Learning Objectives

- To practice computing HMM probabilities
- To run Predict and Observe steps in the forward algorithm

Q1. Hidden Markov Models

Below is the Weather HMM from class. We want to predict the weather given both a distribution for weather today given weather yesterday (transition model) and weather today given people using umbrella today (sensor model). We want to compute the weather on day 4 $P(X_4|e_1, e_2, e_3, e_4)$ assuming that we've witnessed umbrellas each day (all the evidence is True that there is an umbrella). Note that the states are the weather $X_i = W_i$ and the evidence is the umbrella $e_i = U_i$.

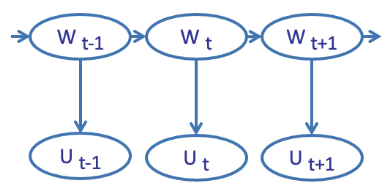
An HMM is defined by:

- Initial distribution: $P(X_1)$
- Transition model: $P(X_t \mid X_{t-1})$
- Sensor model: $P(E_t \mid X_t)$

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Given $P(X_1)$ = {sun:0.5, rain:0.5}
Compute $P(X_4\text{=sun} \mid e_4\text{=} e_3\text{=} e_2\text{=} e_1\text{=True})$



(a) OBSERVE! We are given the initial distribution $P(X_1)$. Now we need to Observe the evidence $e_1 = True$ and compute $P(X_1|e_1)$. We can derive the equation for $P(X_1|e_1)$ directly using Bayes Rule with the probabilities $P(e_1|X_1)$ and $P(X_1)$ or by computing the joint $P(X_1, e_1)$ and normalizing $Z = P(e_1)$. Write the equation below and then compute the probability table $P(X_1|e_1)$.

$P(X_1|e_1) =$
$P(X_1, e_1)/P(e_1) = P(e_1|X_1)P(X_1)/\sum_x P(e_1|x)P(x)$

Table $P(X_1|e_1)$

sun	.1/(.1+.45) = .18
rain	.45/(.1+.45) = .82

(b) PREDICT! Now we have $P(X_1|e_1)$, and we want to Predict $P(X_2|e_1)$. We can do this by summing over X_1: $P(X_2|e_1) = \sum_{x \in X_1} P(X_2|x)P(x|e_1)$. Rewrite this equation below and then compute the probability table using your answer above and the HMM model tables.

$P(X_2|e_1) =$
$\sum_{x \in X_1} P(X_2|x)P(x|e_1)$

Table $P(X_2|e_1)$

sun	.9*.18 + .3*.82 = .41
rain	.1*.18 + .7*.82= .59

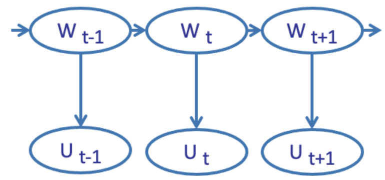
An HMM is defined by:

- Initial distribution: $P(X_1)$
- Transition model: $P(X_t \mid X_{t-1})$
- Sensor model: $P(E_t \mid X_t)$

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Given $P(X_1) = \{sun{:}0.5, rain{:}0.5\}$
Compute $P(X_4{=}sun \mid e_4{=}\ e_3{=}\ e_2{=}\ e_1{=}True)$



(c) OBSERVE! Now that we've predicted X_2, we can update the probability given new evidence $e_2 = True$. Use the Observation update to write the formula for $P(X_2|e_1, e_2)$ using $P(X_2|e_1)$ above and then solve.

$P(X_2|e_1, e_2) =$
$\alpha P(X_2, e_2|e_1) = P(e_2|X_2)P(X_2|e_1)/\sum_{x\in X_2} P(e_2|x)P(x|e_1)$

Table $P(X_2|e_1, e_2)$

sun	$.2 * .41/.613 = .13$
rain	$.9 * .59/.613 = .87$

(d) PREDICT! Compute $P(X_3|e_1, e_2)$ using the transition probabilities and $P(X_2|e_1, e_2)$ above. Write this equation below and then compute the probability table.

$P(X_3|e_1, e_2) =$
$\sum_{x\in X_2} P(X_3|x)P(x|e_1, e_2)$

Table $P(X_3|e_1, e_2)$

sun	$.9 * .13 + .3 * .87 = .38$
rain	$.1 * .13 + .7 * .87 = .62$

(e) OBSERVE! Now that we've predicted X_3, we can update the probability given new evidence $e_3 = True$. Use the Observation update to write the formula for $P(X_3|e_1, e_2, e_3)$ using $P(X_3|e_1, e_2)$ above and then solve.

$P(X_3|e_1, e_2, e_3) =$
$\alpha P(X_3, e_3|e_1, e_2) = \alpha P(e_3|X_3)P(X_3|e_1, e_2)$
$\alpha = 1/\sum_{x\in X_3} P(e_3|x)P(x|e_1, e_2)$

Table $P(X_3|e_1, e_2, e_3)$

sun	$.2 * .38/.634 = .12$
rain	$.9 * .62/.634 = .88$

(f) PREDICT! Compute $P(X_4|e_1, e_2, e_3)$ using the transition probabilities and $P(X_3|e_1, e_2, e_3)$ above. Write the equation below and then compute the probability table.

$P(X_4|e_1, e_2, e_3) =$
$\sum_{x\in X_3} P(X_4|x)P(x|e_1, e_2, e_3)$

Table $P(X_4|e_1, e_2, e_3)$

sun	$.9 * .12 + .3 * .88 = .37$
rain	$.1 * .12 + .7 * .88 = .63$

(g) OBSERVE! Finally, we can update the probability of X_4 given new evidence $e_4 = True$ (and the rest of the evidence). Use the Observation update rule to write the formula for $P(X_4|e_1, e_2, e_3, e_4)$ using $P(X_4|e_1, e_2, e_3)$ above and then solve for the new probability table.

$P(X_4|e_1, e_2, e_3, e_4) =$
$\alpha P(X_4, e_4|e_1, e_2, e_3) = \alpha P(e_4|X_4)P(X_4|e_1, e_2, e_3)$
$\alpha = 1/\sum_{x\in X_4} P(e_4|x)P(x|e_1, e_2, e_3)$

Table $P(X_4|e_1, e_2, e_3, e_4)$

sun	$.2 * .37/.641 = .115$
rain	$.9 * .63/.641 = .885$