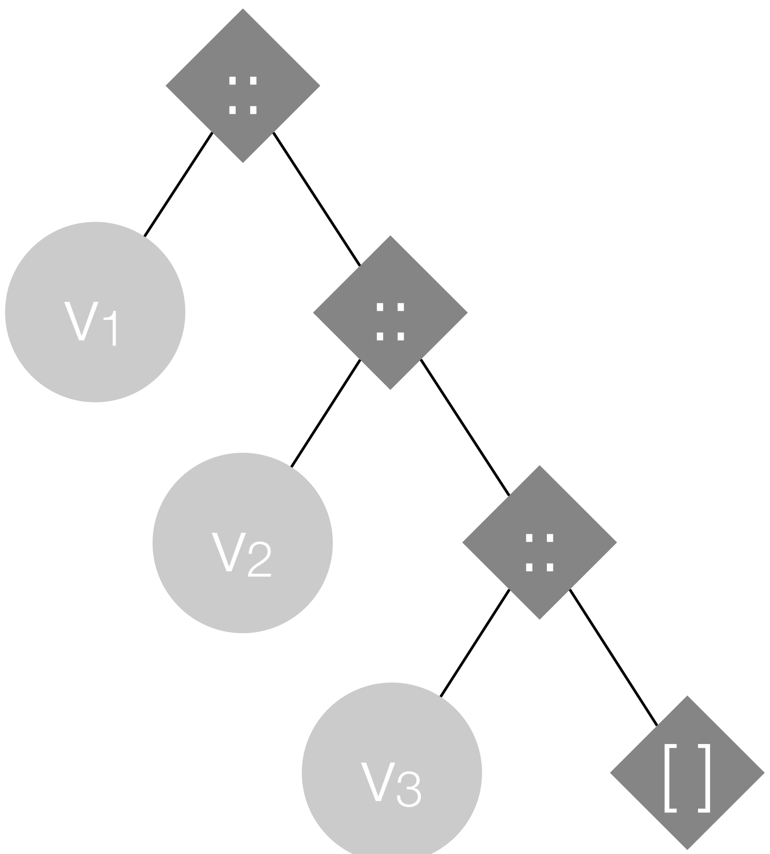

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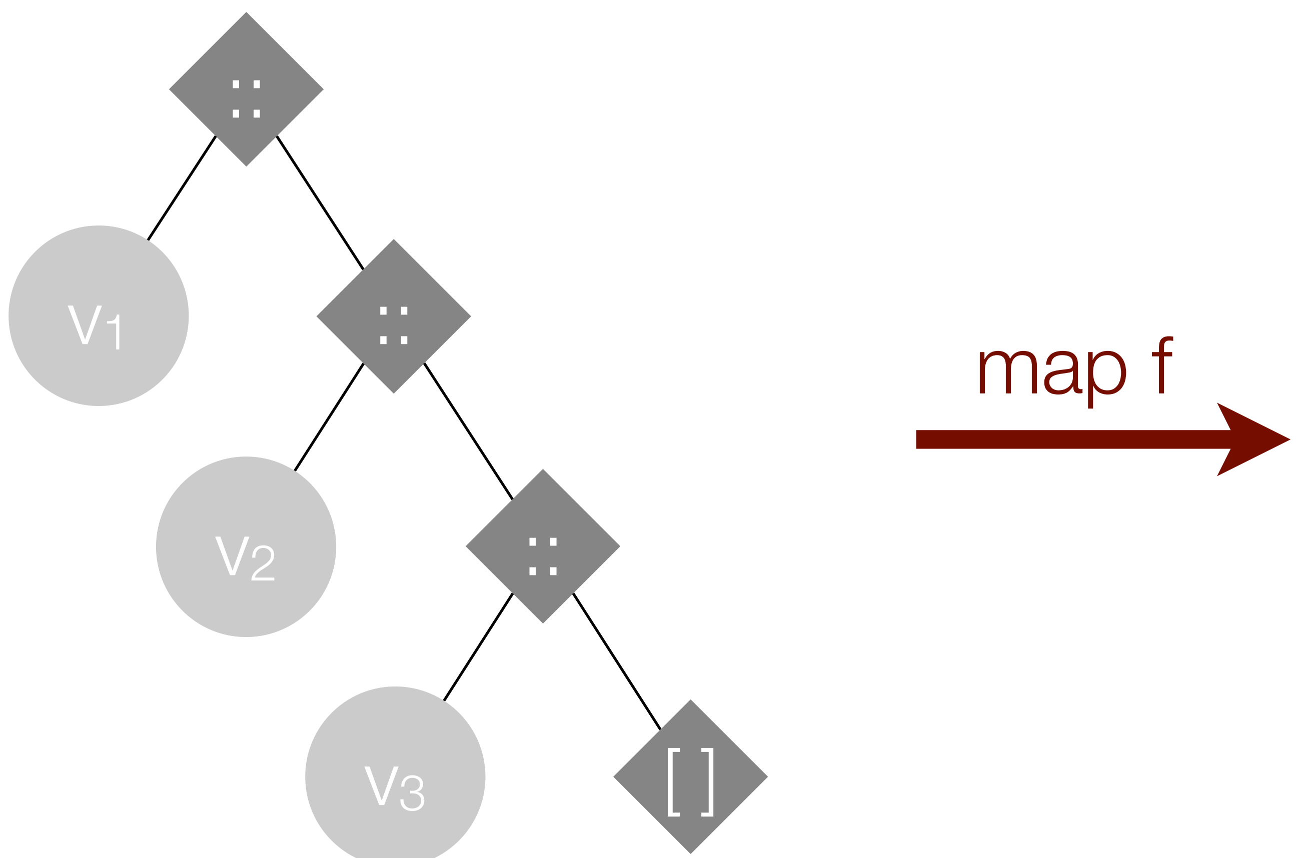

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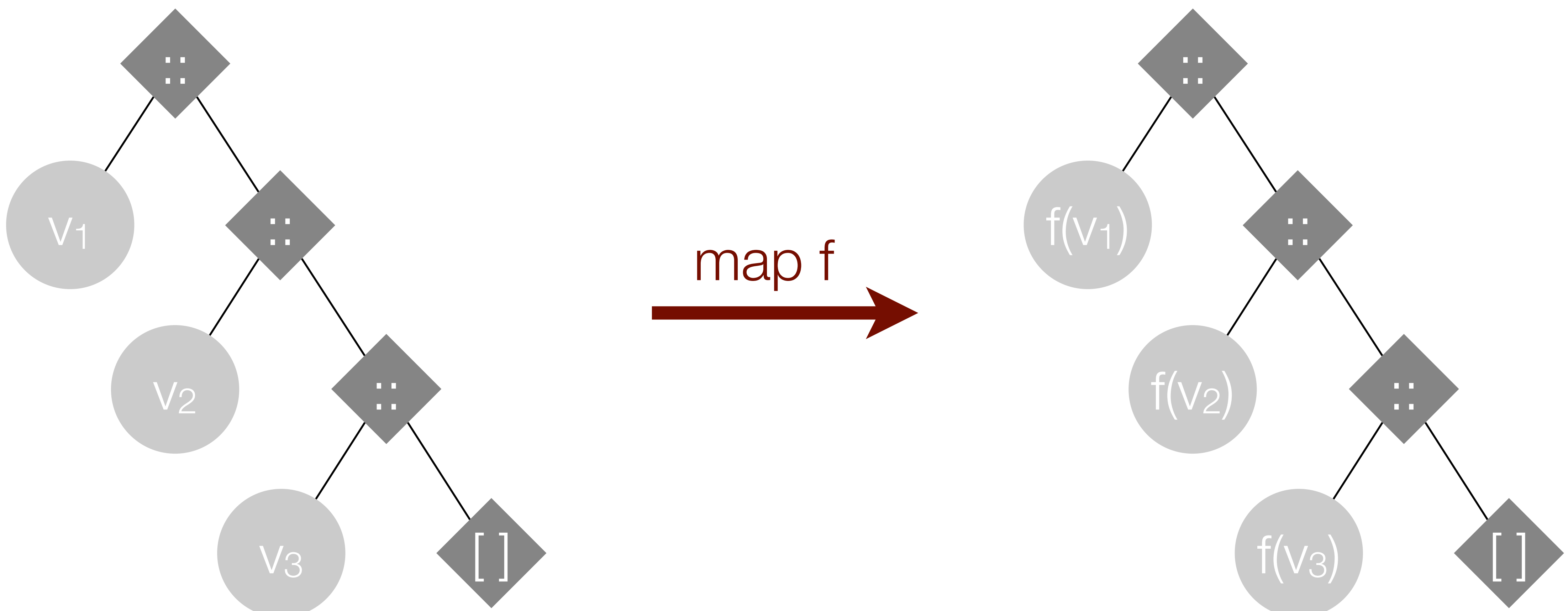

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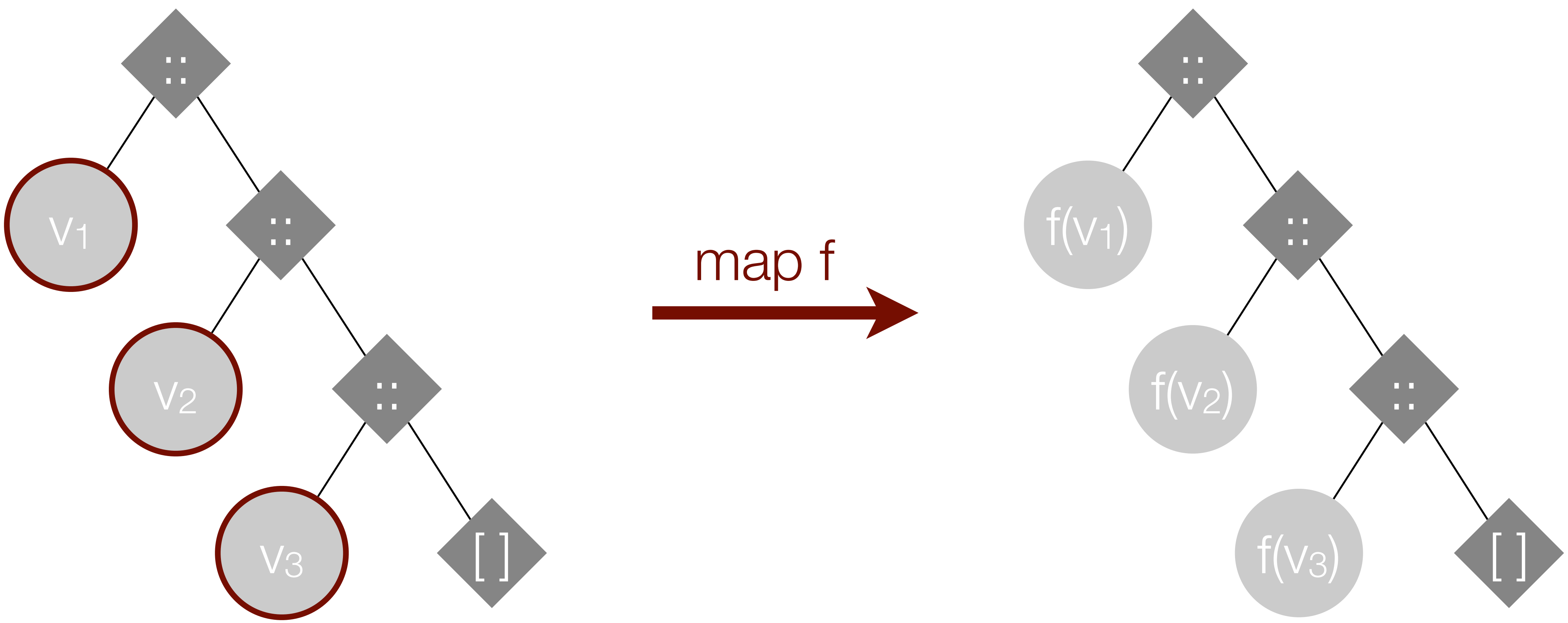

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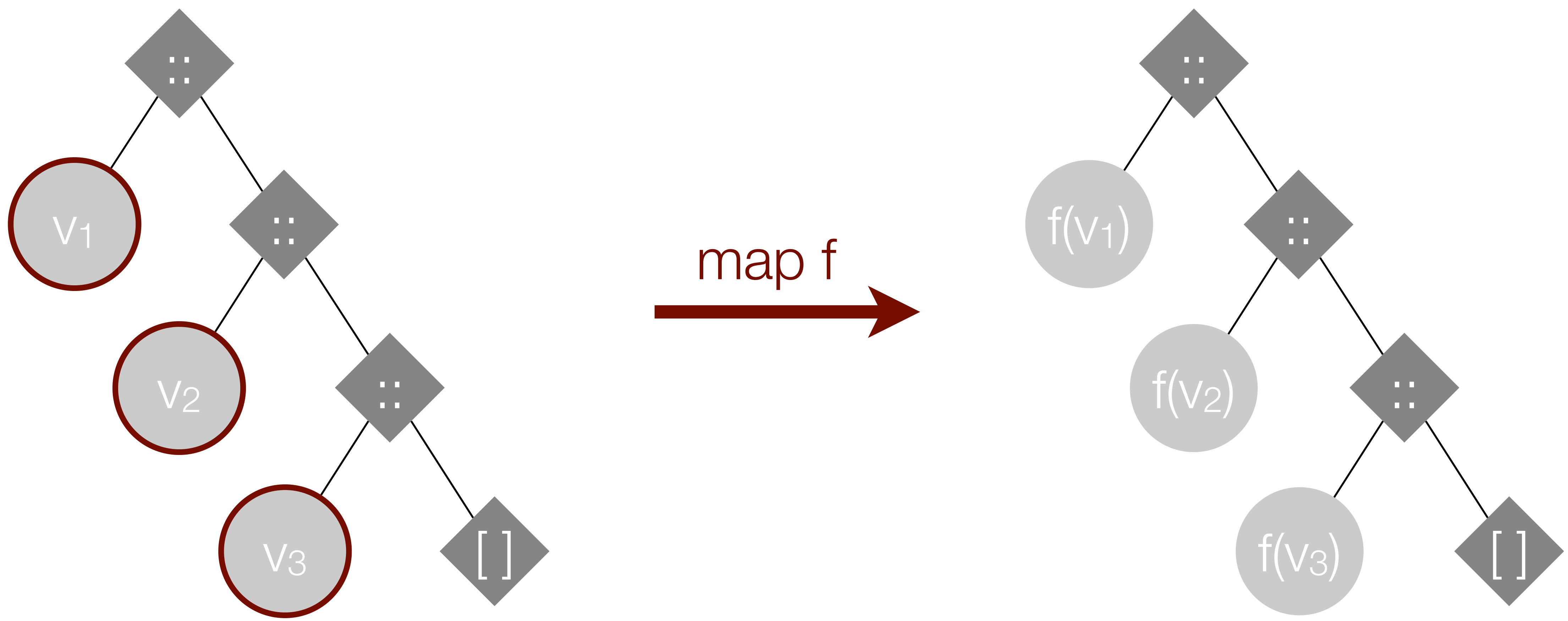

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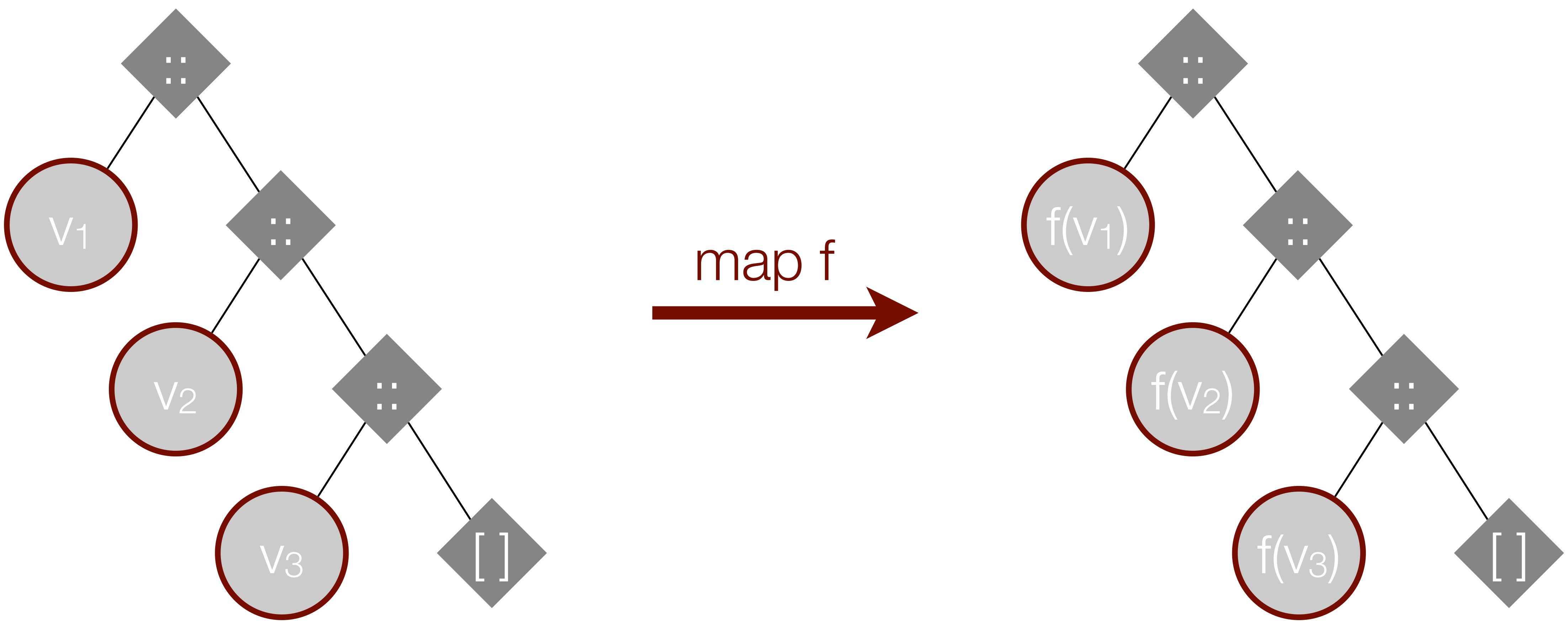

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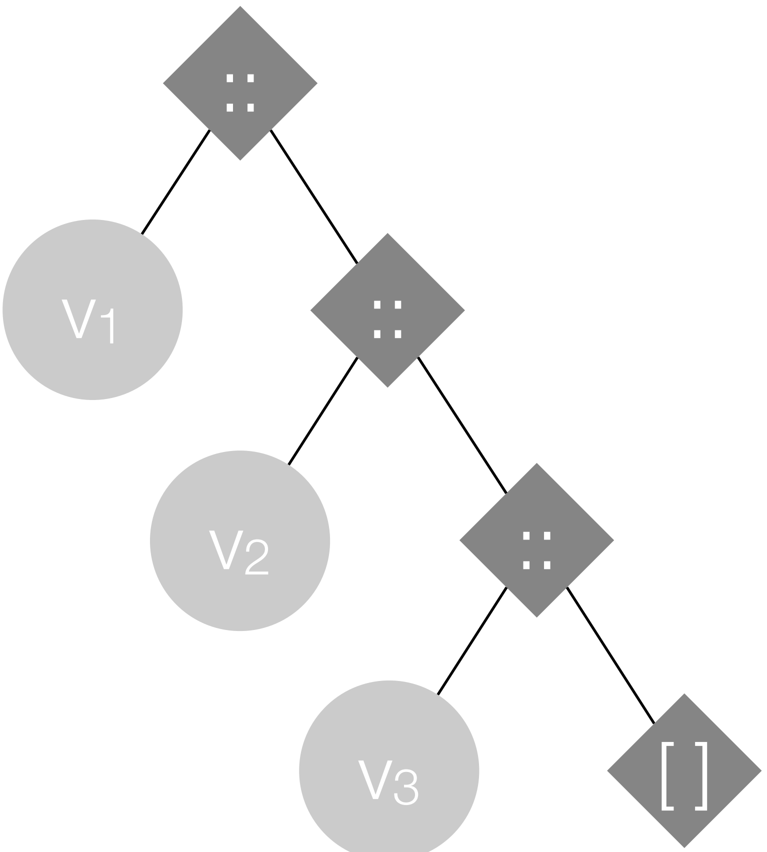

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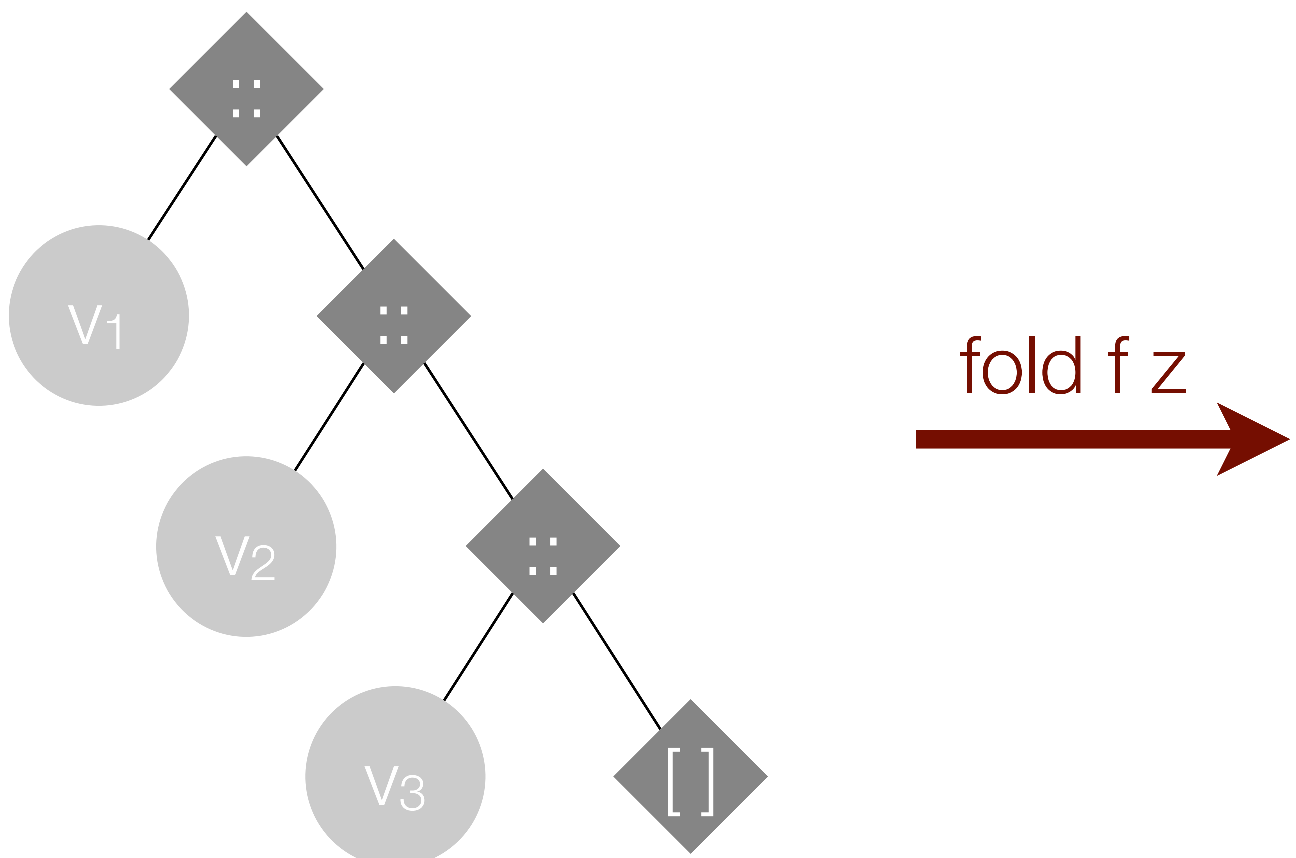

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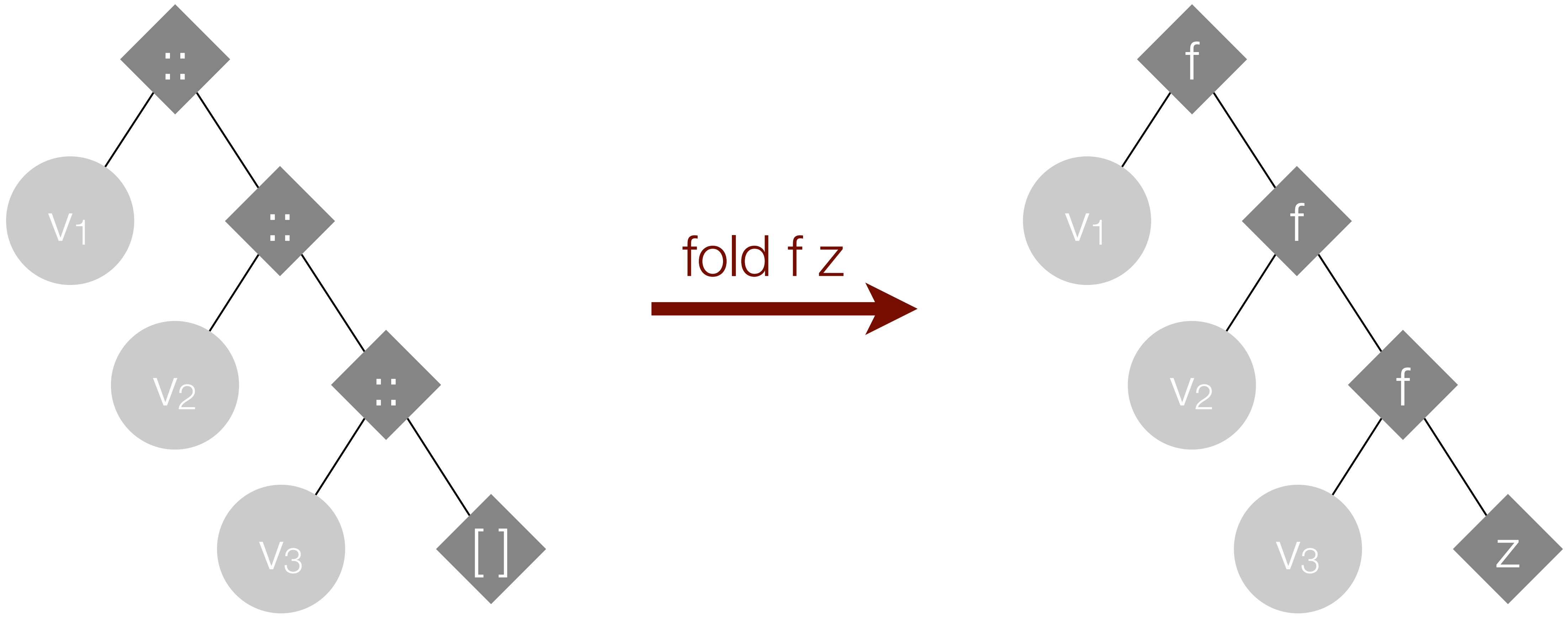

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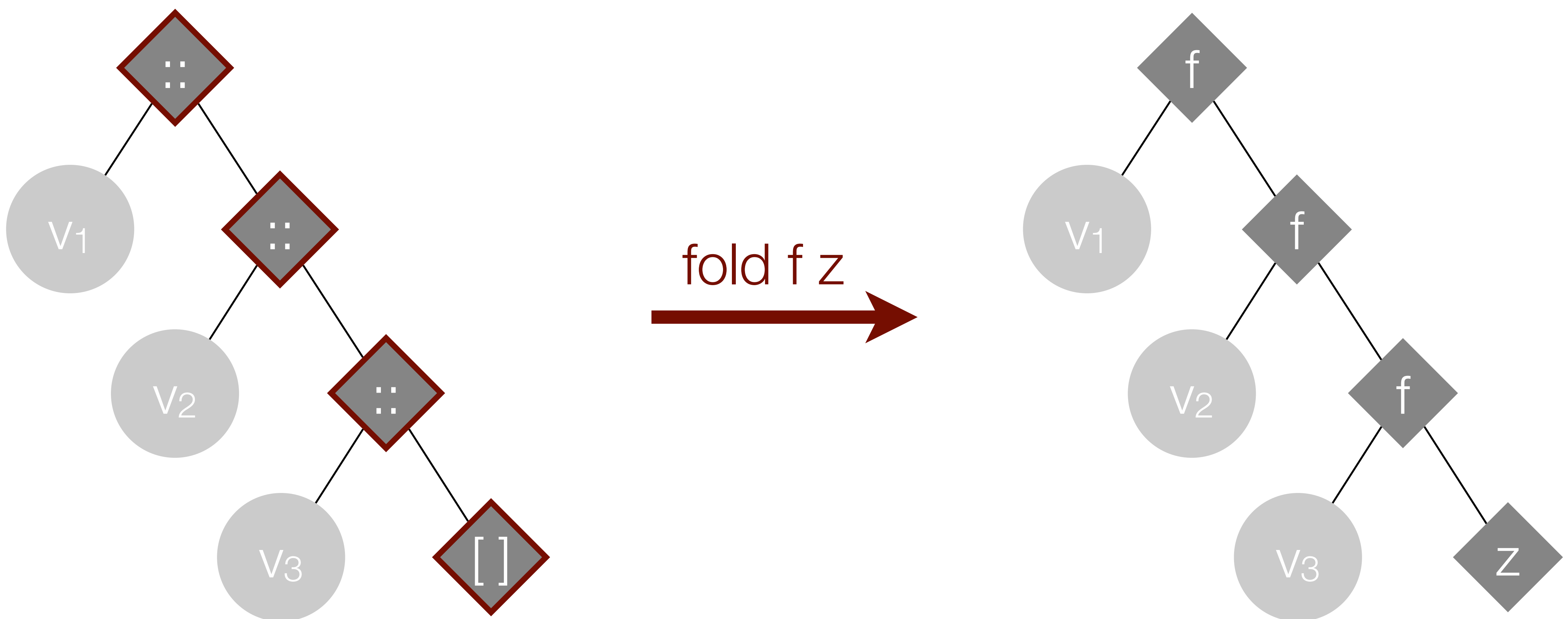

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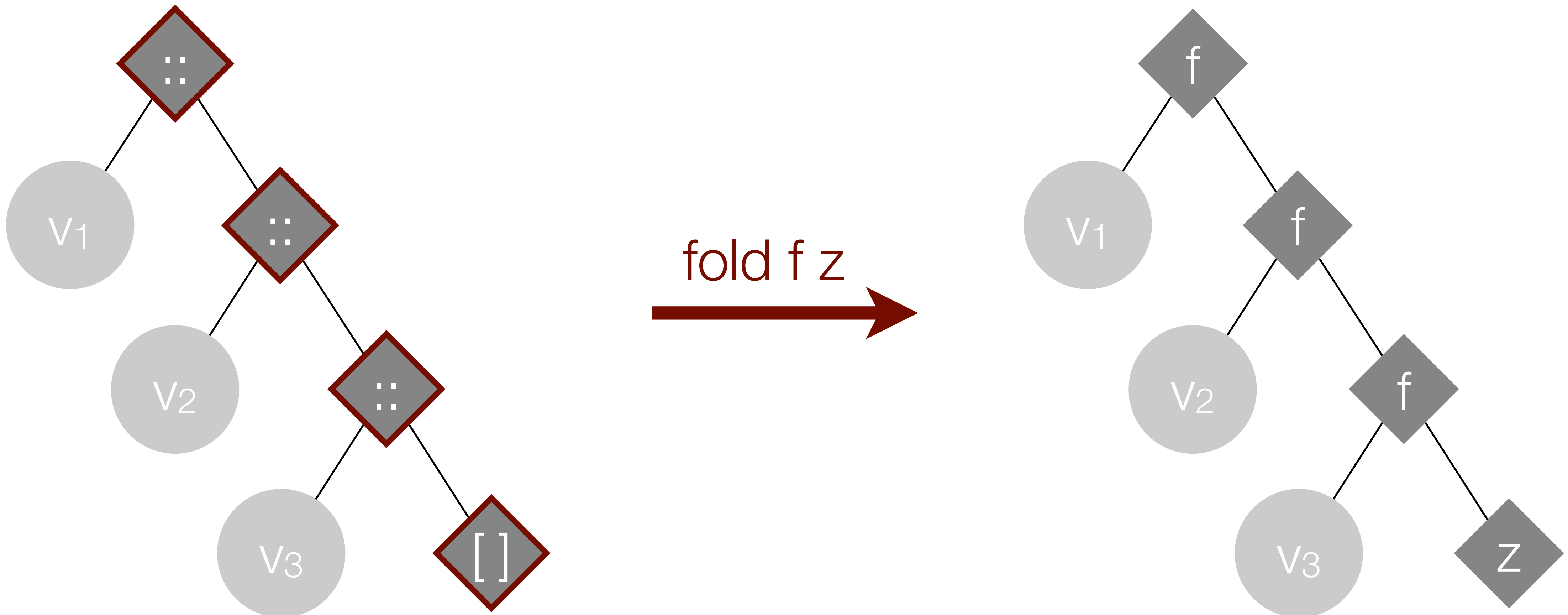

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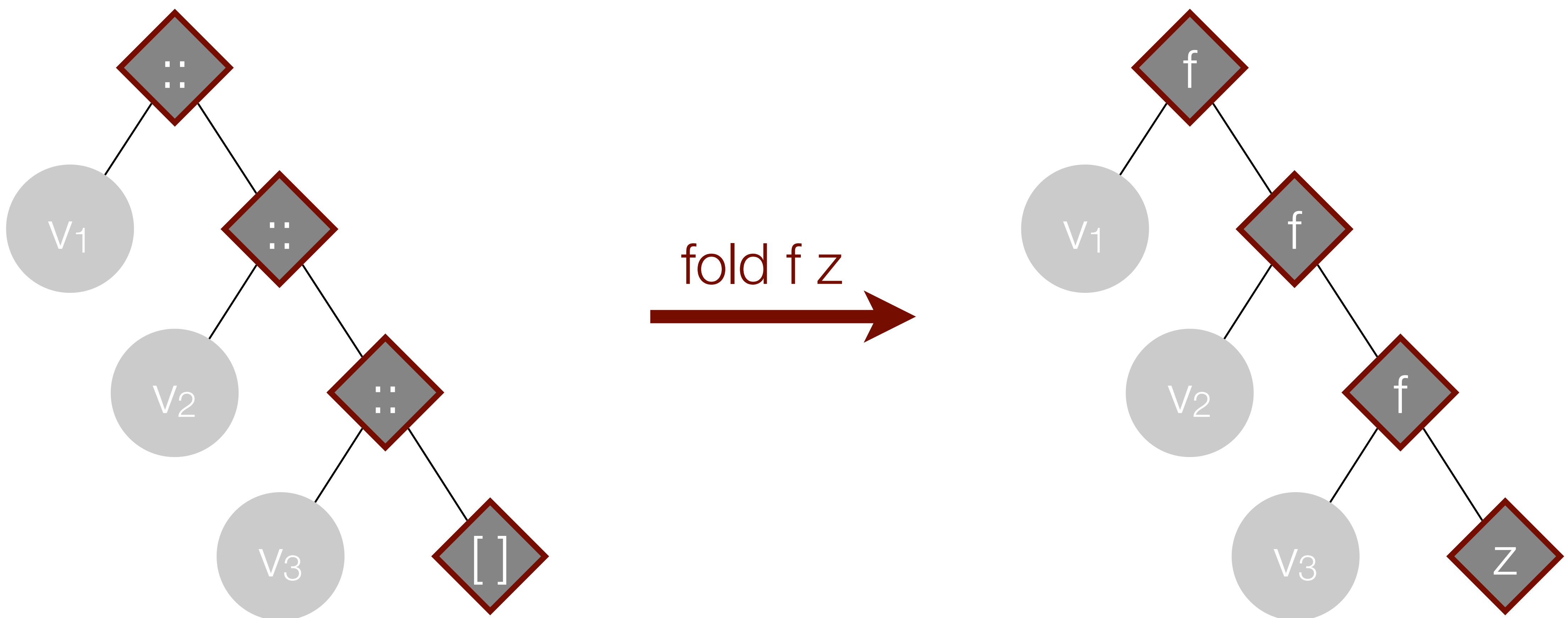

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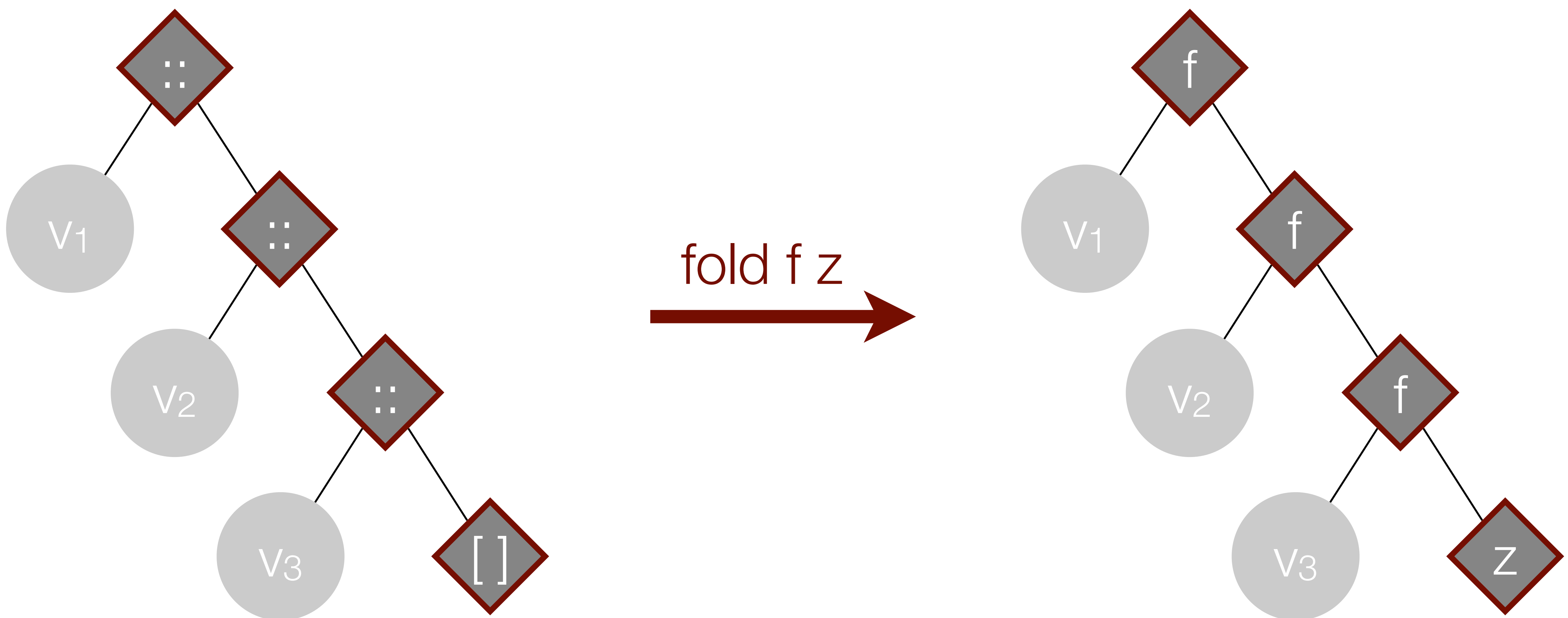

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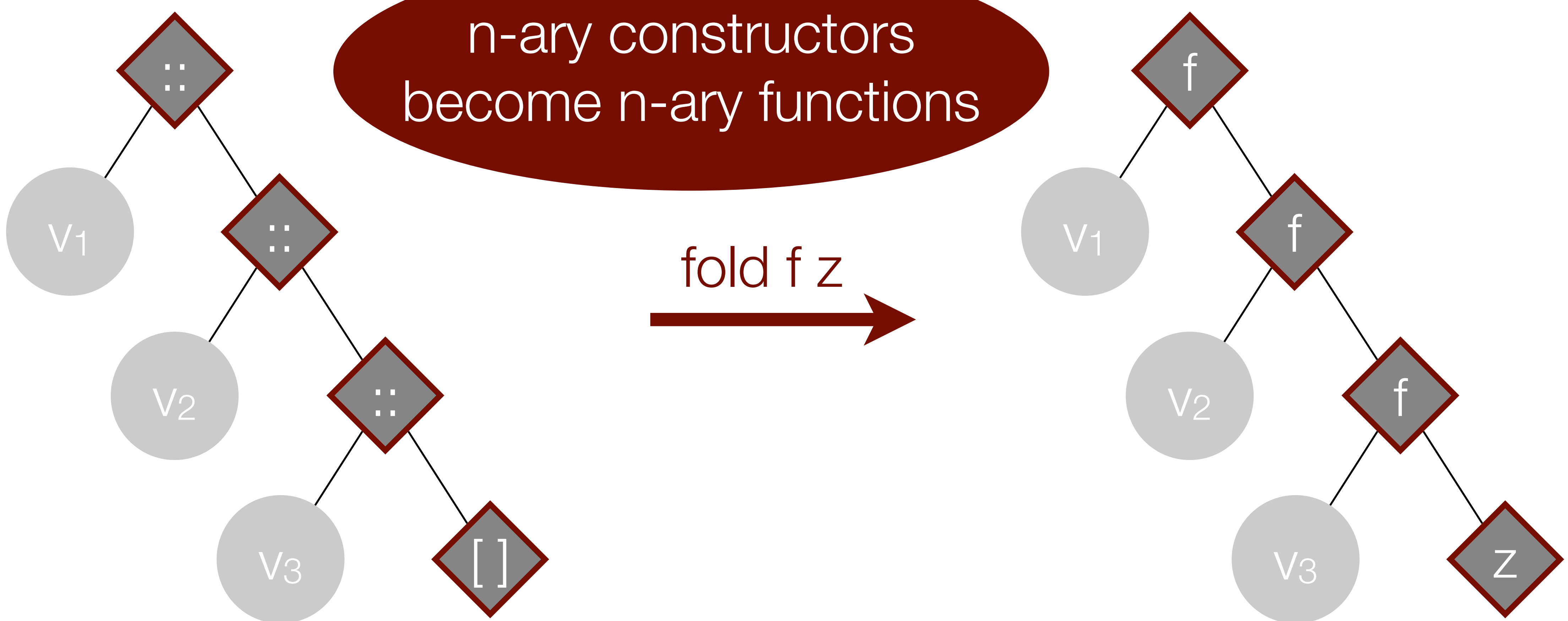
n-ary constructors become n-ary functions



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Map and fold for binary trees

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   | opfold f z (SOME x) =
```

Map and fold for non-recursive datatypes

```
datatype 'a option = NONE | SOME of 'a

(* opmap: ('a -> 'b) -> 'a option -> 'b option *)
fun opmap f NONE = NONE
  | opmap f (SOME x) = SOME (f x)

(* opfold: ('a-> 'b) -> 'b -> 'a option -> 'b *)
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(* opmap: ('a -> 'b) -> 'a option -> 'b option *)

val ostringify = opmap Int.toString

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What are the types of `otringify` and `osum`?

```
(* ostringify : int option -> string option *)

(* osum : int option -> int *)
```

Another use of HOF: Staging

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- Concern: efficiency (“cost”) of evaluation
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  - to factor out expensive part
  - to specialize inexpensive part for specific argument.
- Improves efficiency when specialized function used many times.

Staging

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Suppose the horrible computation takes 10 months.
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Then each of these expressions takes at least 10 months to evaluate:

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f (5,2)
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Curried version of f:

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No application, and thus no evaluation of body!

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The horrible computation has not yet happened :-(

Staging

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[env

```
[5/x]
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Evaluating `g5(2)`

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$$\left[\boxed{\begin{array}{l}\text{env} \\ \texttt{[5/x]} \\ \texttt{fn y => let val z = hc(x) in z+y end}\end{array}} \Big/ \texttt{g5} \right]$$

Evaluating `g5(2)`

```
==> [5/x, 2/y] let val z = hc(x) in z+y end
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```
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(for some integer n)

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10 months!

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Similarly, `g5(3)` will take 10 months.

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Defining `g` in place of `f` has not yet helped!

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Move is valid because the computation does not depend on **y**.

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Such rearrangement of code — putting it in the "right spot" — we refer to as staging.

Staging

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fun h (x:int) : int -> int =
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This is a lambda, and thus s a value!

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(for some integer n)

10 months!

This is the closure returned by `h(5)`.

Staging

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Similarly, `h5(3)` will be very quick.

Staging

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Similarly, `h5(3)` will be very quick.

Factoring `hc(x)` out of the inner lambda has improved efficiency!

Staging

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`val g5 = g(5)`	fast
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Staging

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`val g5 = g(5)`	fast
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`g5 (3)`	> 10 months
`val h5 = h(5)`	> 10 months
`h5 (2)`	fast
`h5 (3)`	fast

More combinators!

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Recall function composition:

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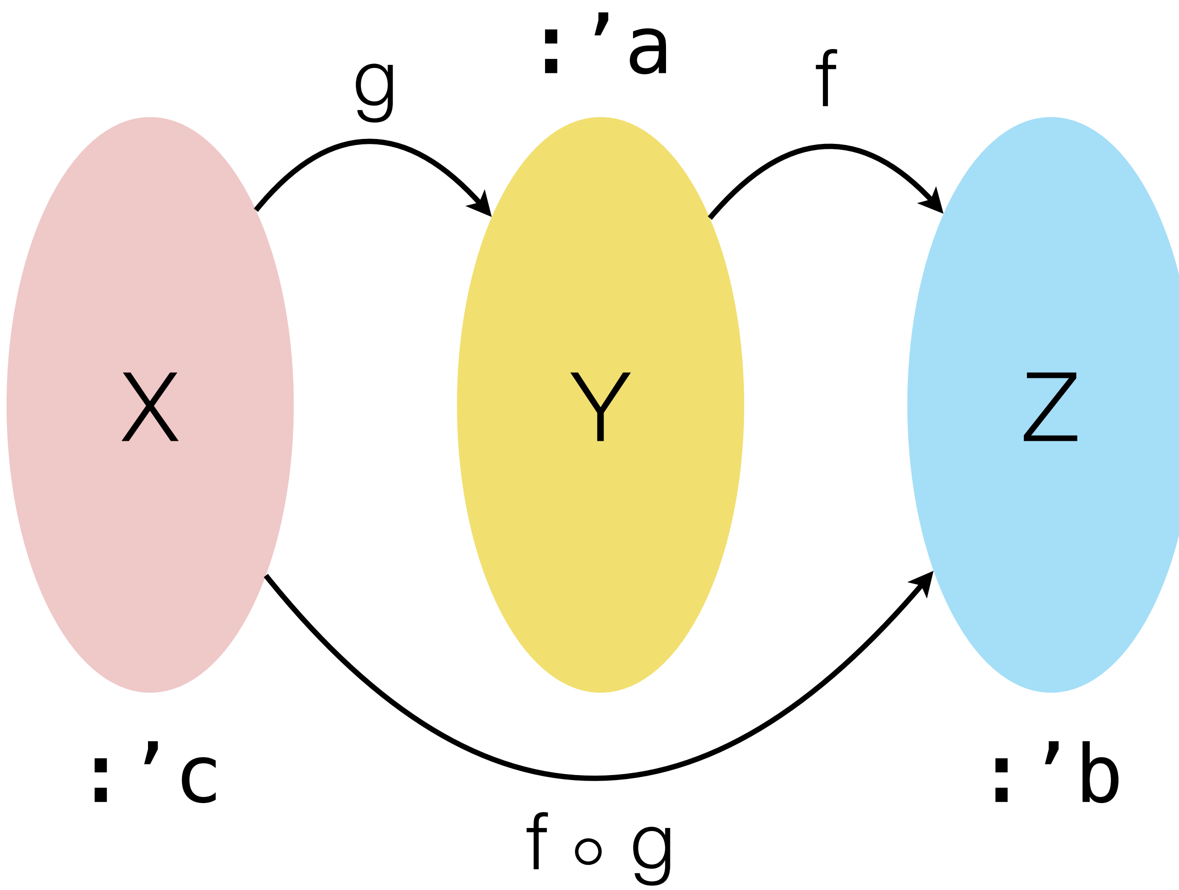

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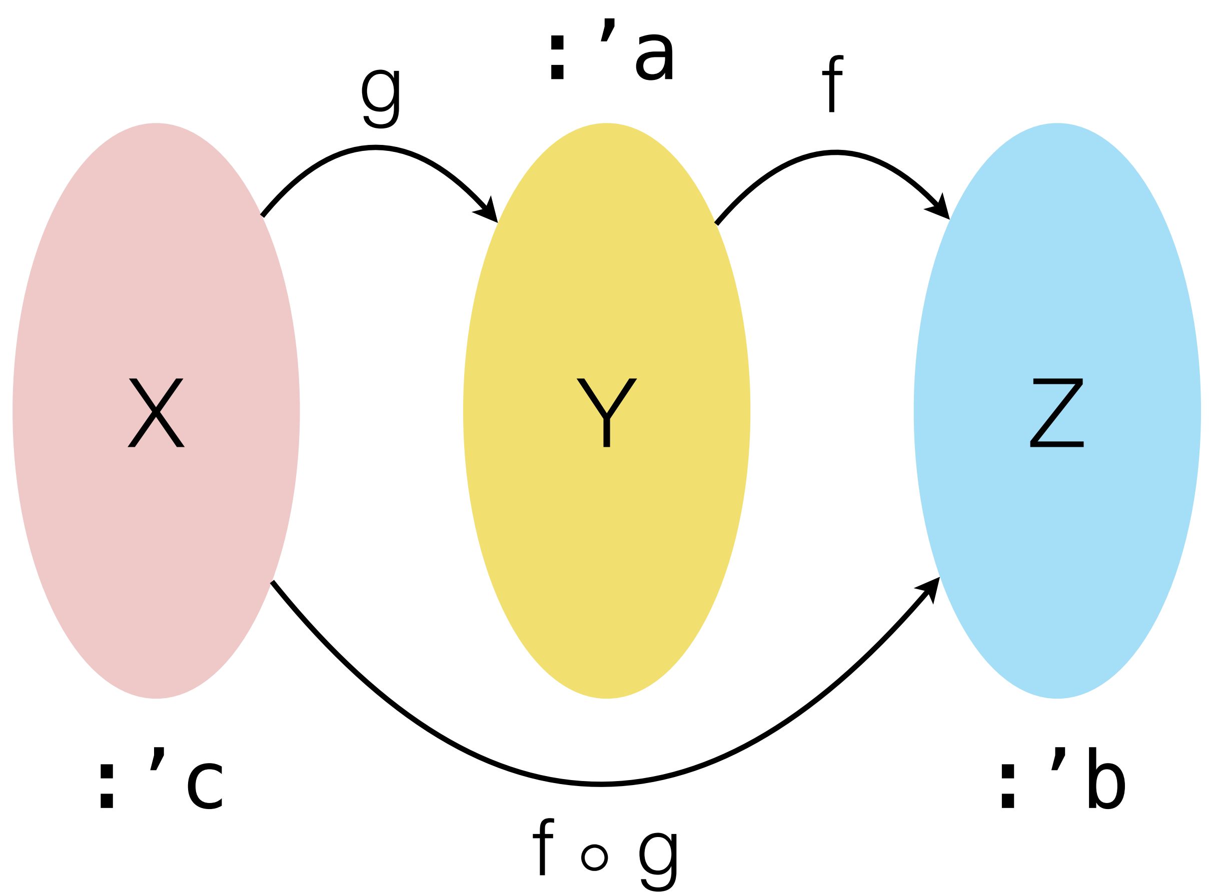

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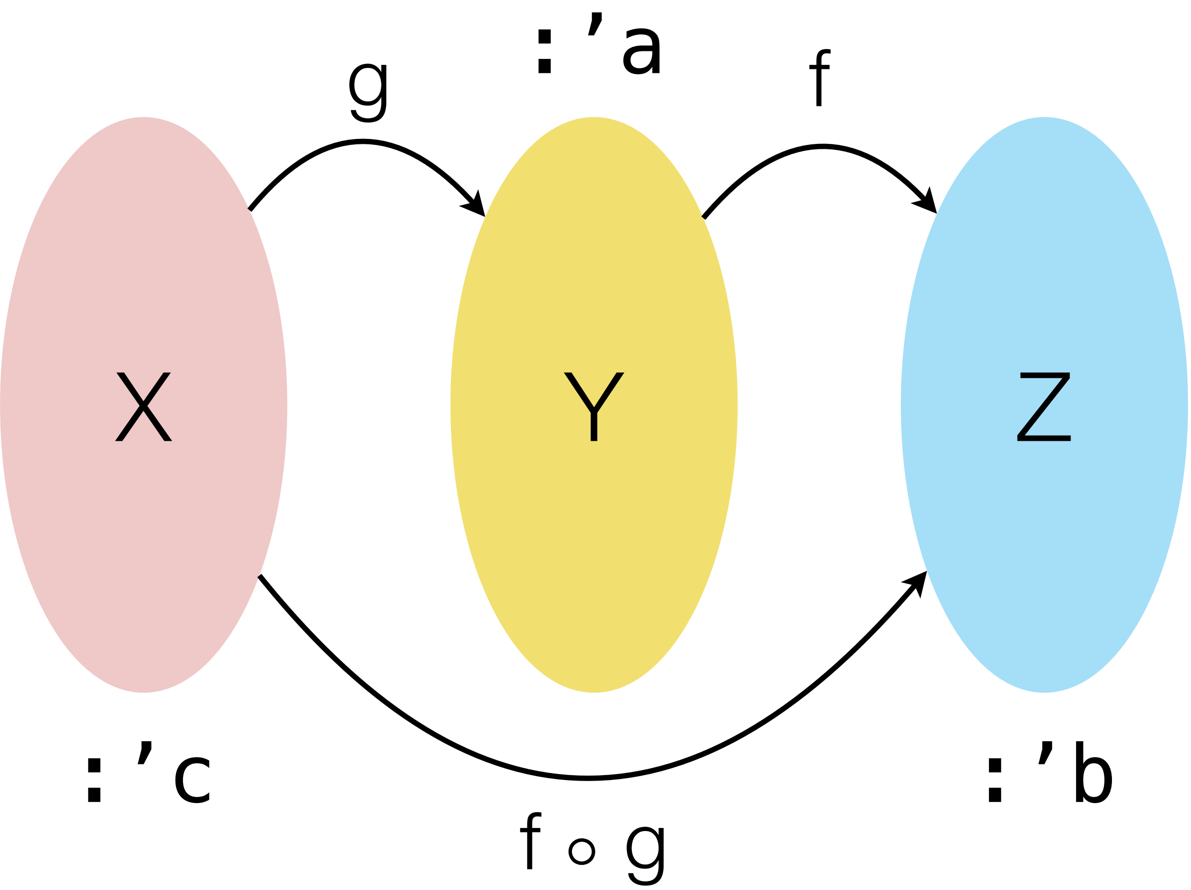

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Combinators are functions that combine small pieces of code into larger pieces of code.

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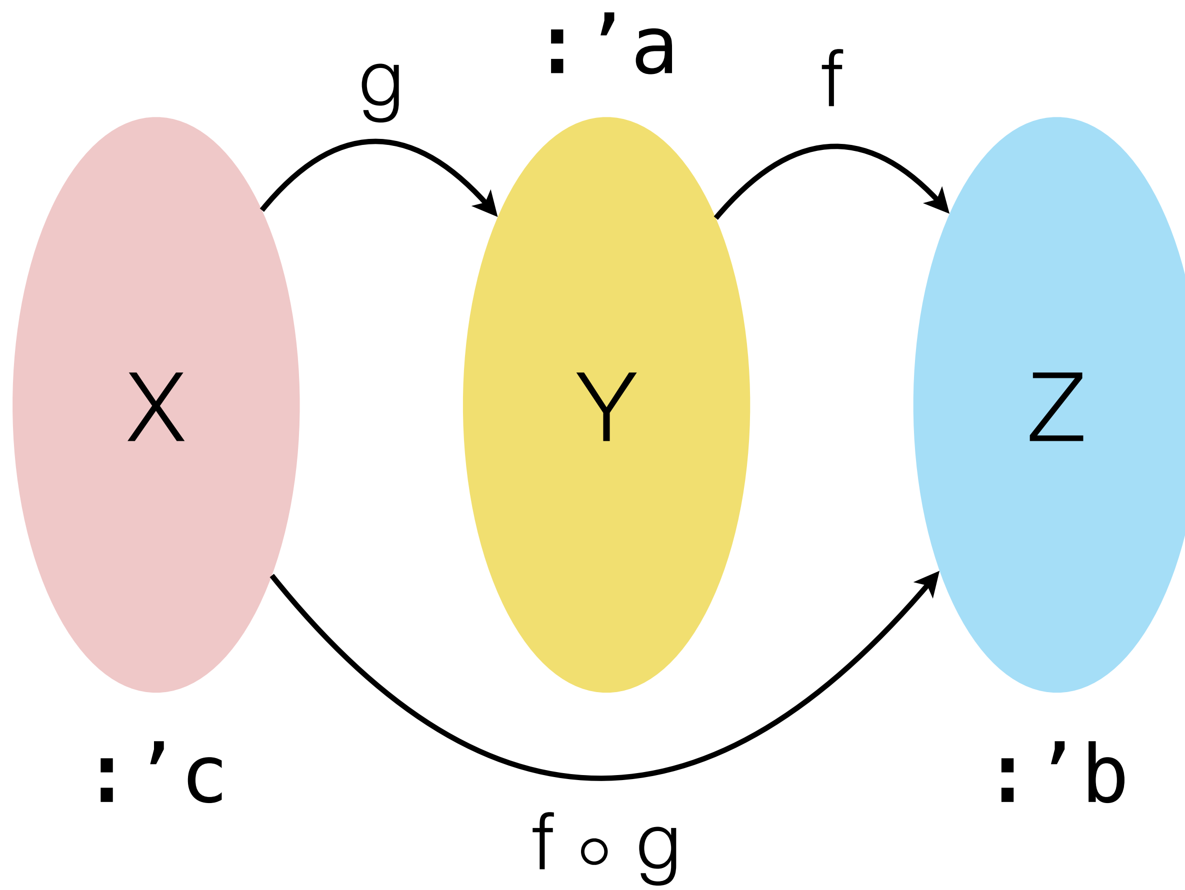

- Combinators are functions that combine small pieces of code into larger pieces of code.
- We will view combinators are higher-order functions that expect functions and return functions.

More combinators!

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An abstract view of combinators:

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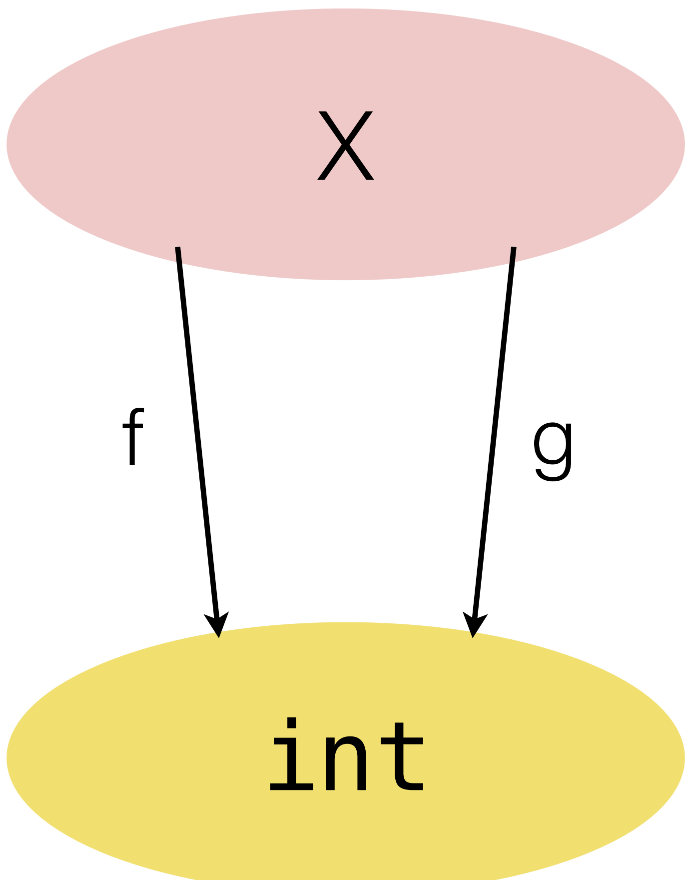

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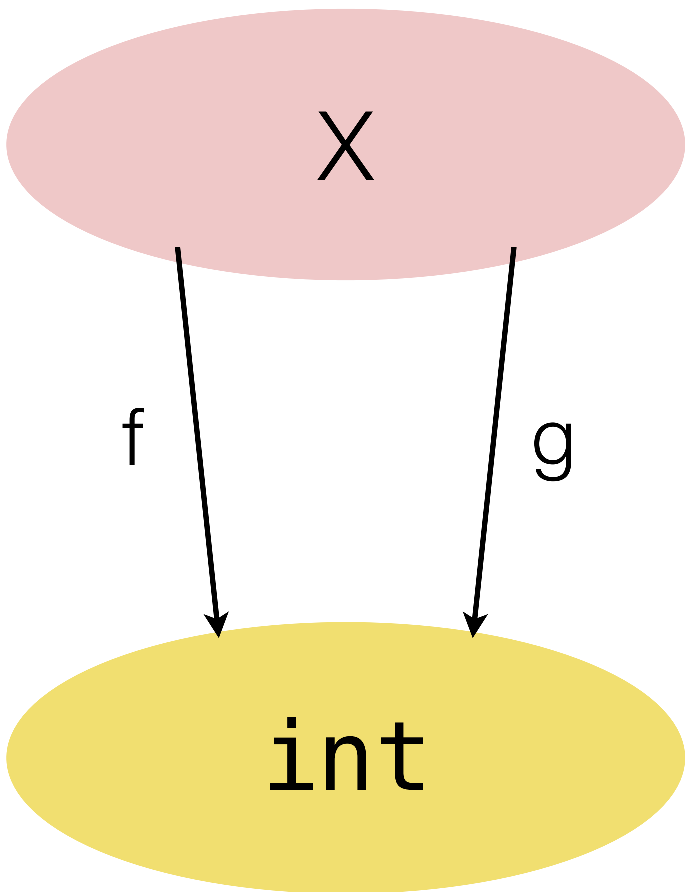
An abstract view of combinators:

Space (set):



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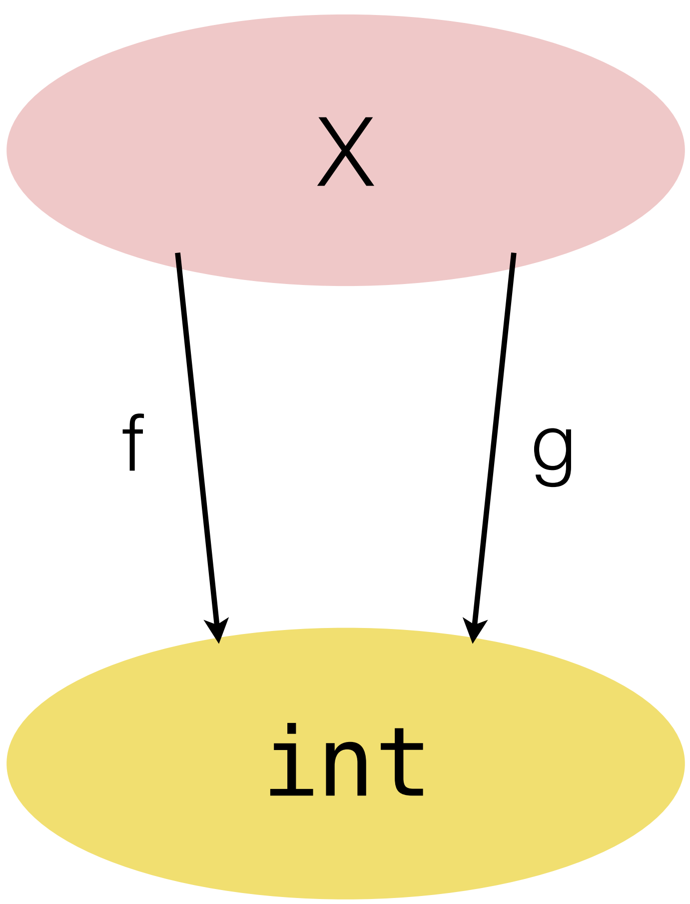

Integers

More combinators!

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Integer functions

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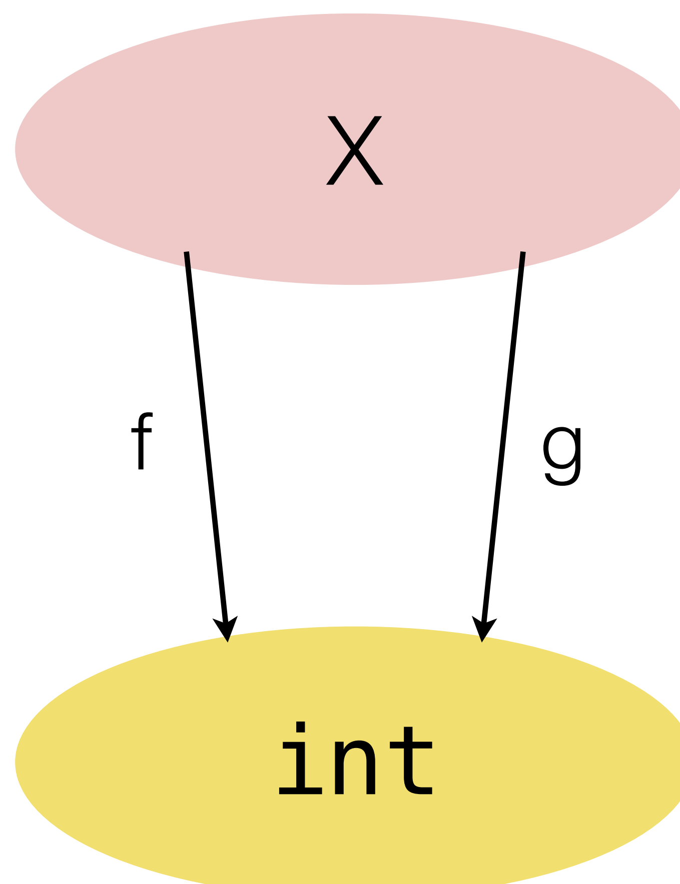


More combinators!

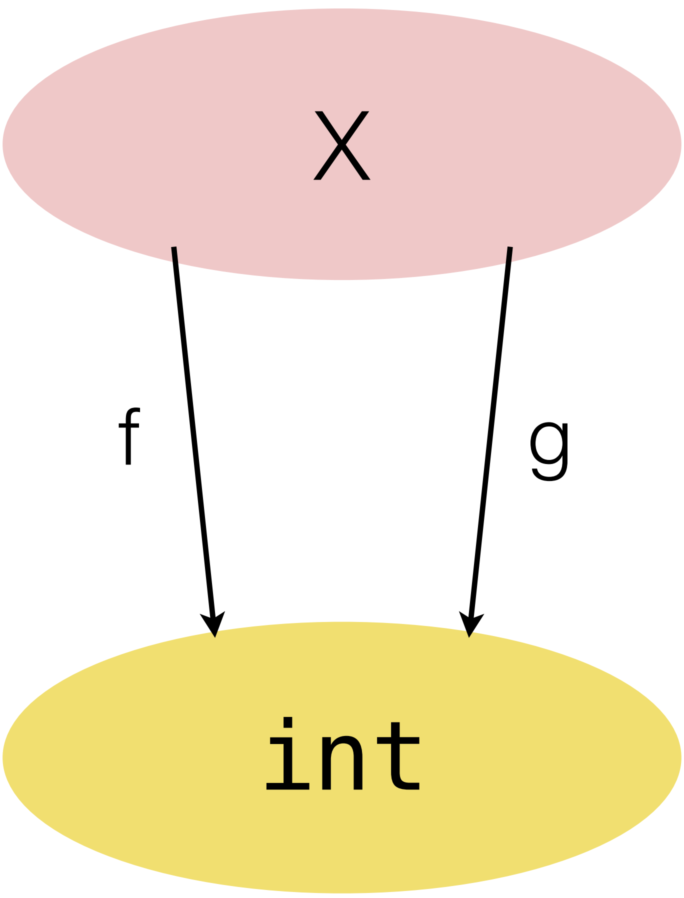
An abstract view of combinators:

Space (set):

Operations on elements:

Integer functions

Integers



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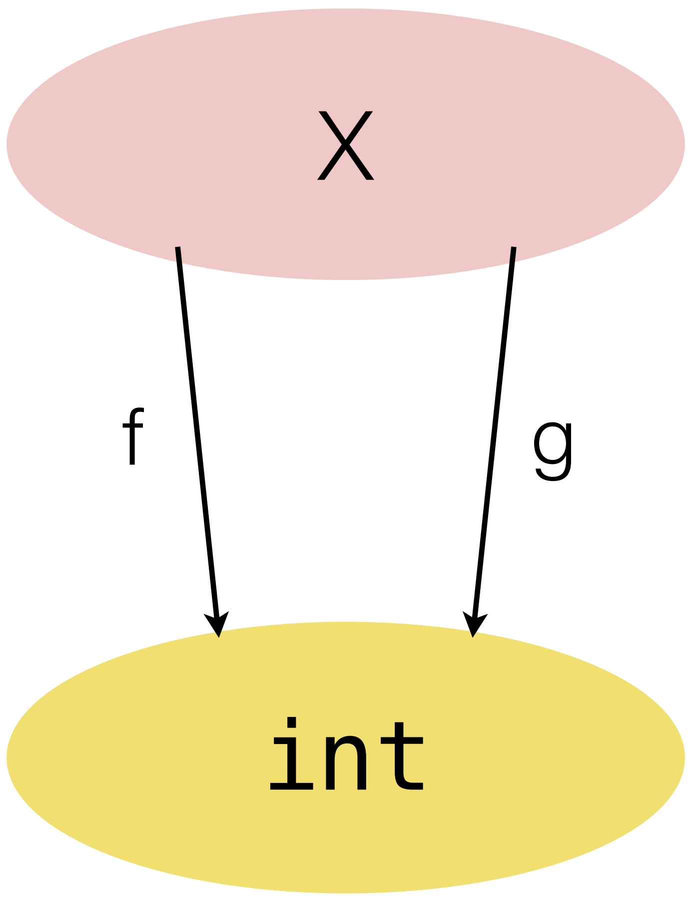

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Operations on integers:
`+`, `*`, `Int.min`, …

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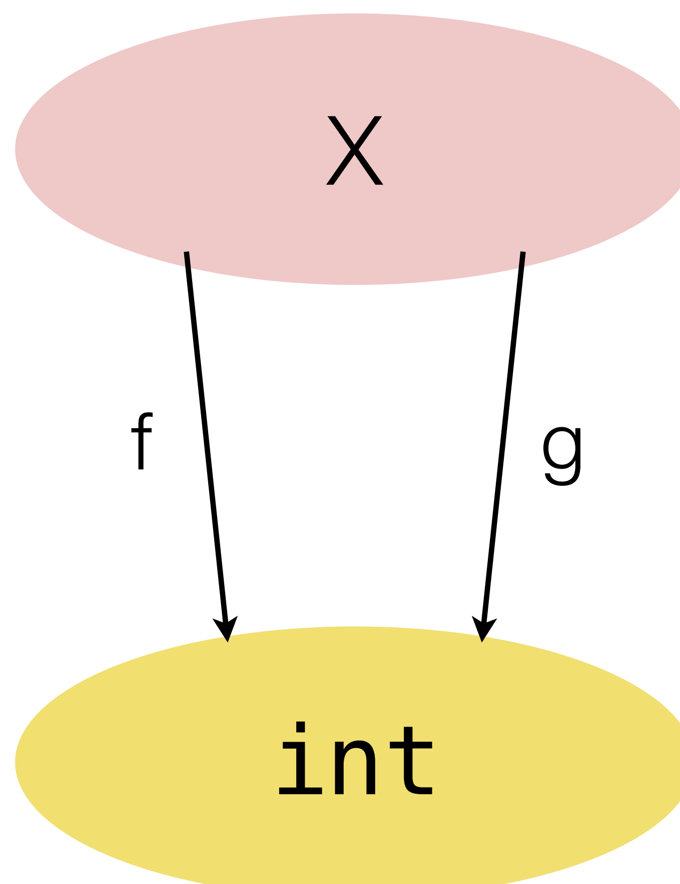

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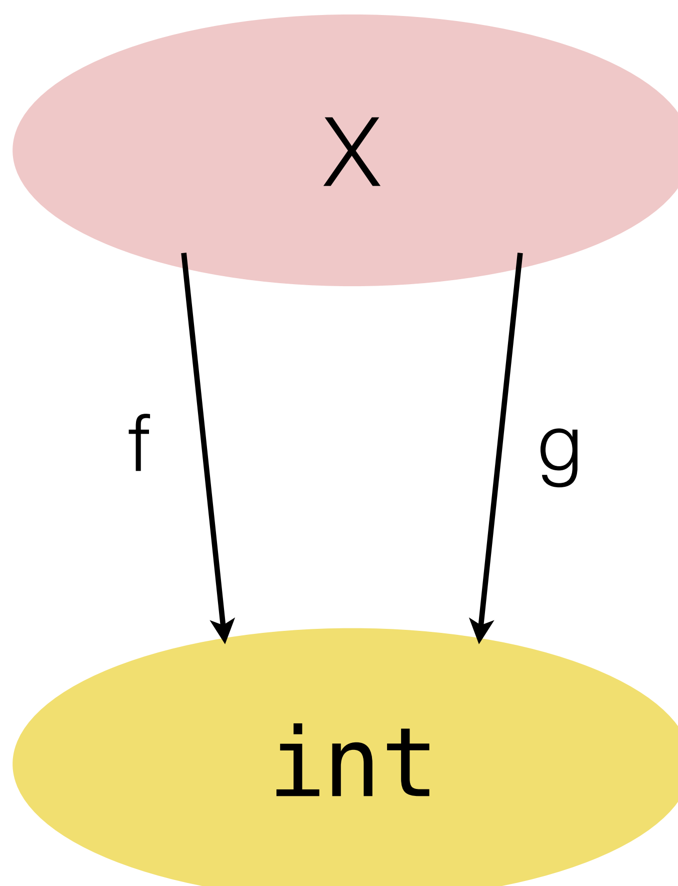

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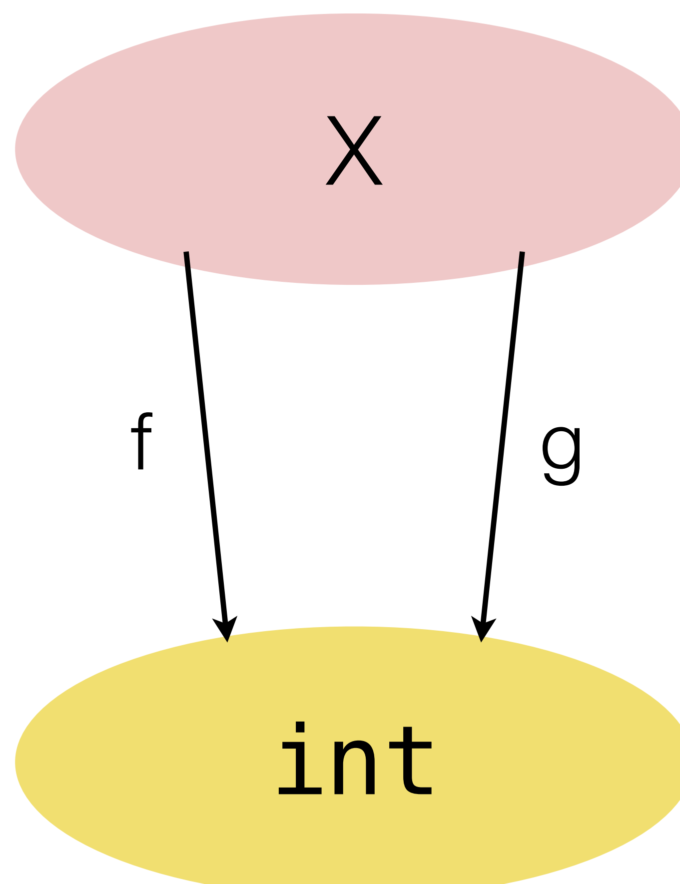

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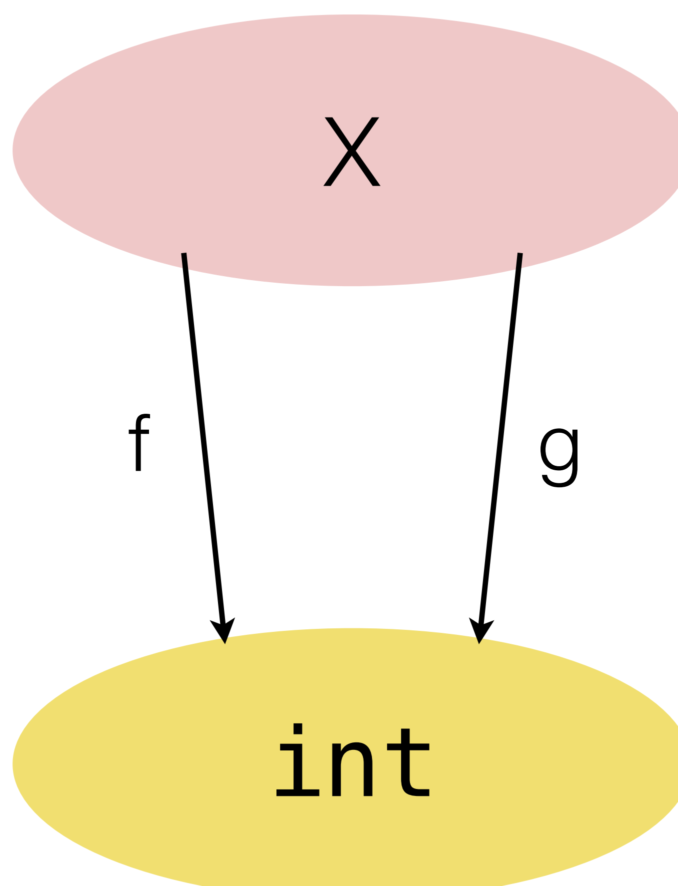

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➡ **Combinators facilitate point-free programming.**

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In math, one may write the sum of two integer-valued functions in a **point-free** way:

$$f + g.$$

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does not involve function arguments

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In SML, we define combinators using point-specific equations and use them for point-free programming.

Examples of combinators

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Addition of functions:

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(* (op ++) : ('a -> int) * ('a -> int) -> 'a -> int *)
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quadratic (3) $\hookrightarrow$ 15

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See lecture notes for more examples!