Warm-up as You Log In

Given

- Set `actions` (persistent/static)
- Set `states` (persistent/static)
- Function `T(s,a,s_prime)`

Write the pseudo code for:

- `function V(s) return value`

that implements:

$$V(s) = \max_{a \in actions} \sum_{s' \in states} T(s,a,s')V(s')$$

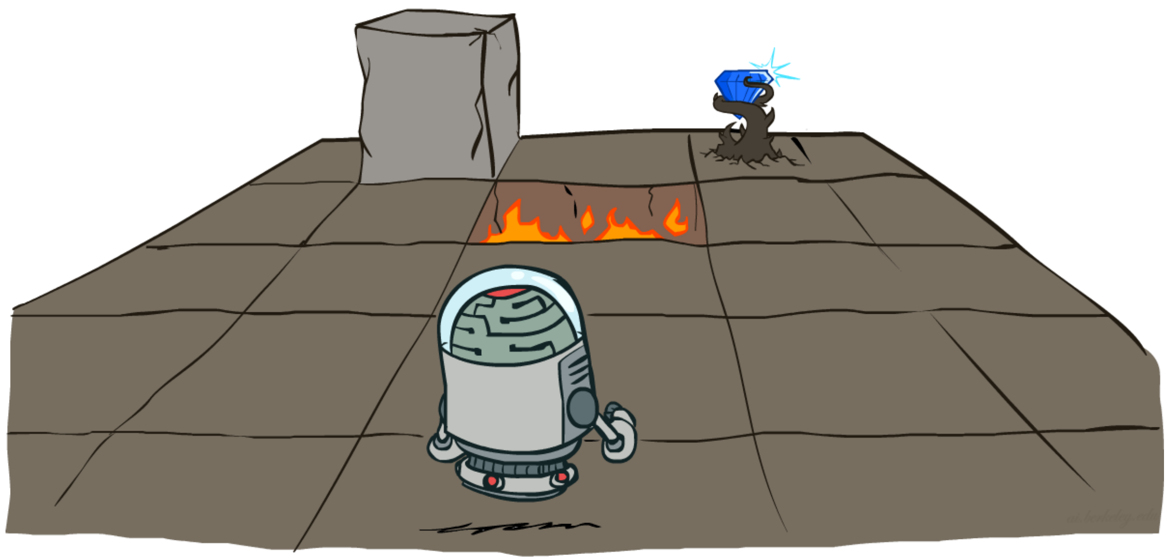
Announcements

Assignments:

- HW5 (written)
  - Due Tonight, 10 pm
- P3 Checkpoint
  - Due Friday 3/3, 10 pm
- HW6 (online)
  - Due Tues 3/14, 10 pm (Tuesday after spring break)
- P3 All
  - Due Friday 3/17, 10 pm
- Midsemester grades submitted by end of Fall break
- Midsemester feedback!

AI: Representation and Problem Solving

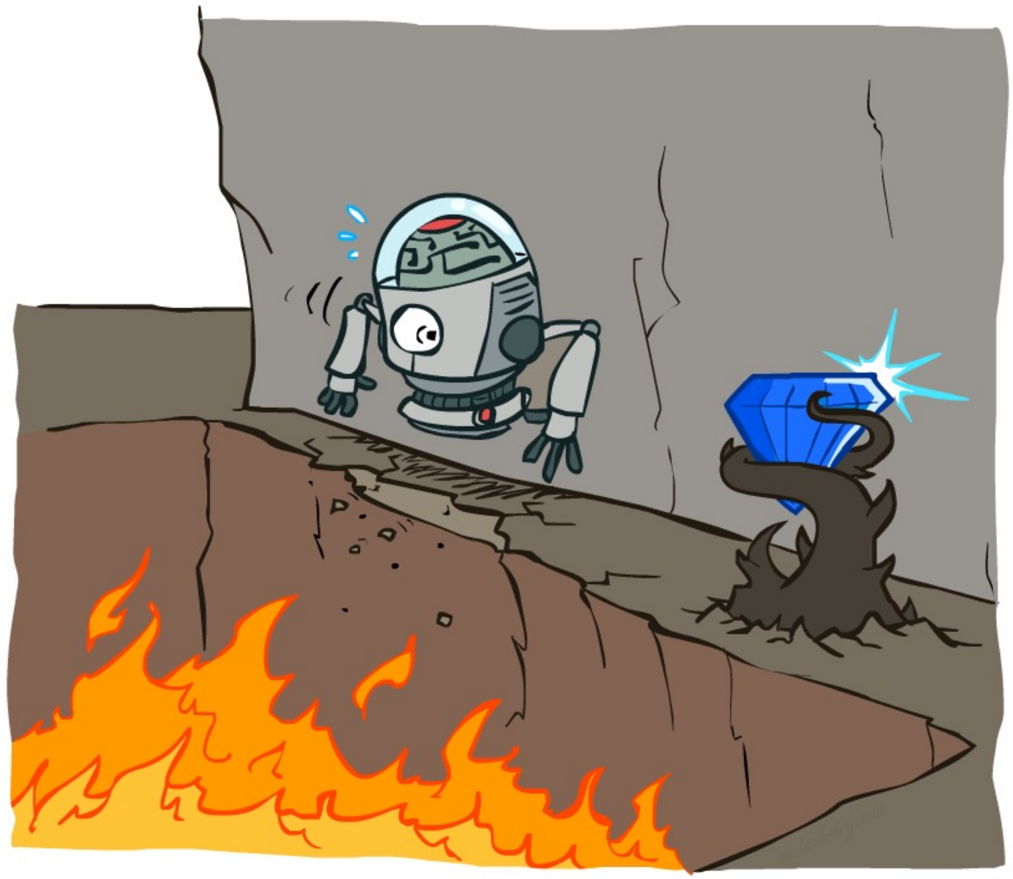
Markov Decision Processes



Instructor: Stephanie Rosenthal

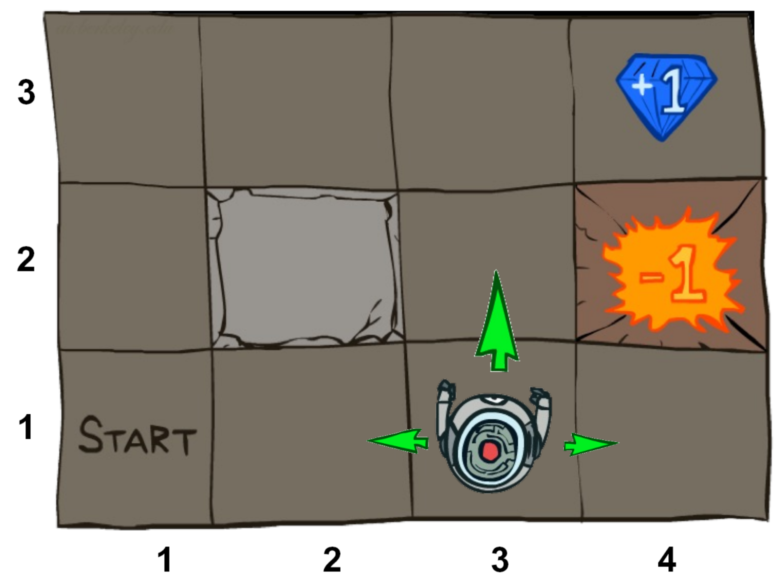
Slide credits: CMU AI and http://ai.berkeley.edu

Non-Deterministic Search



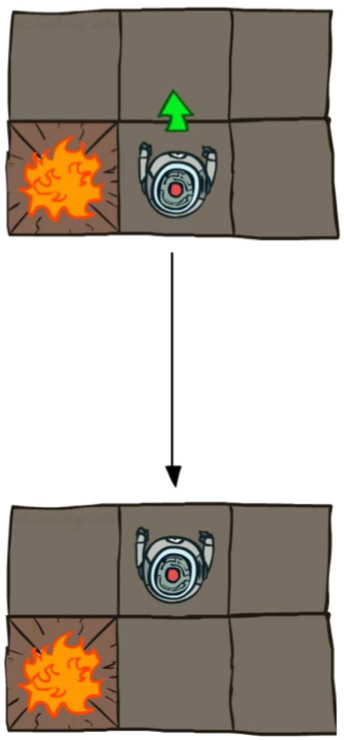
Example: Grid World

- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent's path
- Noisy movement: actions do not always go as planned
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
  - Small "living" reward each step (can be negative)
  - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards

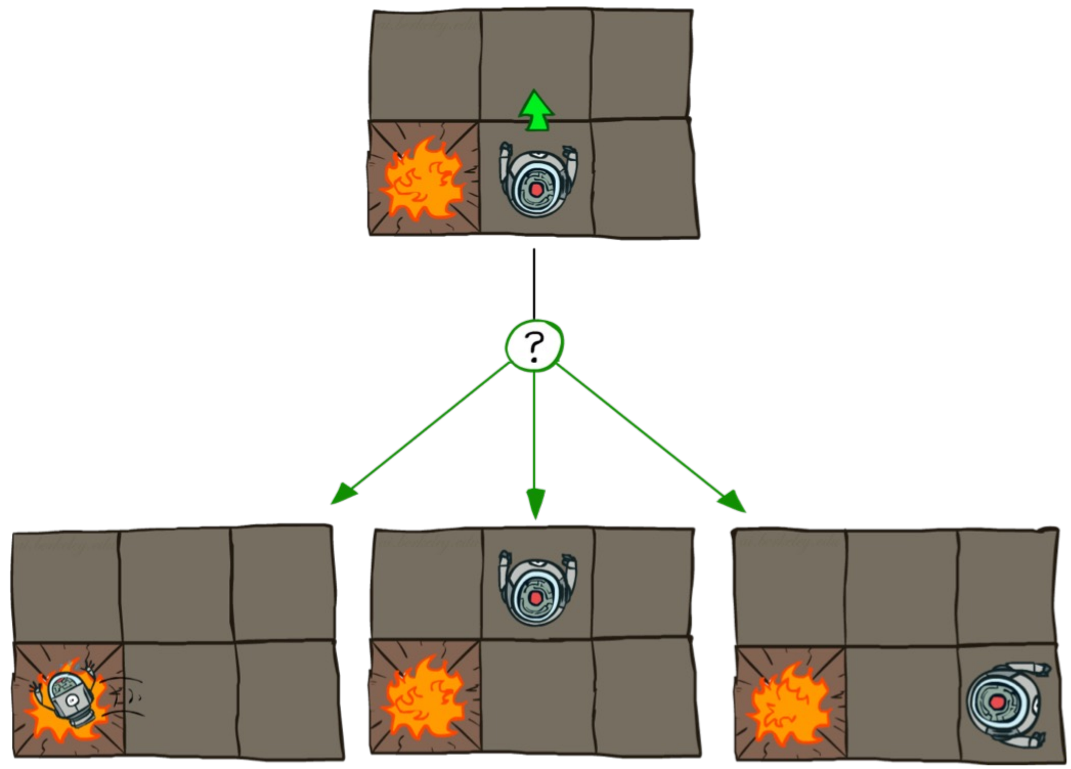


Grid World Actions

Deterministic Grid World



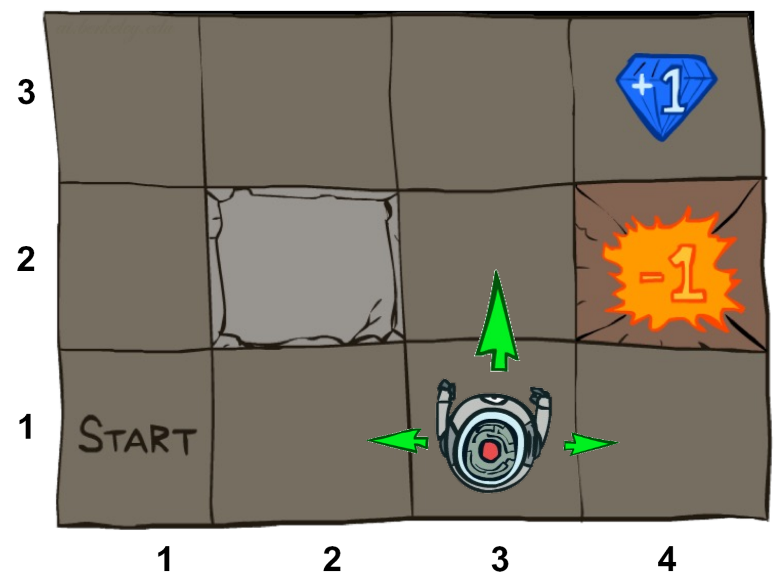
Stochastic Grid World



Markov Decision Processes

An MDP is defined by:

- A set of states $s \in S$
- A set of actions $a \in A$
- A transition function T(s, a, s')
  - Probability that a from s leads to s', i.e., P(s' | s, a)
  - Also called the model or the dynamics
- A reward function R(s, a, s')
  - Sometimes just R(s) or R(a,s')
- Maybe a terminal state



What is Markov about MDPs?

"Markov" generally means that given the present state, the future and the past are independent

For Markov decision processes, "Markov" means action outcomes depend only on the current state

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \dots S_0 = s_0)$$

$$=$$

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

Andrey Markov
(1856-1922)

This is just like search, where the successor function could only depend on the current state (not the history)

Markov Decision Processes

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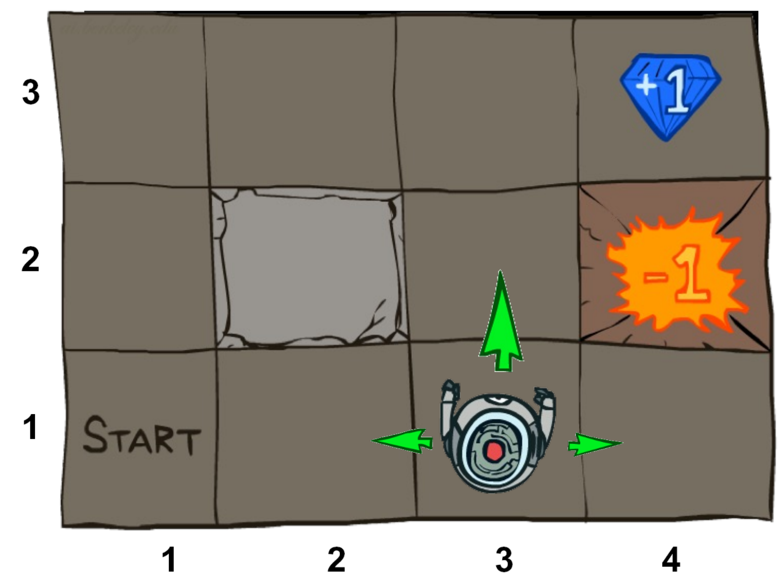
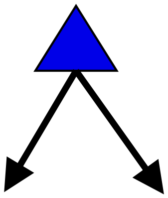
  - Sometimes just $R(s)$ or $R(s')$
- Maybe a terminal state



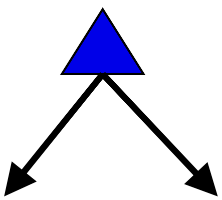
MDPs are non-deterministic search problems

- One way to solve them is with expectimax search
- We'll have a new tool soon

Minimax Notation



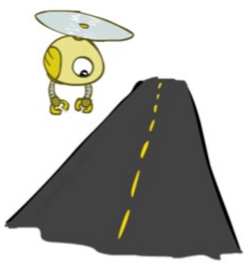
$$V(s) = \max_a V(s'),$$
where $s' = result(s, a)$

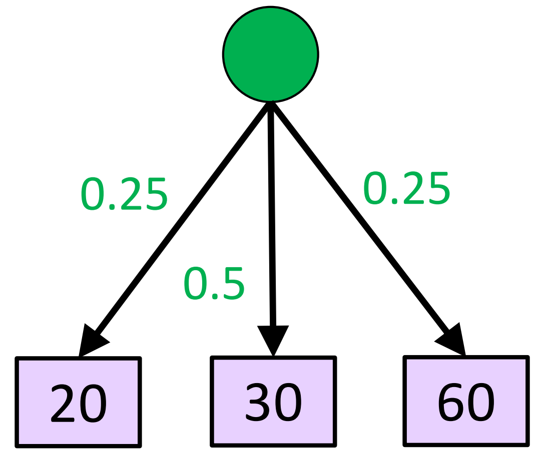


$$\hat{a} = \underset{a}{\operatorname{argmax}}\, V(s'),$$
where $s' = result(s, a)$

Expectations

Time:	20 min		30 min		60 min
	x	+	x	+	x
Probability:	0.25		0.50		0.25





Max node notation

$$V(s) = \max_{a} V(s'),$$

where $s' = result(s, a)$

Chance node notation

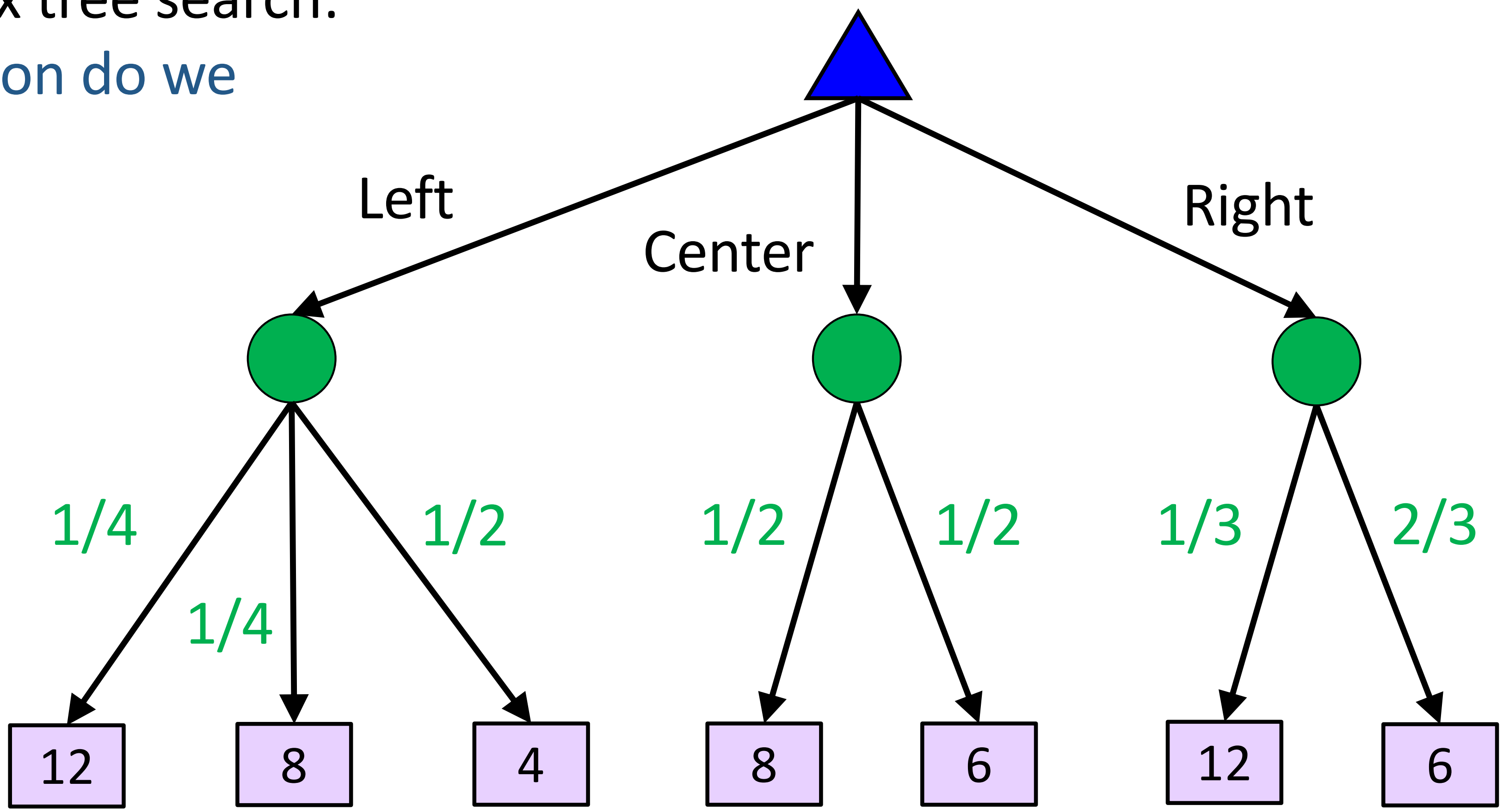
$$V(s) = \sum_{s'} [P(s')\, V(s')]$$

Question from Lecture 4

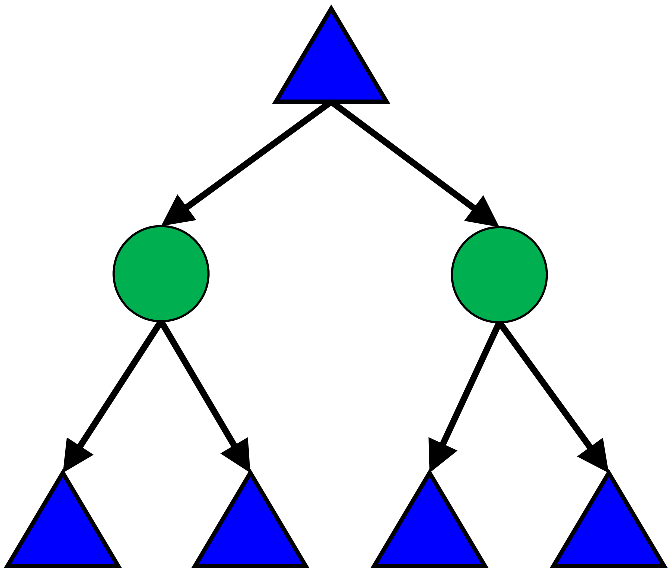
Expectimax tree search:

Which action do we choose?

A) Left
B) Center
C) Right
D) Eight



Expectimax Notation



$$V(s) = \max_a \sum_{s'} [P(s'|s, a)\, V(s')]$$

MDP Notation

Standard expectimax: $$V(s) = \max_a \sum_{s'} P(s'|s,a)V(s')$$

Bellman equations: $$V^*(s) = \max_a \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma V^*(s')]$$

Value iteration: $$V_{k+1}(s) = \max_a \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma V_k(s')], \quad \forall\, s$$

Q-iteration: $$Q_{k+1}(s,a) = \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma \max_{a'} Q_k(s',a')], \quad \forall\, s,a$$

Policy extraction: $$\pi_V(s) = \underset{a}{\operatorname{argmax}} \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma V(s')], \quad \forall\, s$$

Policy evaluation: $$V^{\pi}_{k+1}(s) = \sum_{s'} P(s'|s,\pi(s))[R(s,\pi(s),s') + \gamma V^{\pi}_k(s')], \quad \forall\, s$$

Policy improvement: $$\pi_{new}(s) = \underset{a}{\operatorname{argmax}} \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma V^{\pi_{old}}(s')], \quad \forall\, s$$

MDP Notation

Standard expectimax: $V(s) = \max_a \sum_{s'} P(s'|s,a)V(s')$

Bellman equations: $V^*(s) = \max_a \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma V^*(s')]$

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Policy extraction: $\pi_V(s) = \operatorname{argmax}_a \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma V(s')], \quad \forall\, s$

Policy evaluation: $V_{k+1}^{\pi}(s) = \sum_{s'} P(s'|s,\pi(s))[R(s,\pi(s),s') + \gamma V_k^{\pi}(s')], \quad \forall\, s$

Policy improvement: $\pi_{new}(s) = \operatorname{argmax}_a \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma V^{\pi_{old}}(s')], \quad \forall\, s$

Policies

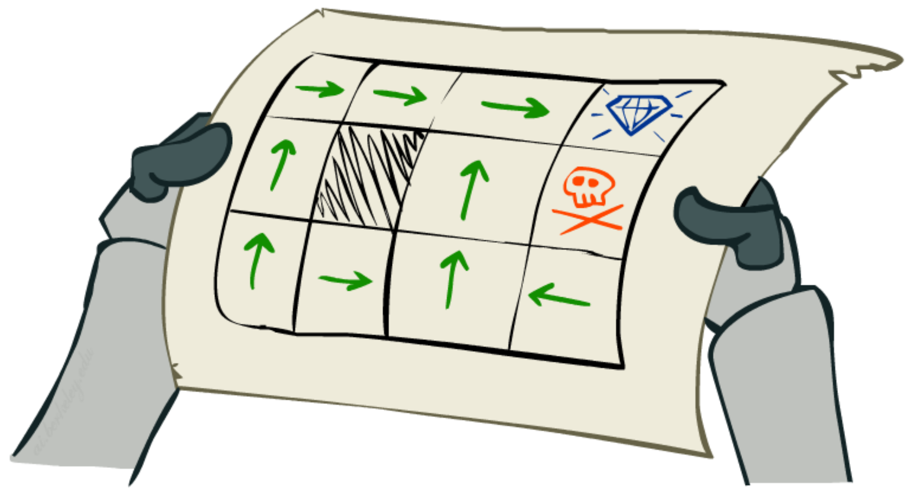
In deterministic single-agent search problems, we wanted an optimal plan, or sequence of actions, from start to a goal

For MDPs, we want an optimal policy $\pi^*: S \rightarrow A$

- A policy π gives an action for each state
- An optimal policy is one that maximizes expected utility if followed
- An explicit policy defines a reflex agent

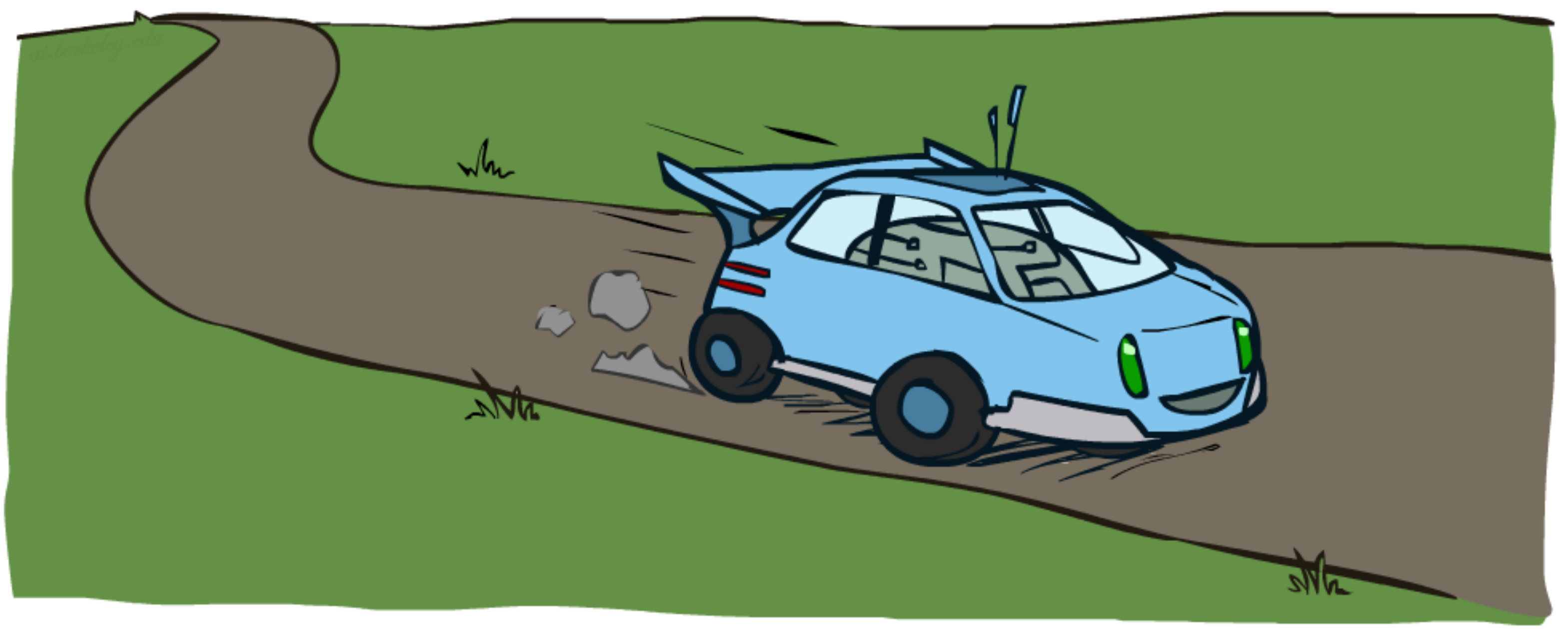
Expectimax didn't compute entire policies

- It computed the action for a single state only



Optimal policy when R(s, a, s') = -0.03 for all non-terminals s

Example: Racing



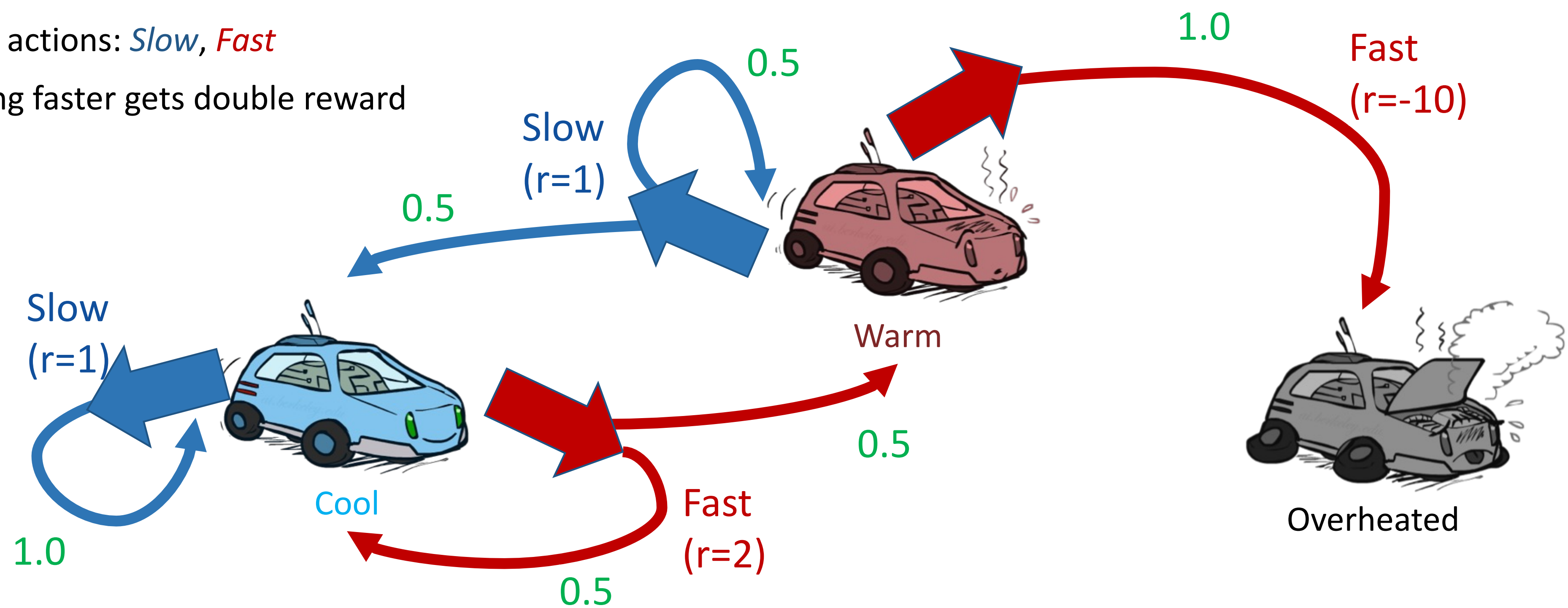
Example: Racing

A robot car wants to travel far, quickly

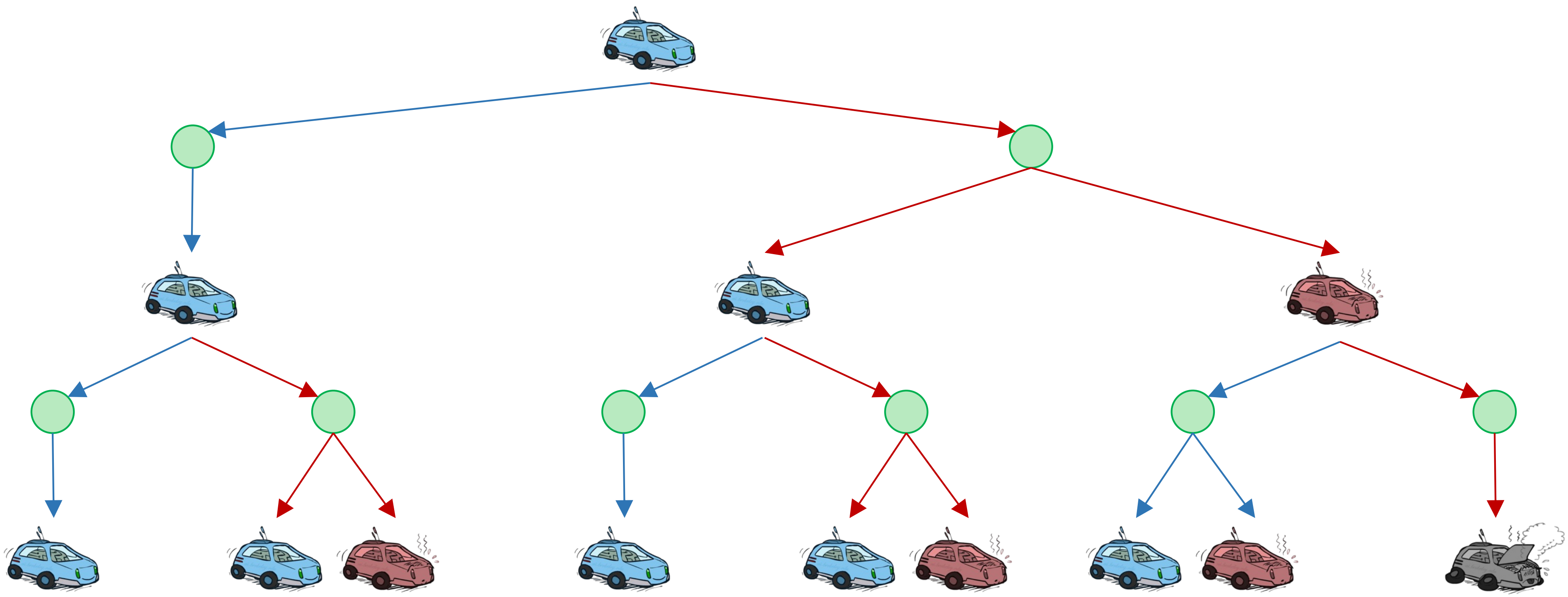
Three states: Cool, Warm, Overheated

Two actions: *Slow*, *Fast*

Going faster gets double reward

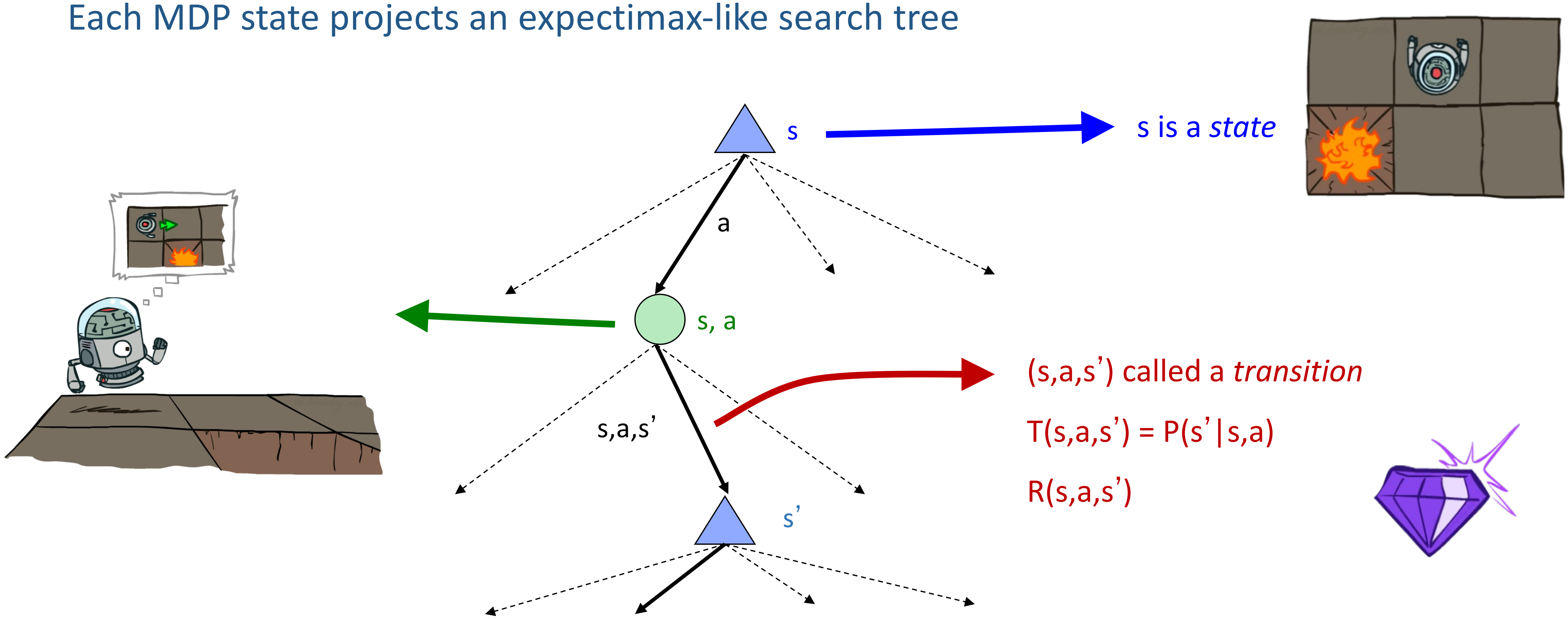


Racing Search Tree



MDP Search Trees

Each MDP state projects an expectimax-like search tree

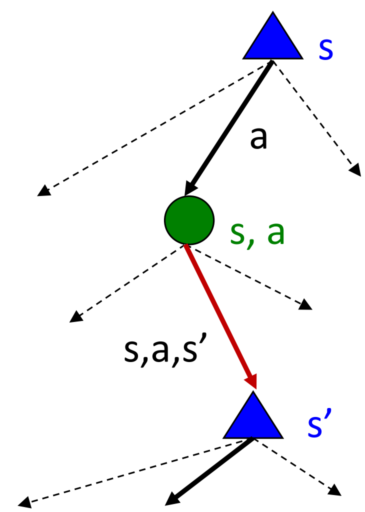


Optimal Quantities

- The value (utility) of a state s:
  $V^*(s)$ = expected utility starting in s and acting optimally

- The value (utility) of a q-state (s,a):
  $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally

- The optimal policy:
  $\pi^*(s)$ = optimal action from state s



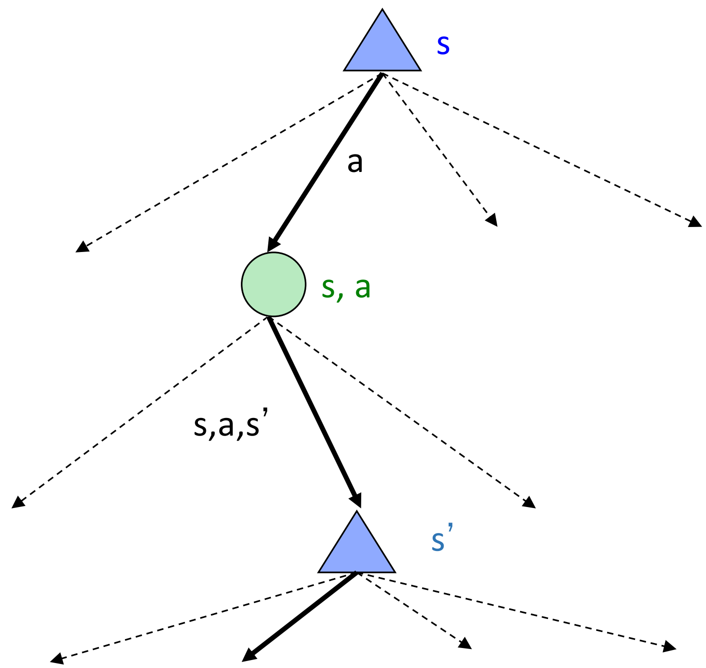
s is a *state*

(s, a) is a *q-state*

(s,a,s') is a *transition*

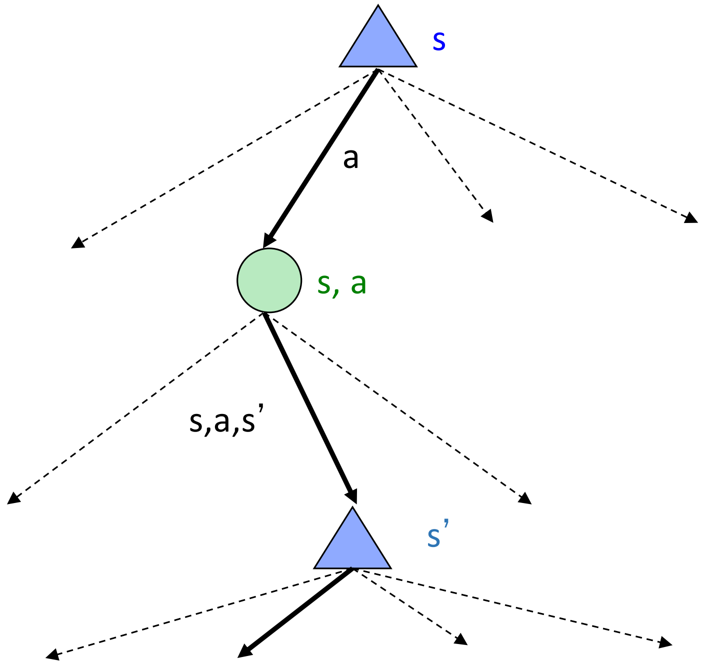
Recursive Expectimax

$$V(s) = \max_a \sum_{s'} P(s'|s,a)\, V(s')$$



Recursive Expectimax

$$V(s) = \max_{a} \sum_{s'} P(s'|s,a)\,[R(s,a,s') + V(s')]$$

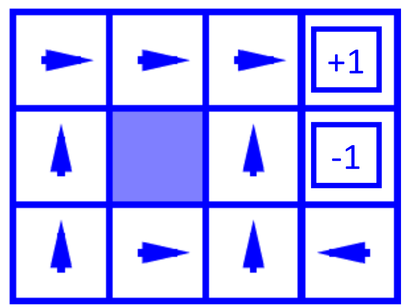


Poll 1

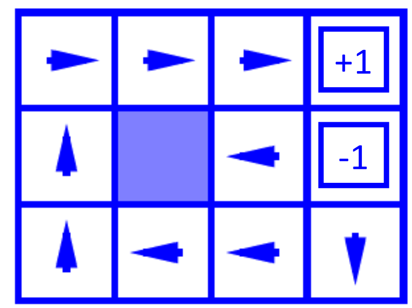
Which sequence of optimal policies matches the following sequence of living rewards:
{-0.01, -0.03, -0.4, -2.0}

I. {A, B, C, D}
II. {B, C, A, D}
III. {D, C, B, A}
IV. {D, A, C, B}

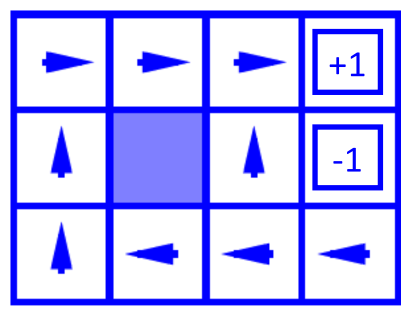
A)



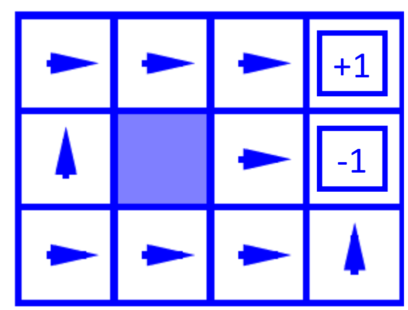
B)



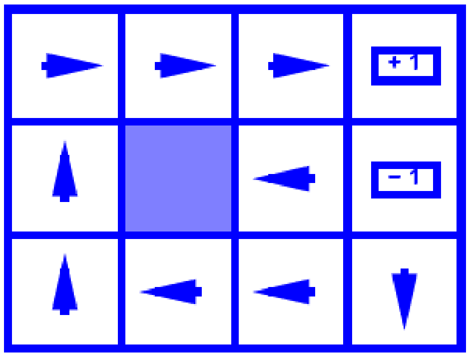
C)



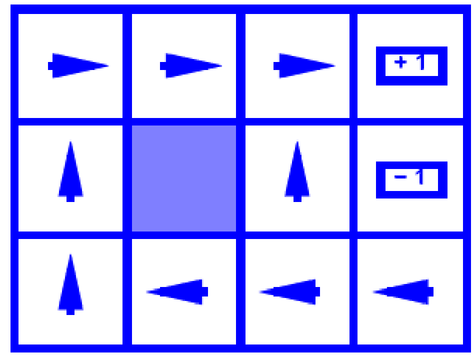
D)



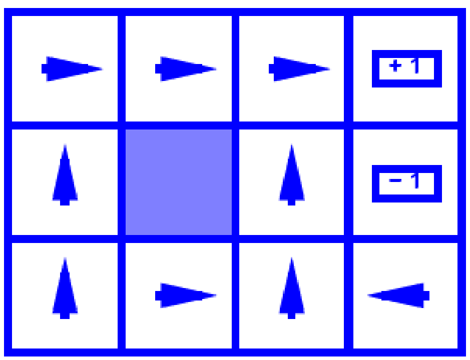
Optimal Policies



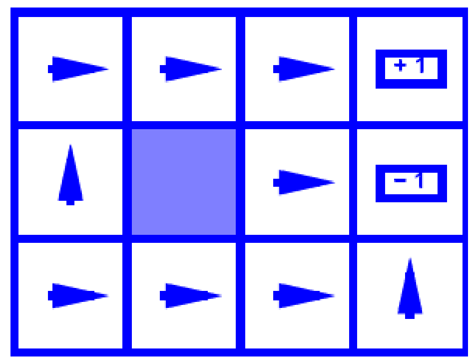
R(s) = -0.01



R(s) = -0.03

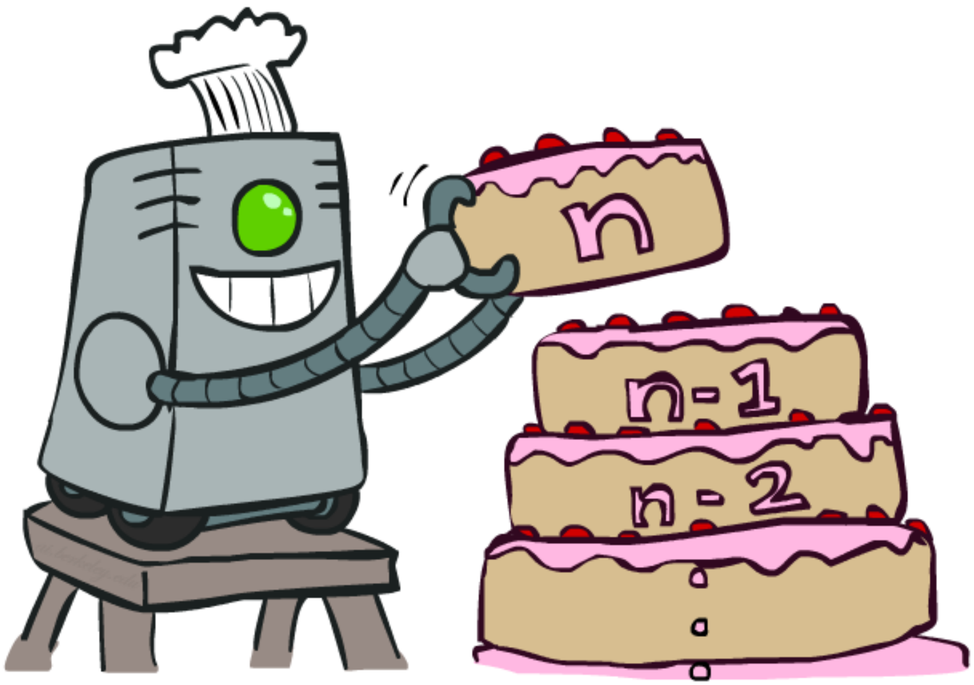


R(s) = -0.4



R(s) = -2.0

Value Iteration

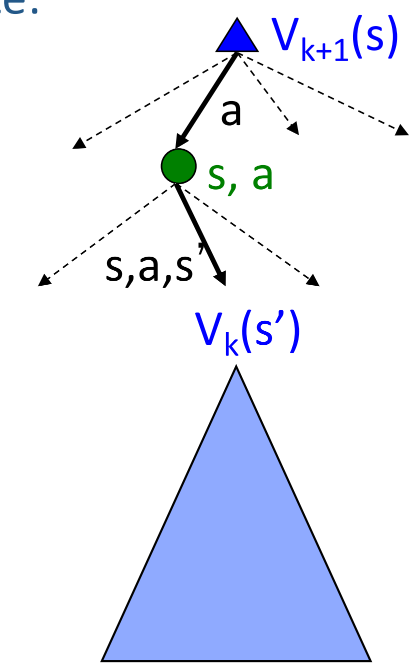


Value Iteration

Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero

Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s') \right]$$



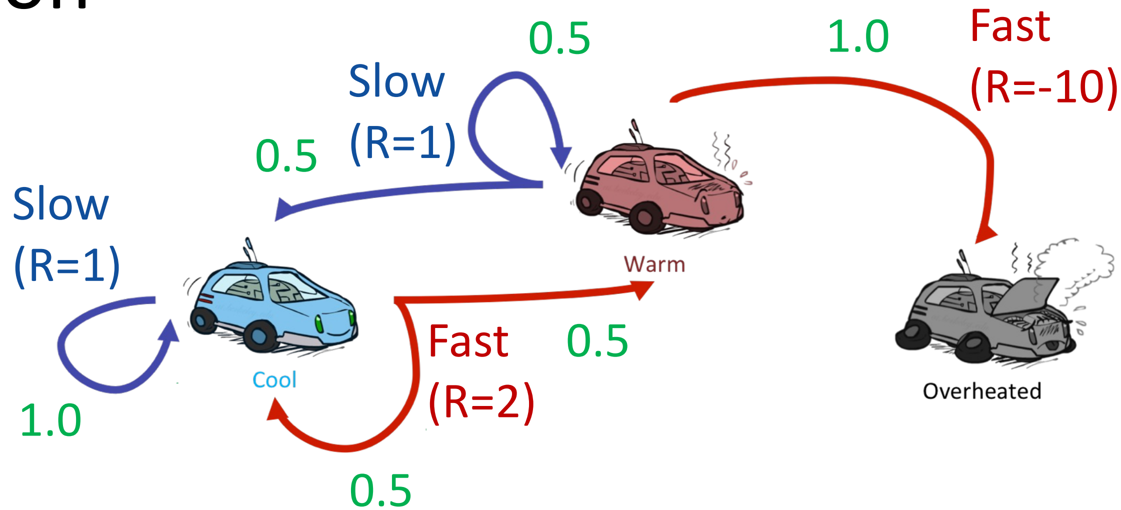
Repeat until convergence

Complexity of each iteration: $O(S^2A)$

Theorem: will converge to unique optimal values
- Basic idea: approximations get refined towards optimal values
- Policy may converge long before values do

Example: Value Iteration

V_2			
V_1	2	1	0
V_0	0	0	0

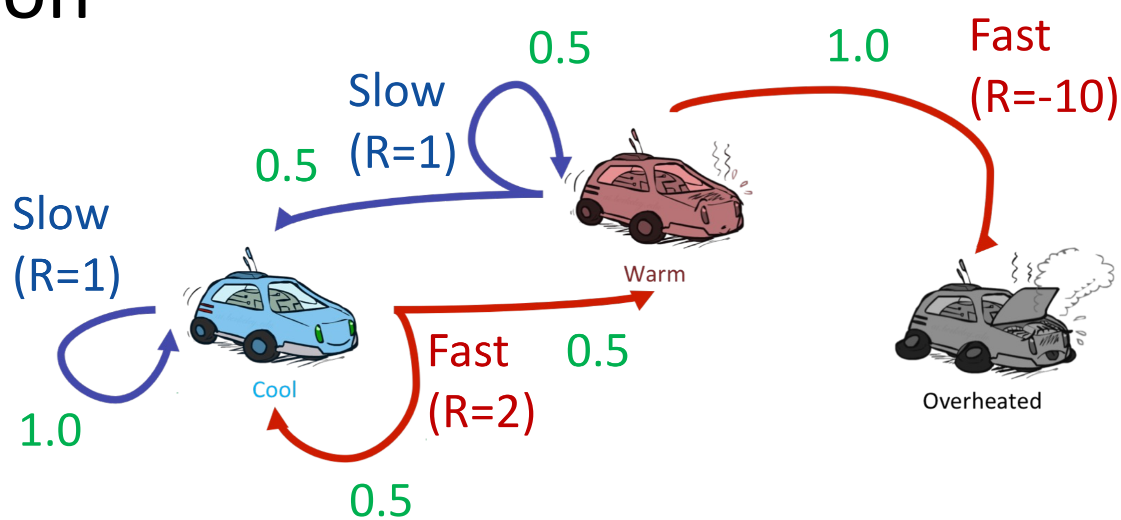


Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s') \right]$$

Example: Value Iteration

V_2	3.5	2.5	0
V_1	2	1	0
V_0	0	0	0



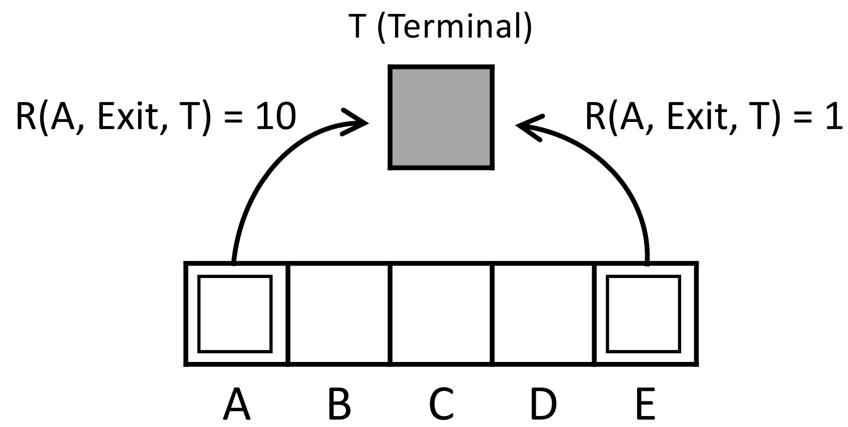
Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

Simple Deterministic Example

- Actions: B, C, D: East, West
- Actions: A, E: Exit
- Transitions: deterministic
- Rewards only for transitioning to terminal state

$$V(s) = \max_a [R(s,a,s') + V(s')]$$



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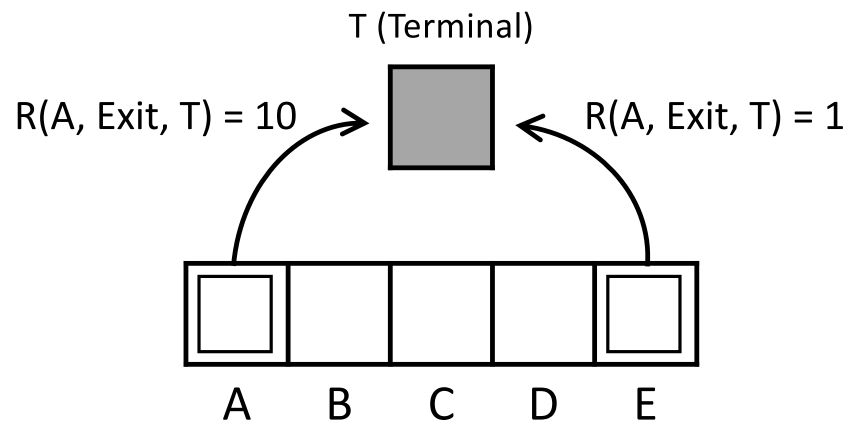

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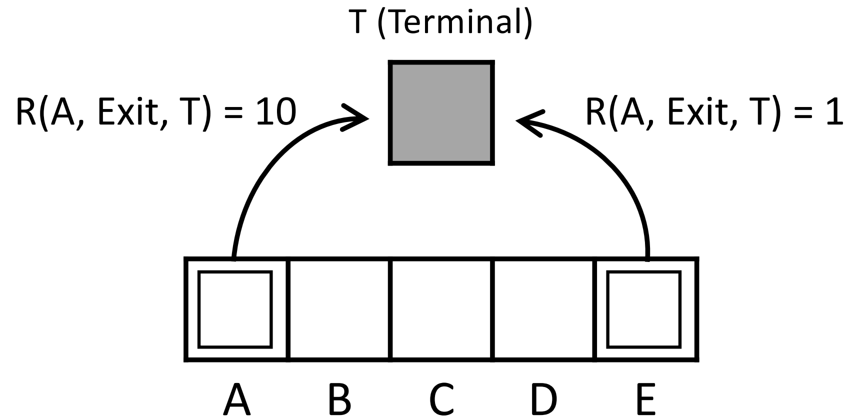
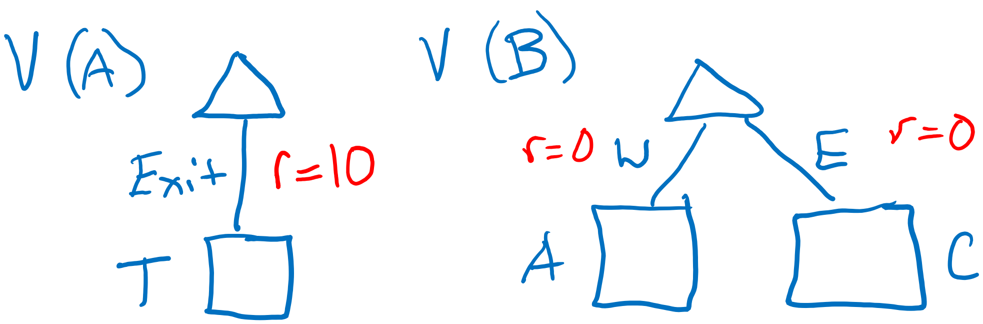


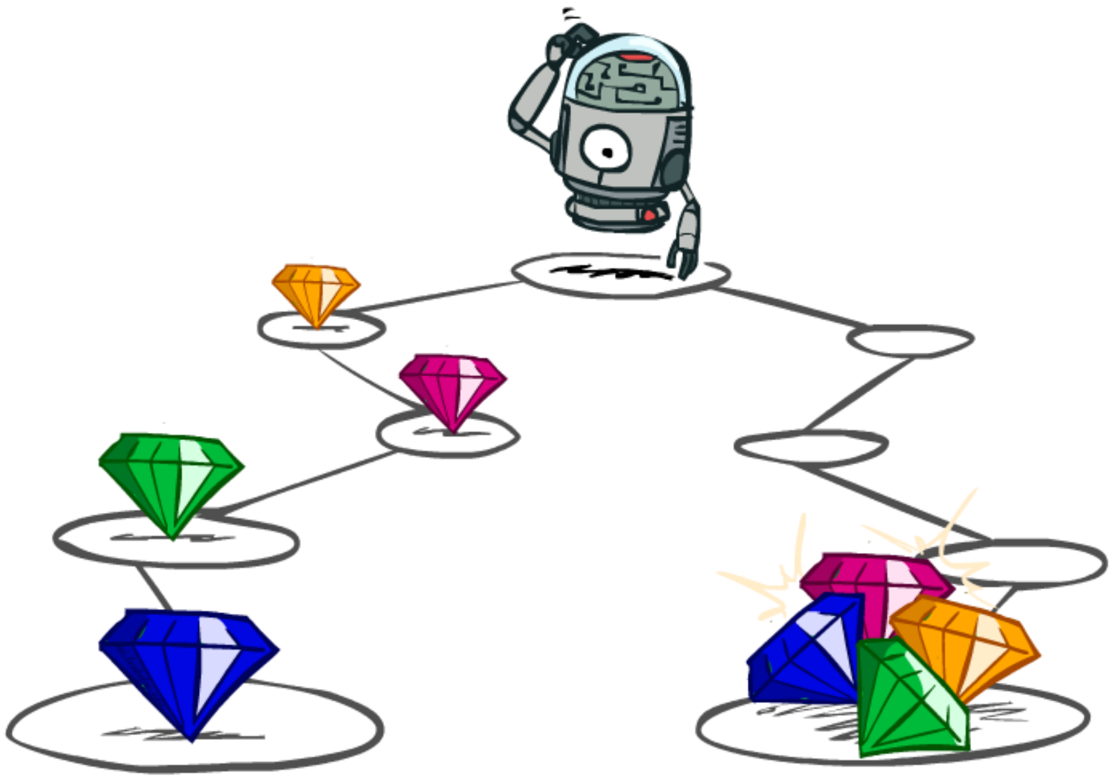
$$V_{k+1}(s) = \max_a [R(s,a,s') + V_k(s')]$$

$V_0(s) = 0 \quad \forall s$



	T	A	B	C	D	E
V_0	0	0	0	0	0	0
V_1	0	10	0	0	0	1
V_2	0					
V_3						
V_4						

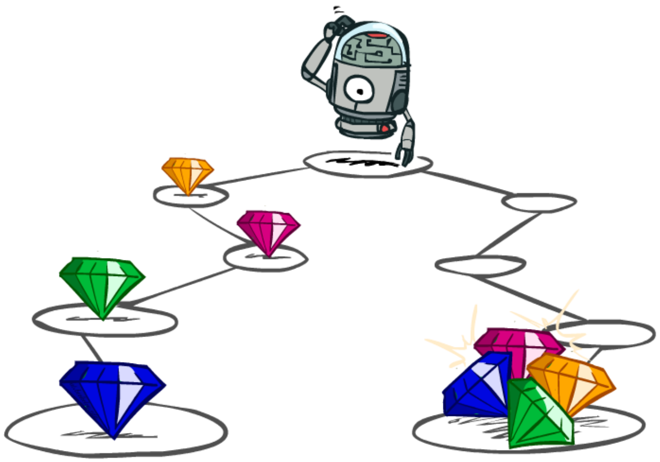
Utilities of Sequences



Utilities of Sequences

What preferences should an agent have over reward sequences?

More or less?	[1, 2, 2]	or	[2, 3, 4]
Now or later?	[0, 0, 1]	or	[1, 0, 0]



Discounting

It’s reasonable to maximize the sum of rewards

It’s also reasonable to prefer rewards now to rewards later

One solution: values of rewards decay exponentially

1

Worth Now

γ

Worth Next Step

γ^2

Worth In Two Steps

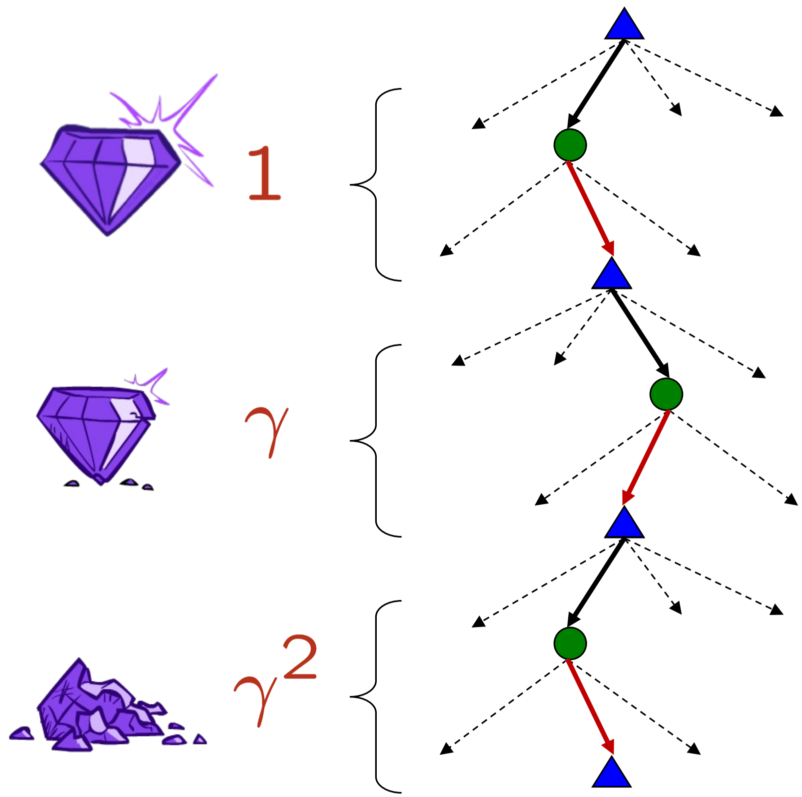
Discounting

How to discount?

- Each time we descend a level, we multiply in the discount once

Why discount?

- Sooner rewards probably do have higher utility than later rewards
- Also helps our algorithms converge



Poll 2

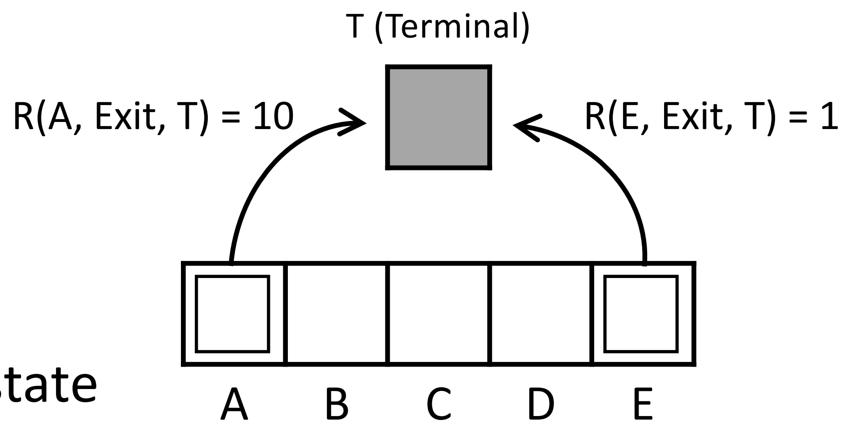
If an agent will get 2 reward now at t=0, 4 at t=1, and 8 at t=2, what is the expected value of the current state if $\gamma = 0.5$?

A. 3
B. 6
C. 7
D. 14

Bonus: What is the value of U[8,4,2] with $\gamma = 0.5$?

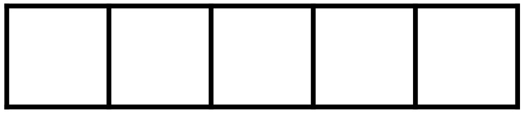
Discounting

- Actions: B, C, D: East, West
- Actions: A, E: Exit
- Transitions: deterministic
- Rewards only for transitioning to terminal state

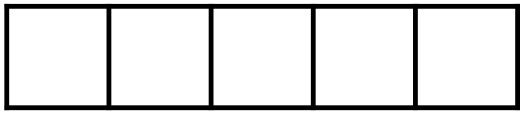


$$V_{k+1}(s) = \max_a [R(s,a,s') + \gamma\, V_k(s')]$$

For γ = 1, what is the optimal policy?



For γ = 0.1, what is the optimal policy?

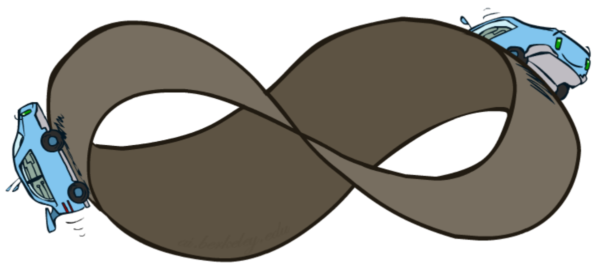


For which γ are West and East equally good when in state D?

Infinite Utilities?!

- Problem: What if the game lasts forever? Do we get infinite rewards?

- Solutions:
- Finite horizon: (similar to depth-limited search)
  - Terminate episodes after a fixed T steps (e.g. life)
  - Gives nonstationary policies (π depends on time left)



- Discounting: use $0 < \gamma < 1$

$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\text{max}}/(1-\gamma)$$

  - Smaller γ means smaller "horizon" – shorter term focus

- Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like "overheated" for racing)

In–Class Activity

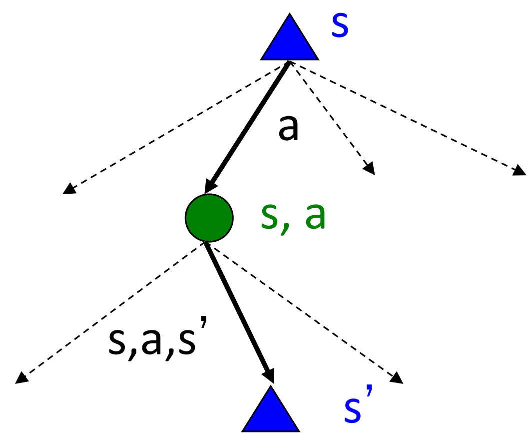
Practice Value iteration in our Trivia Game MDP example

Poll: What is the policy π_3 after 3 iterations?

Recap: Defining MDPs

Markov decision processes:

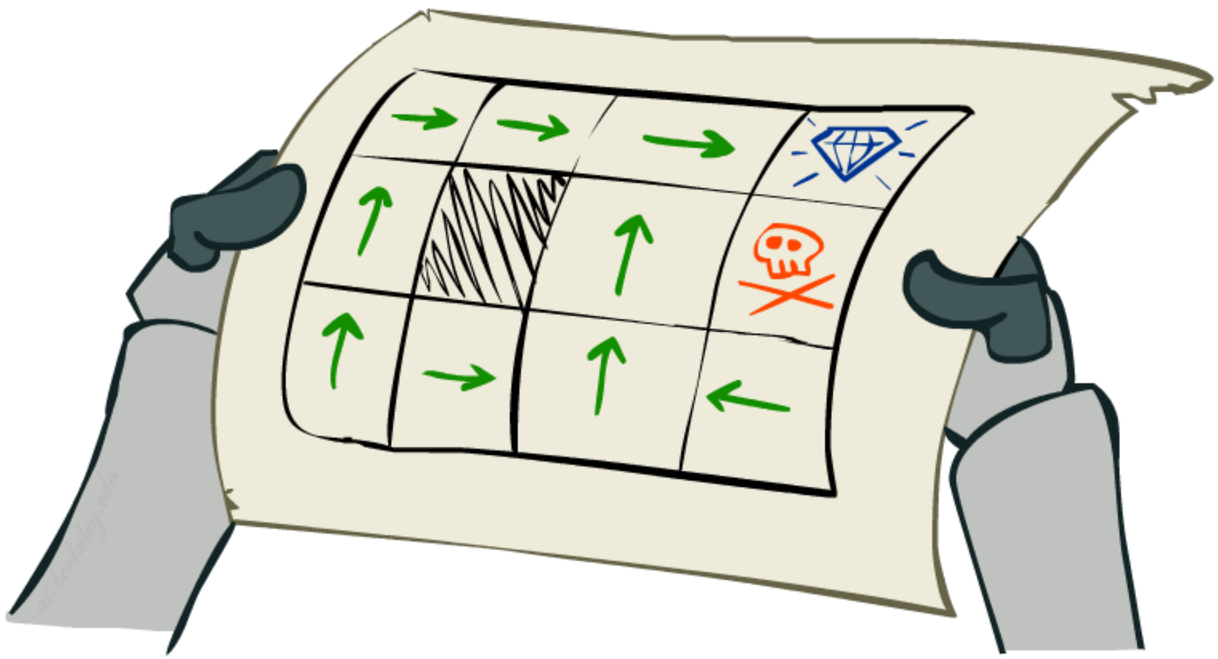
- Set of states S
- Start state s_0
- Set of actions A
- Transitions P(s'|s,a) (or T(s,a,s'))
- Rewards R(s,a,s') (and discount γ)



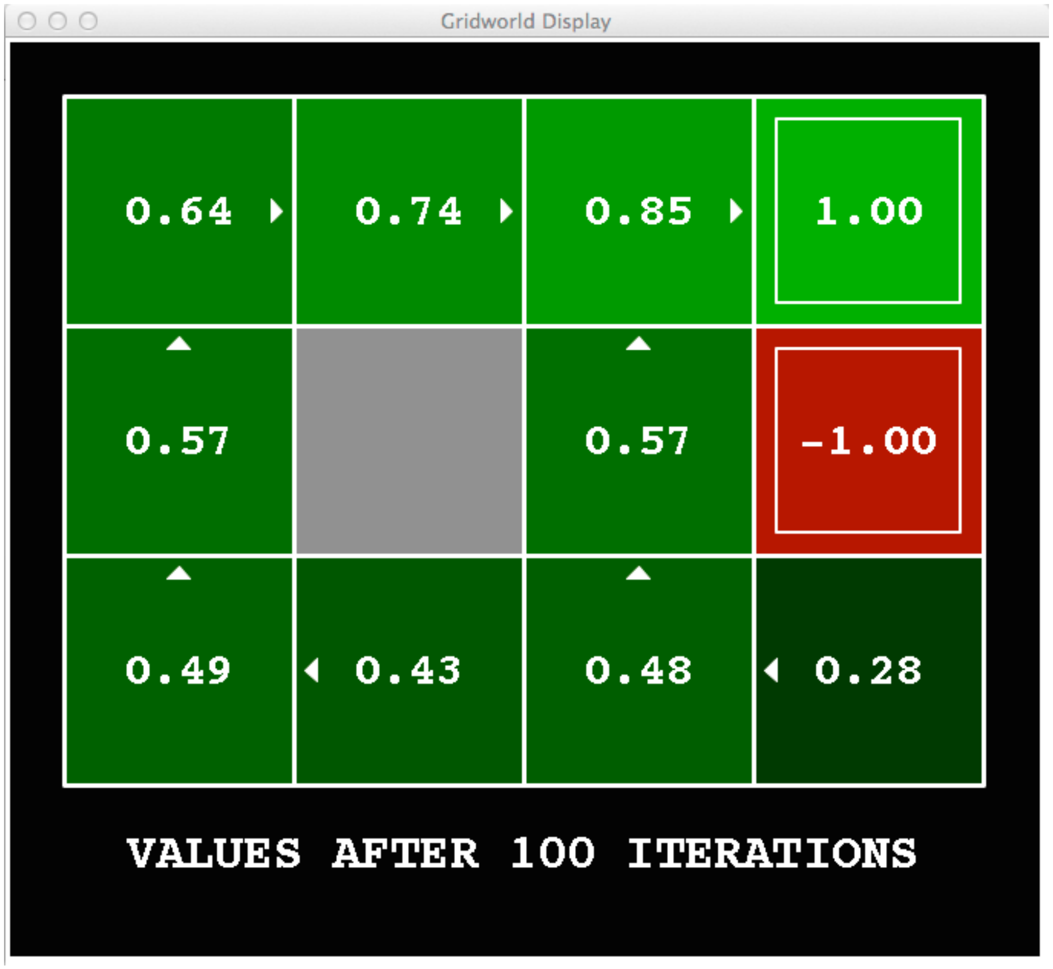
MDP quantities so far:

- Policy = Choice of action for each state
- Utility = sum of (discounted) rewards

Solving MDPs

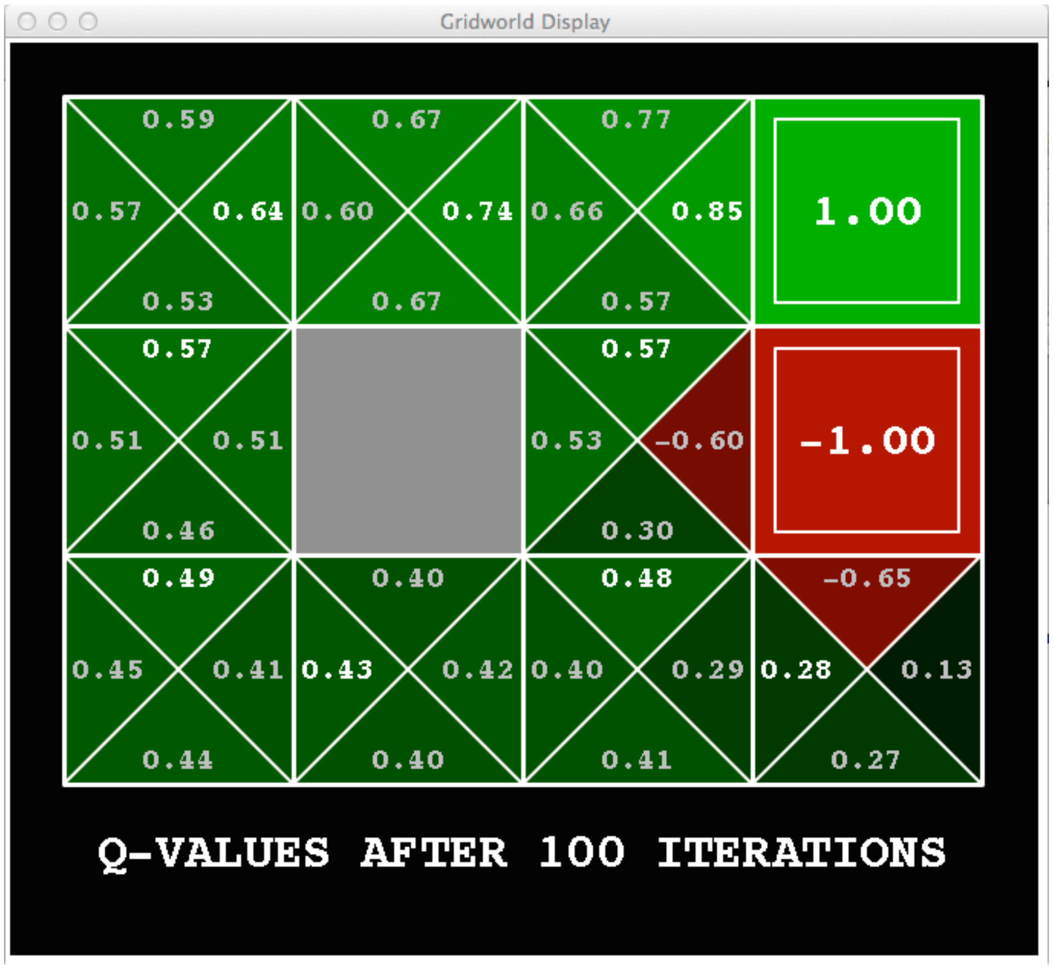


Snapshot of Demo – Gridworld V Values



Noise = 0.2
Discount = 0.9
Living reward = 0

Snapshot of Demo – Gridworld Q Values

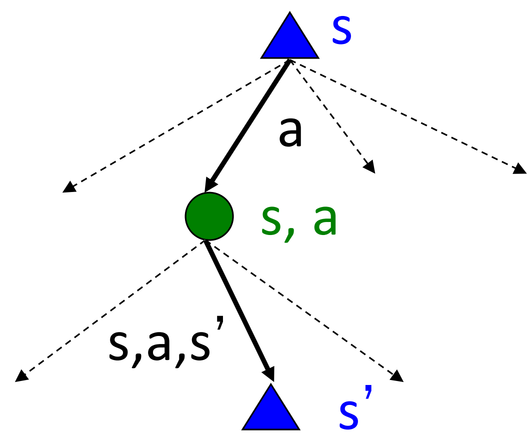


Noise = 0.2
Discount = 0.9
Living reward = 0

Values of States

Fundamental operation: compute the (expectimax) value of a state

- Expected utility under optimal action
- Average sum of (discounted) rewards
- This is just what expectimax computed!



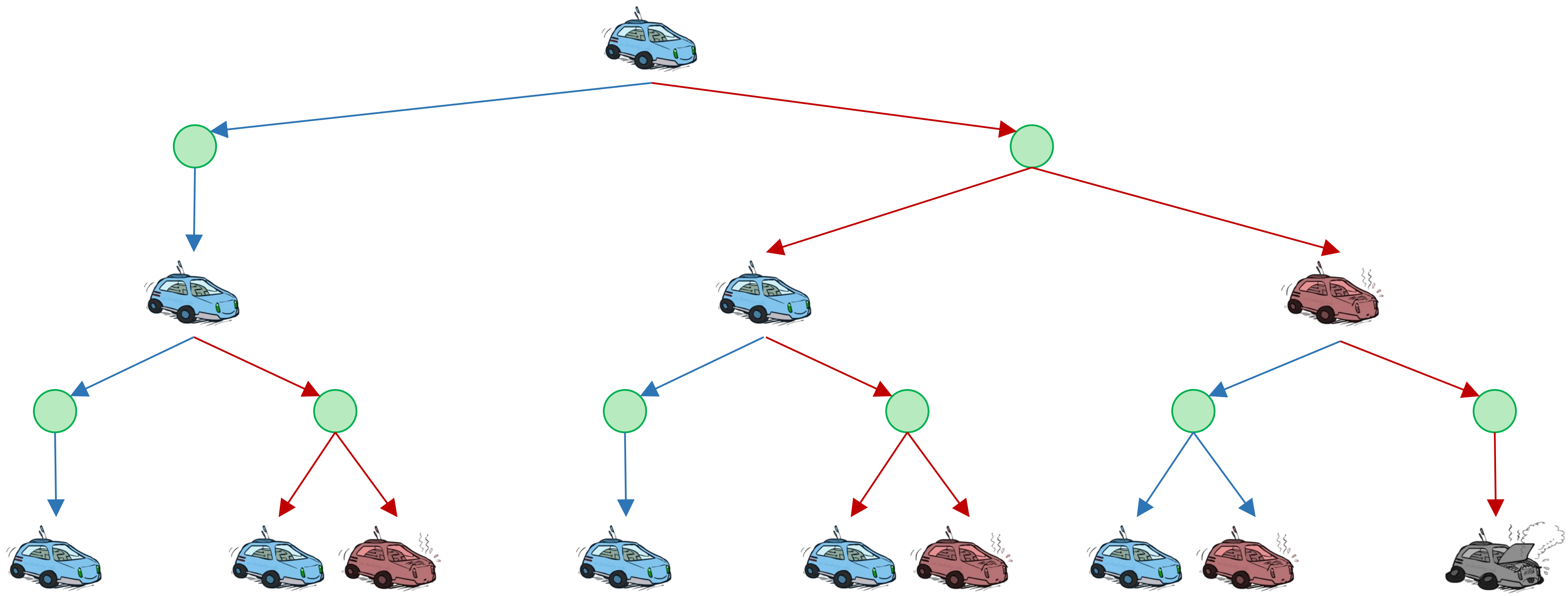
Recursive definition of value:

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

Racing Search Tree



Racing Search Tree

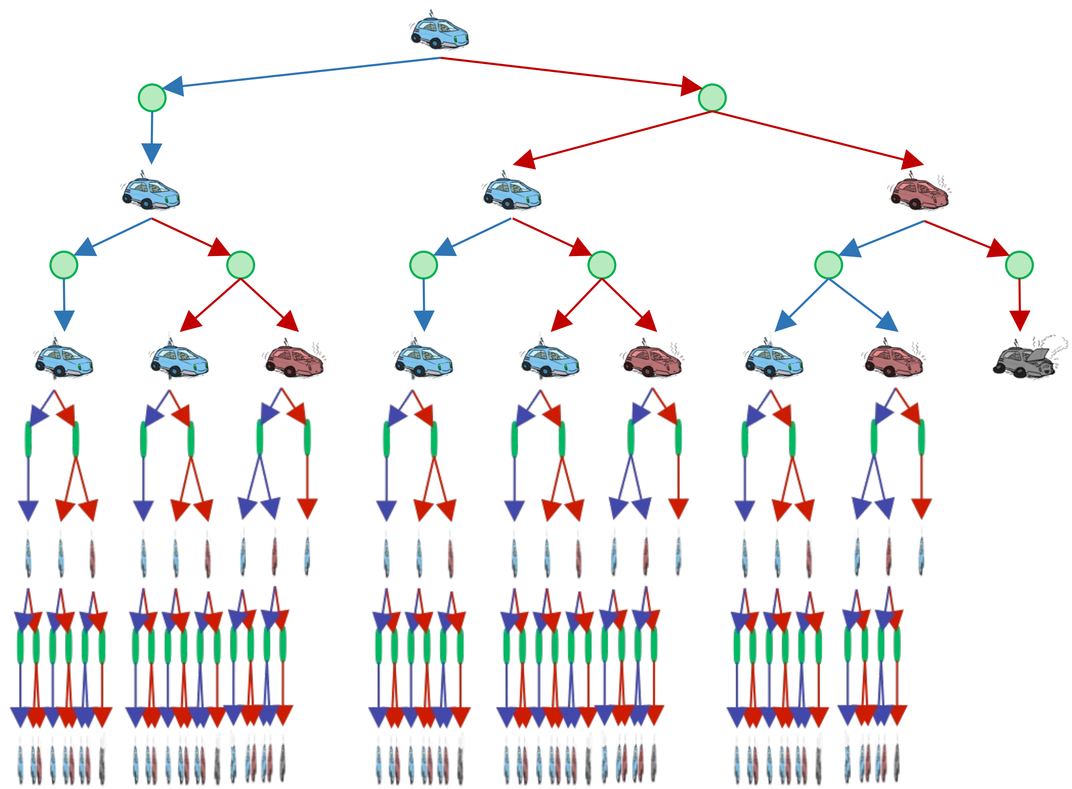
We're doing way too much work with expectimax!

Problem: States are repeated

- Idea: Only compute needed quantities once

Problem: Tree goes on forever

- Idea: Do a depth-limited computation, but with increasing depths until change is small
- Note: deep parts of the tree eventually don't matter if $\gamma < 1$

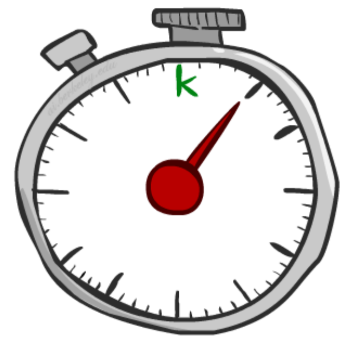


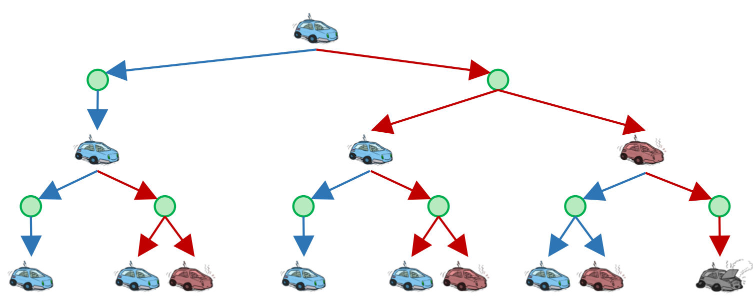
Time-Limited Values

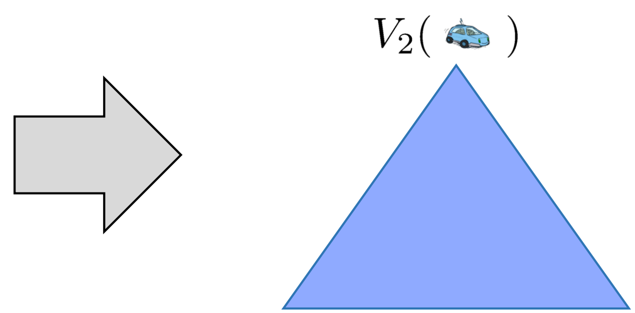
Key idea: time-limited values

Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps

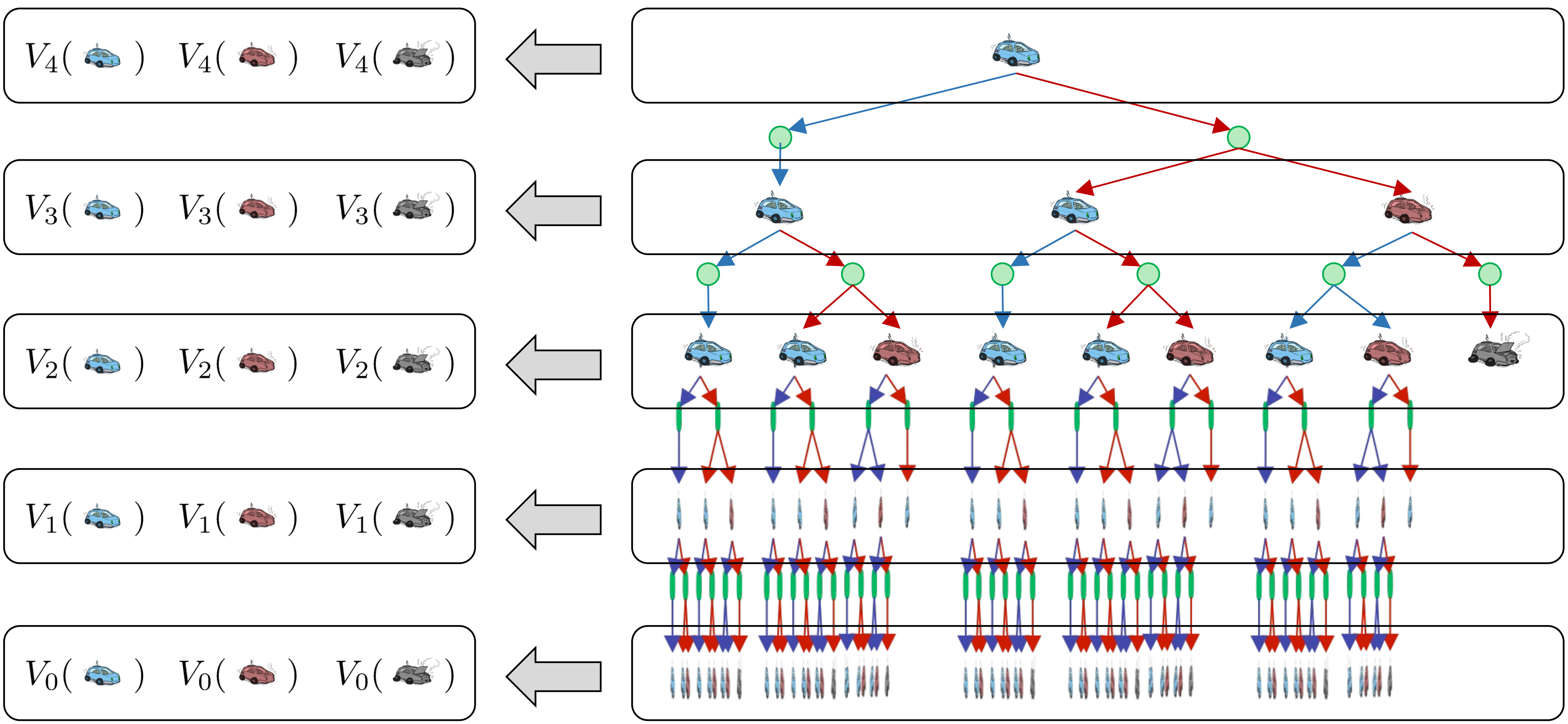
- Equivalently, it's what a depth-k expectimax would give from s



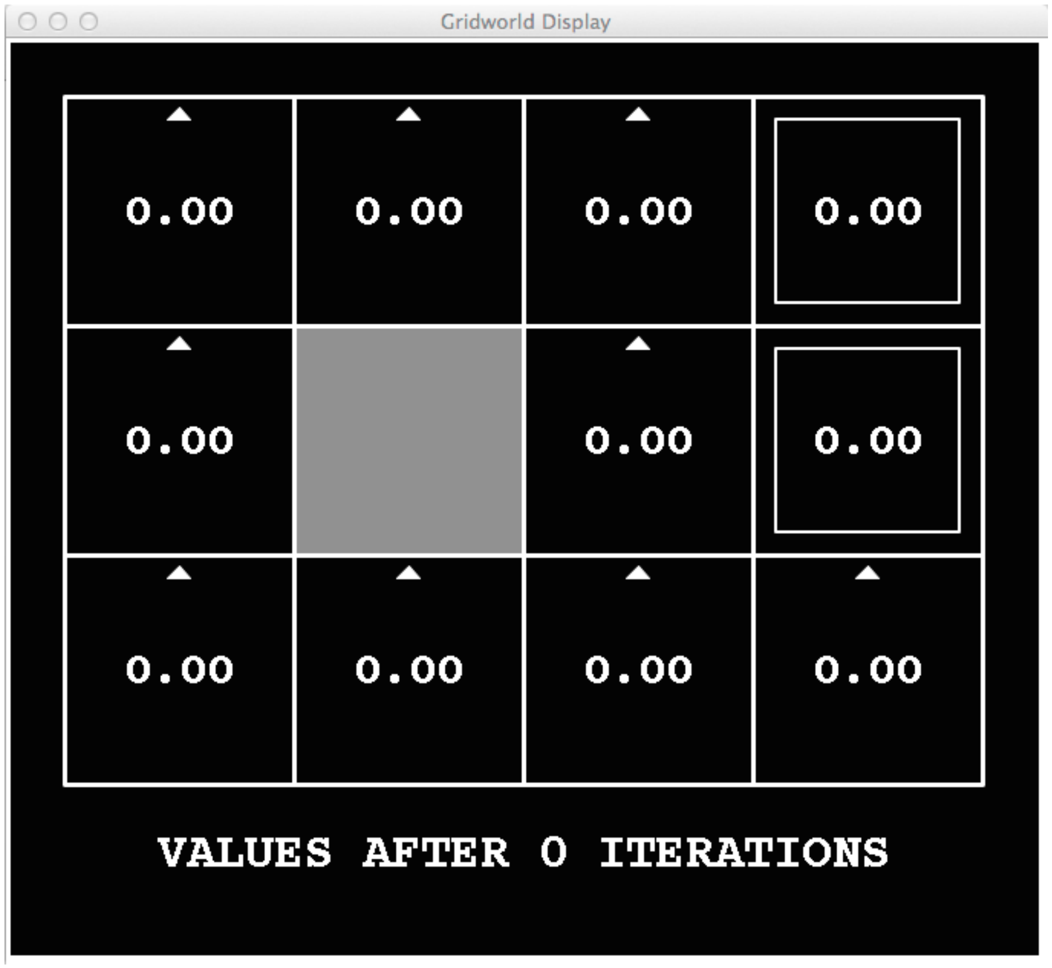




Computing Time-Limited Values

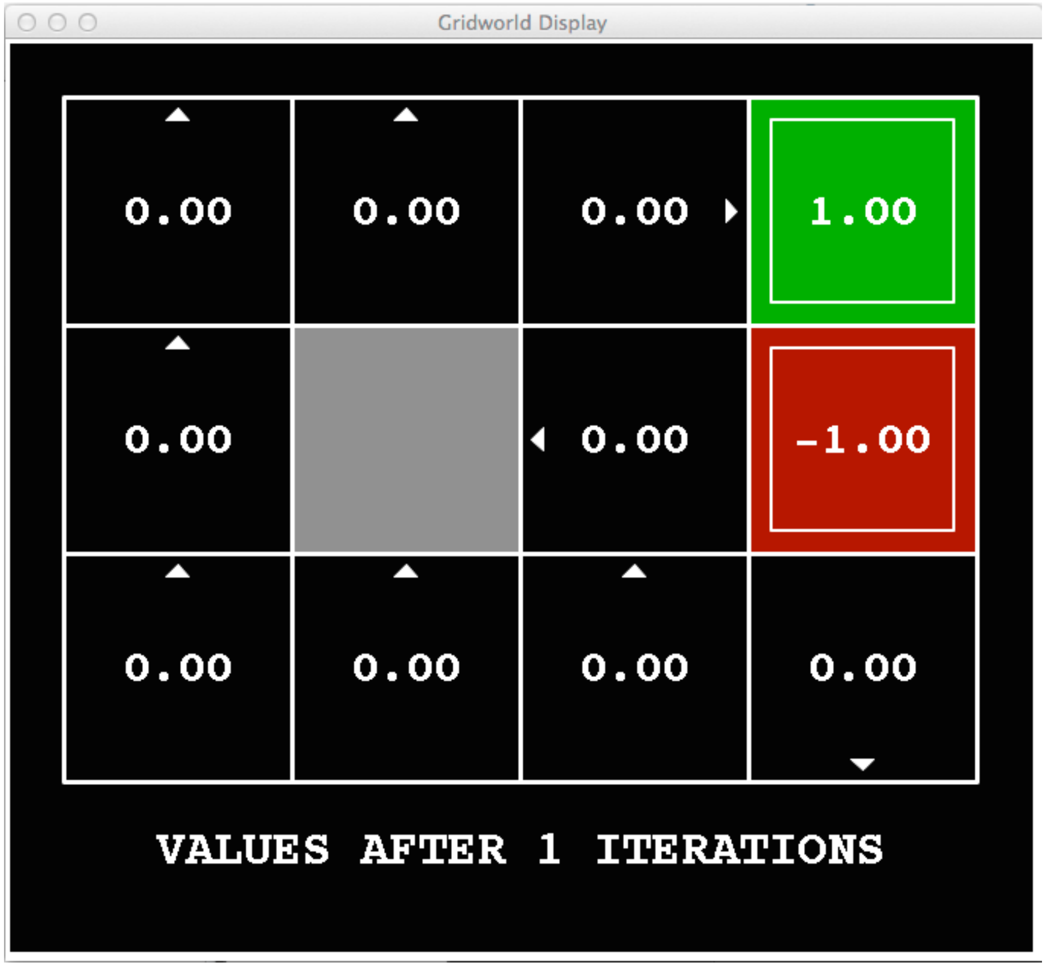


k=0



Noise = 0.2
Discount = 0.9
Living reward = 0

k=1



Noise = 0.2
Discount = 0.9
Living reward = 0

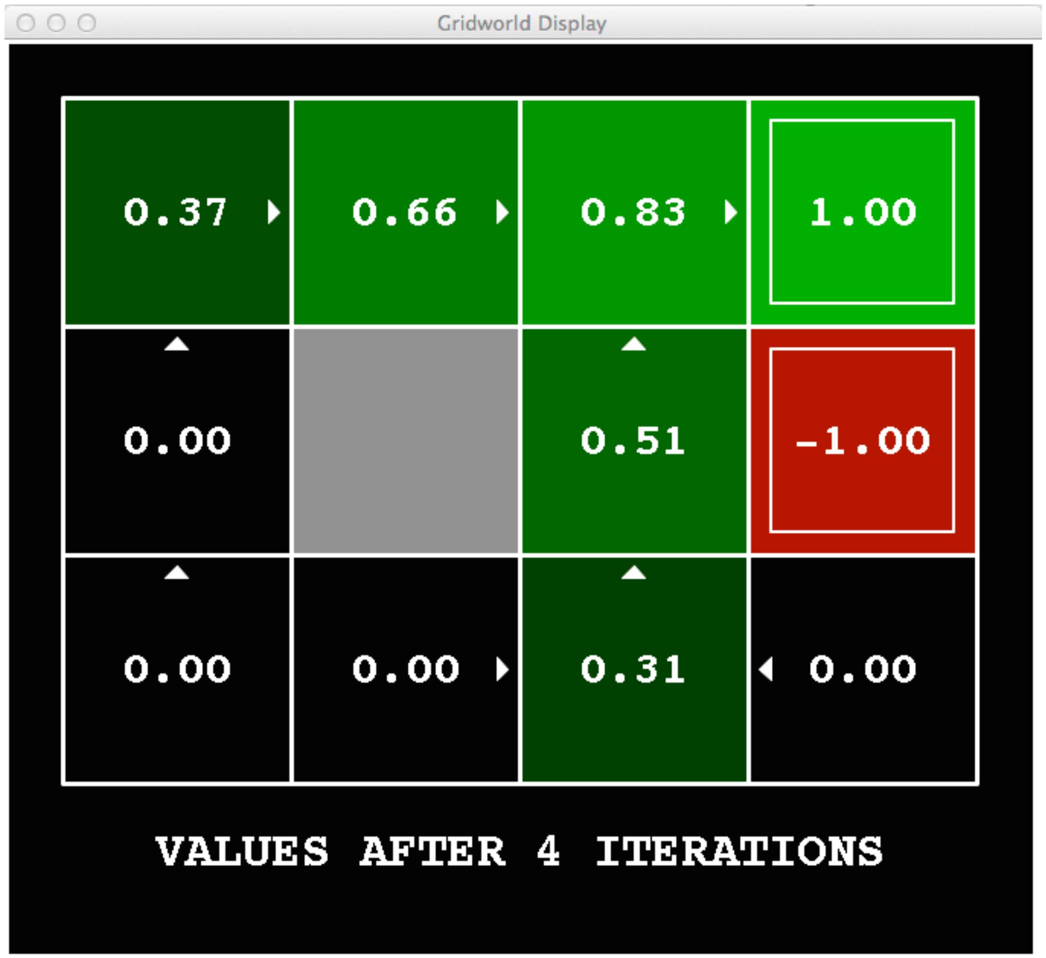
k=2

Noise = 0.2
Discount = 0.9
Living reward = 0

k=3

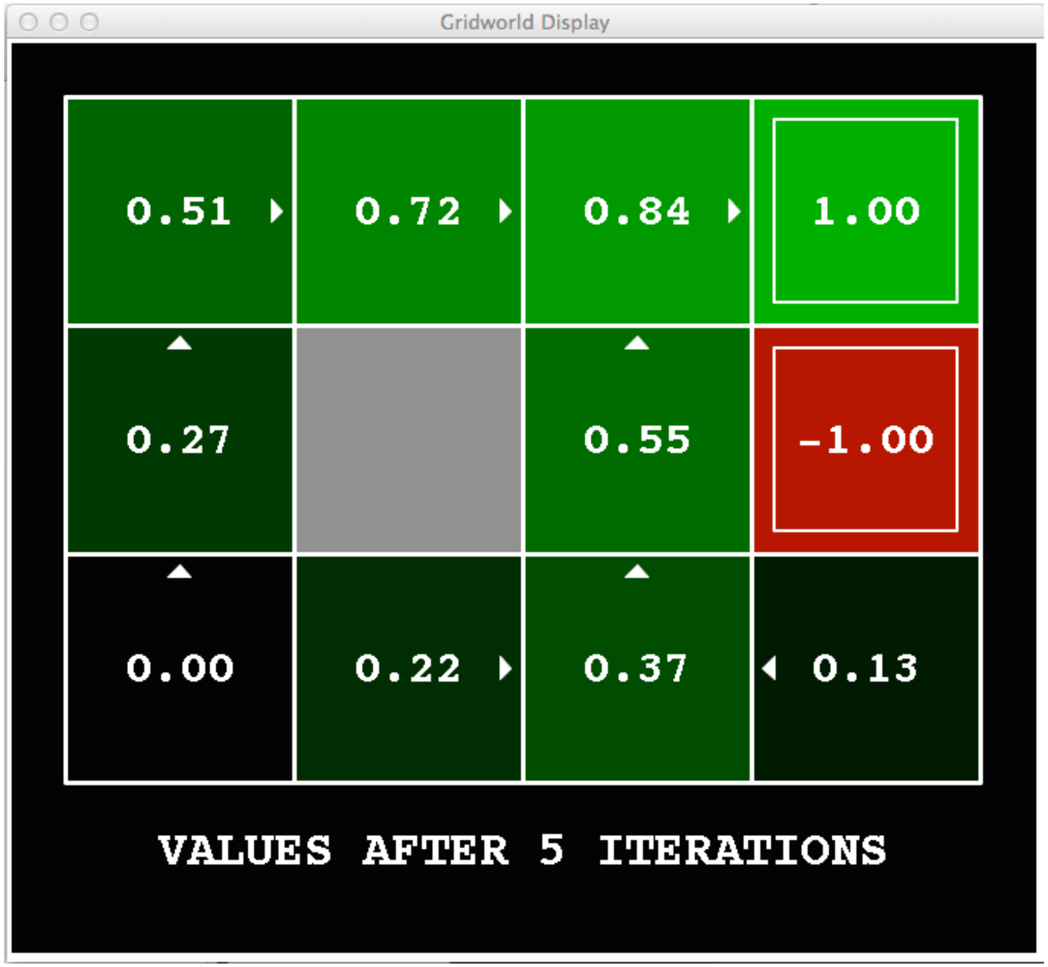
Noise = 0.2
Discount = 0.9
Living reward = 0

k=4



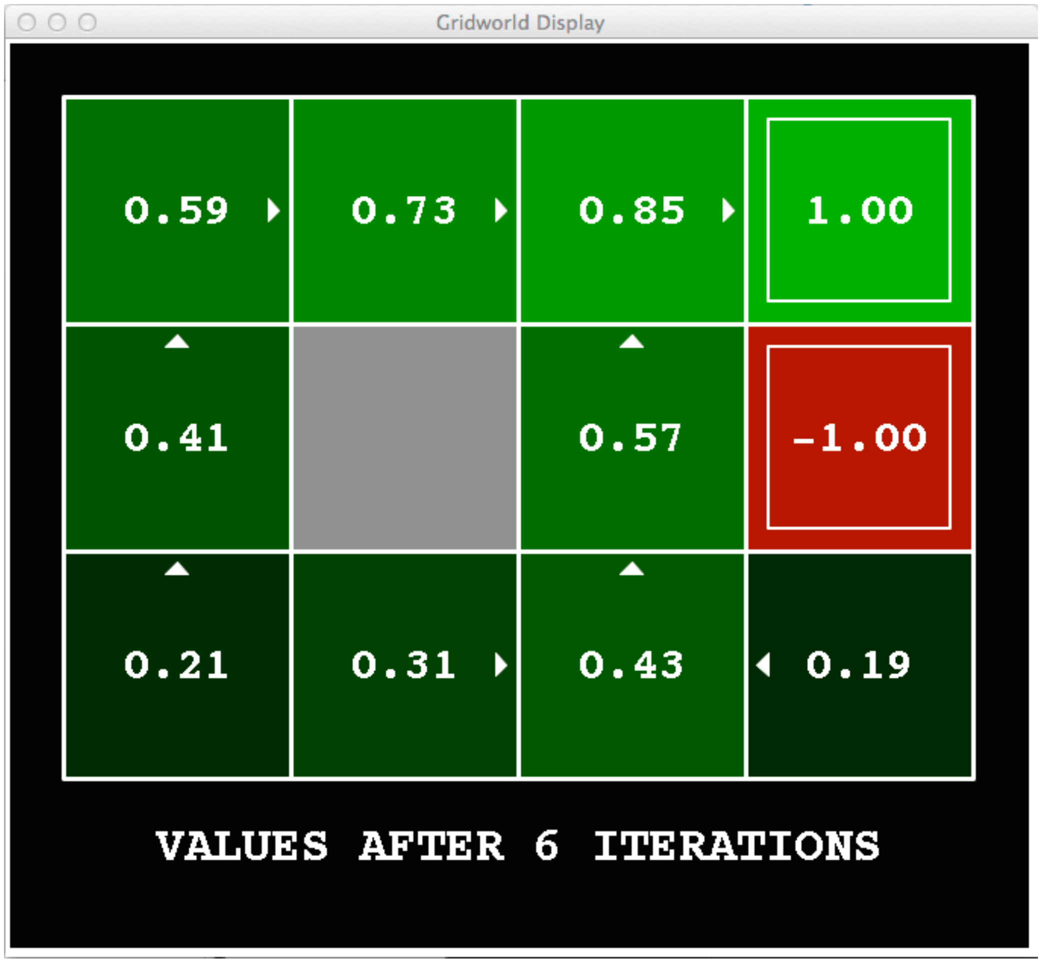
Noise = 0.2
Discount = 0.9
Living reward = 0

k=5



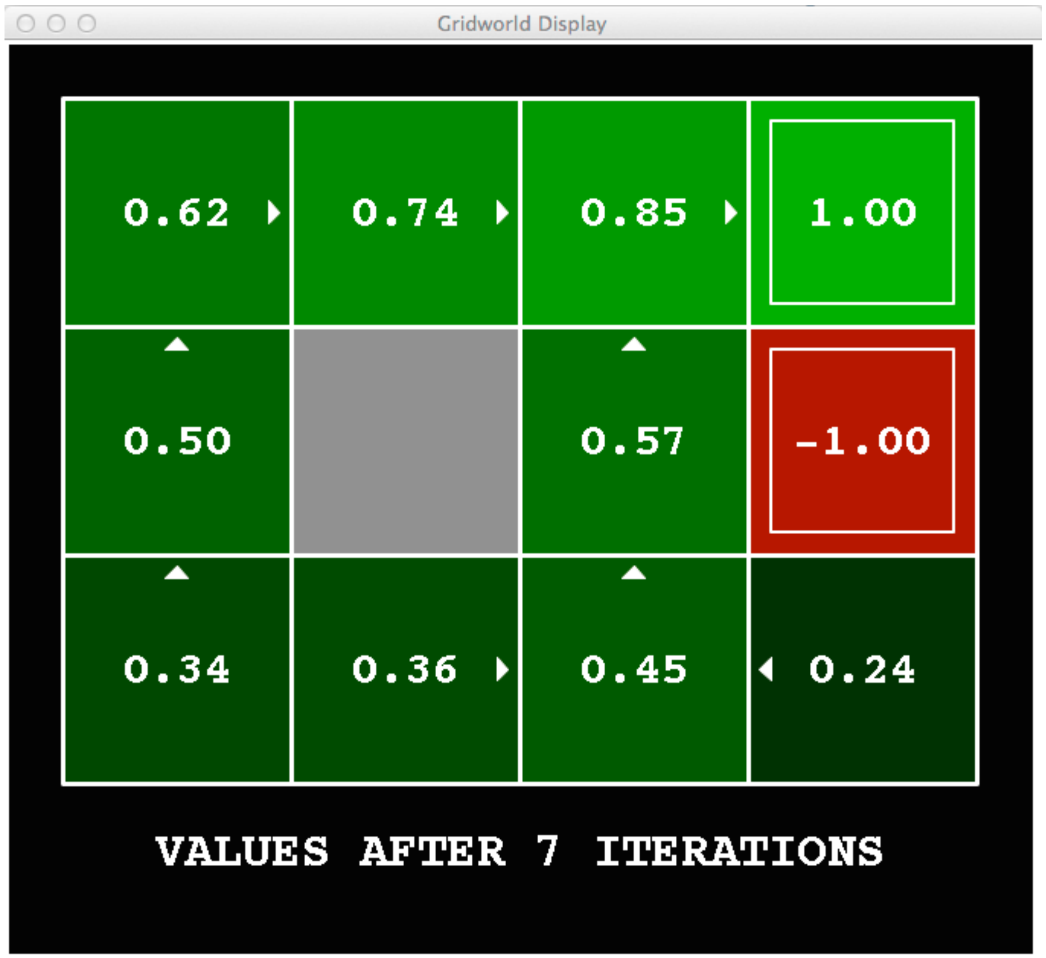
Noise = 0.2
Discount = 0.9
Living reward = 0

k=6



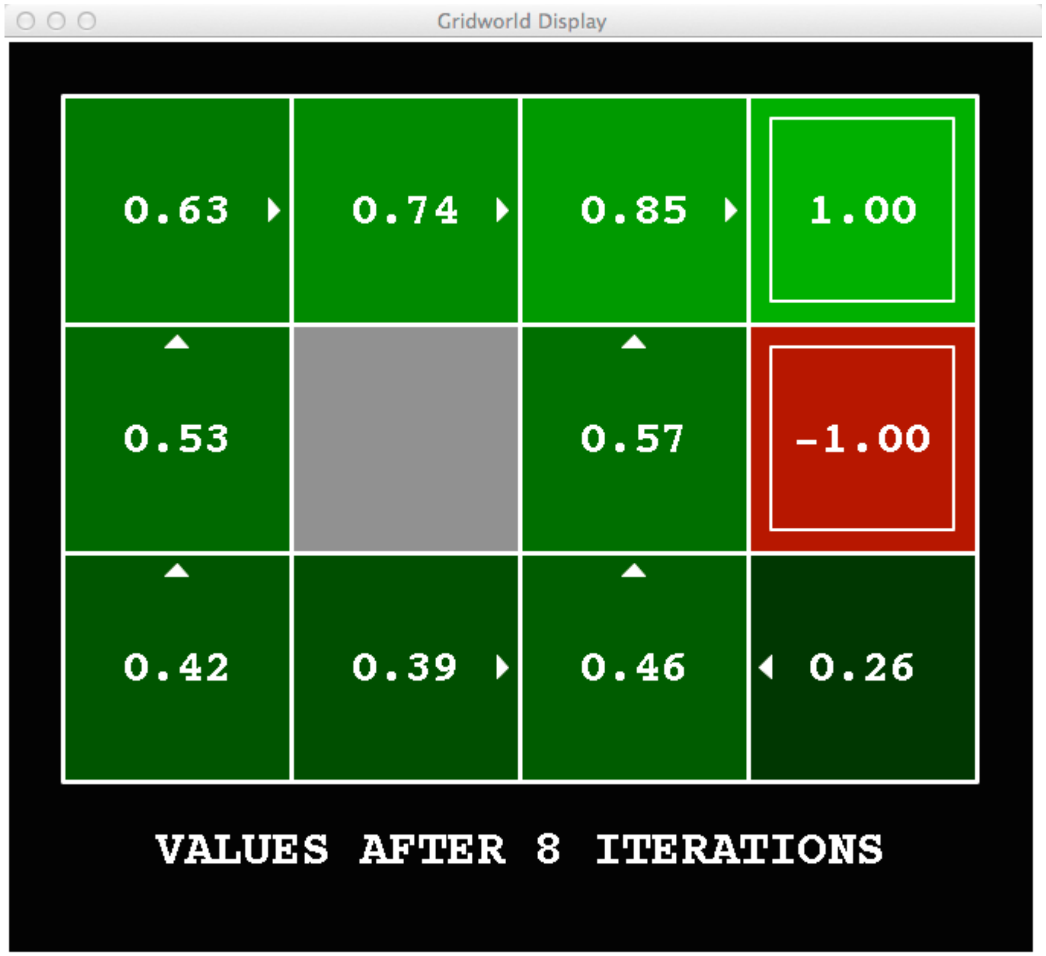
Noise = 0.2
Discount = 0.9
Living reward = 0

k=7



Noise = 0.2
Discount = 0.9
Living reward = 0

k=8

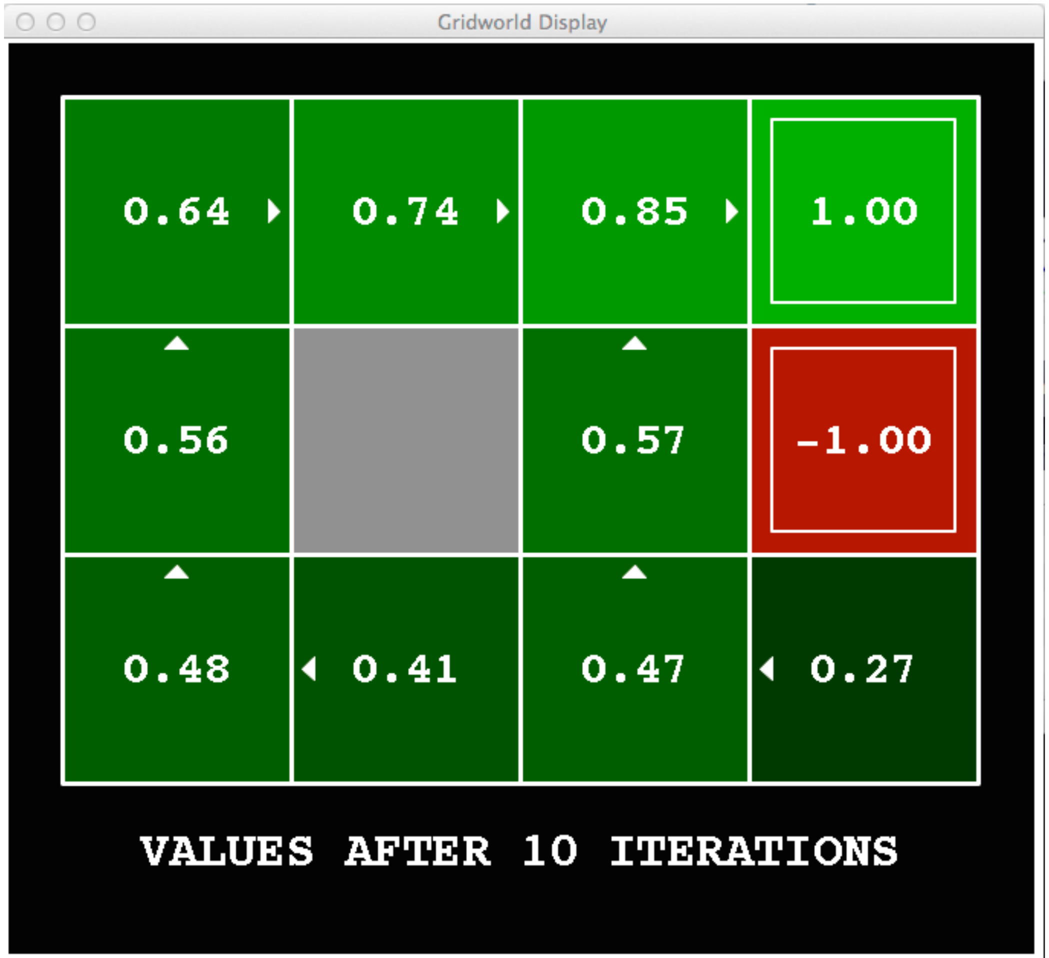


Noise = 0.2
Discount = 0.9
Living reward = 0

k=9

Noise = 0.2
Discount = 0.9
Living reward = 0

k=10

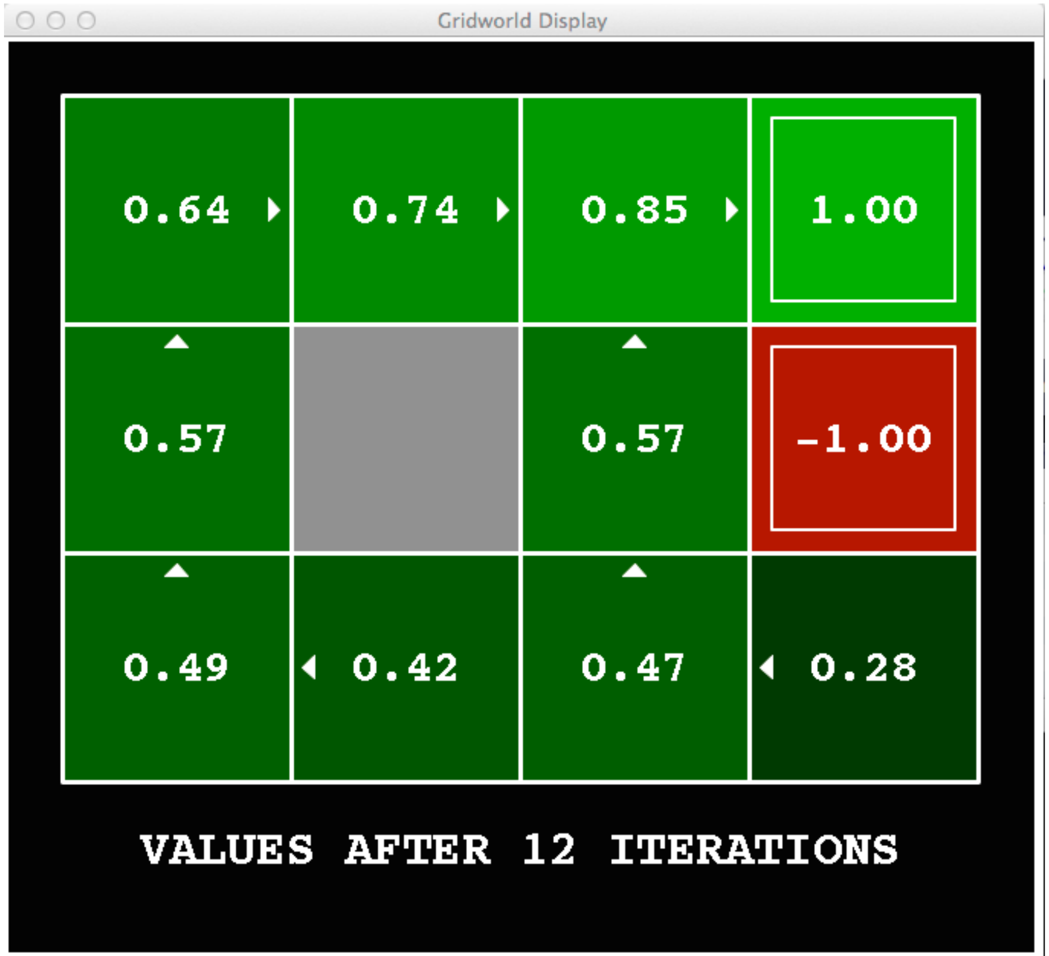


Noise = 0.2
Discount = 0.9
Living reward = 0

k=11

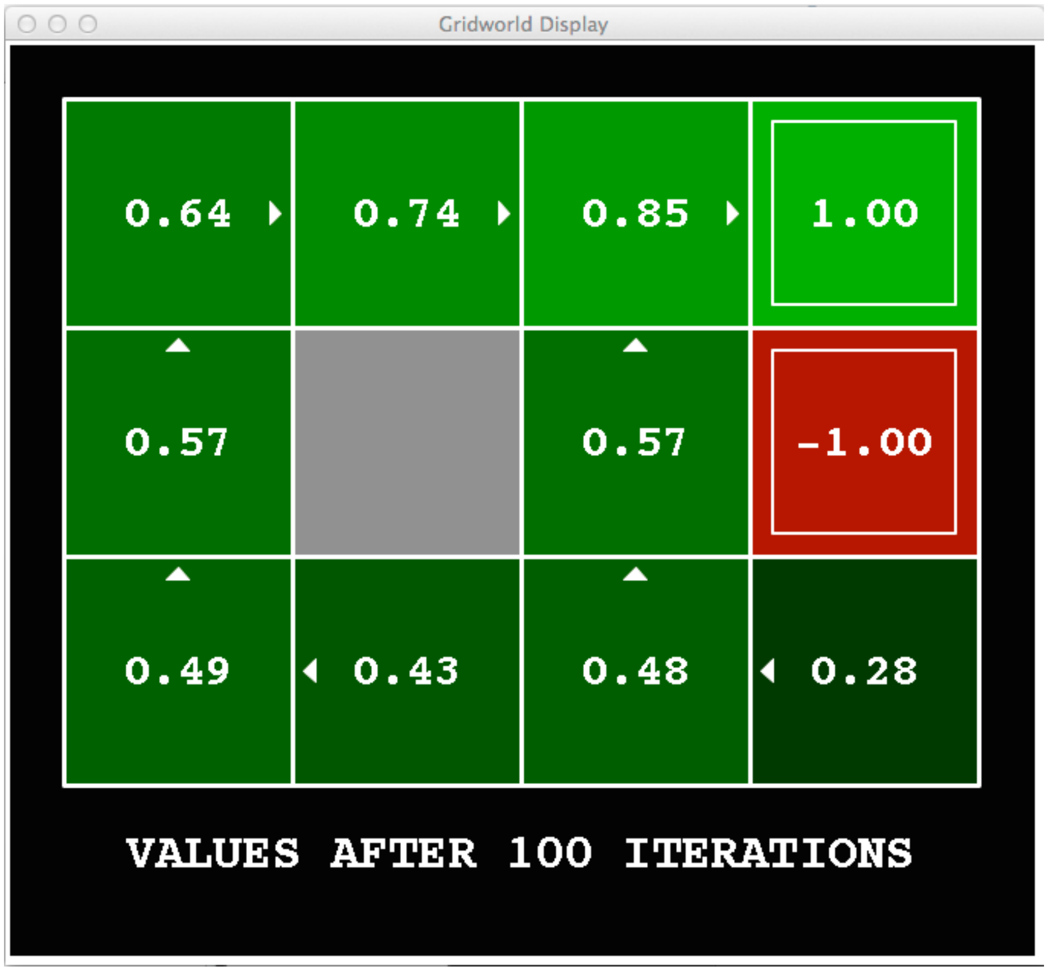
Noise = 0.2
Discount = 0.9
Living reward = 0

k=12



Noise = 0.2
Discount = 0.9
Living reward = 0

k=100



Noise = 0.2
Discount = 0.9
Living reward = 0

Convergence

How do we know the V_k vectors are going to converge?

Case 1: If the tree has maximum depth M, then V_M holds the actual untruncated values

Case 2: If the discount is less than 1

- Sketch: For any state V_k and V_{k+1} can be viewed as depth k+1 expectimax results in nearly identical search trees
- The difference is that on the bottom layer, V_{k+1} has actual rewards while V_k has zeros
- That last layer is at best all R_{MAX}
- It is at worst R_{MIN}
- But everything is discounted by γ^k that far out
- So V_k and V_{k+1} are at most γ^k max|R| different
- So as k increases, the values converge