Warm-up as you walk in

Given these N=10 observations of the world:

What is the approximate value for
$P(-c \mid -a, +b)$?

A. 1/10
B. 5/10
C. 1/4
D. 1/5
E. I'm not sure

$-a, -b, +c$
$+a, -b, +c$
$-a, -b, +c$
$-a, +b, +c$
$+a, -b, +c$
$-a, +b, -c$
$-a, +b, +c$
$-a, +b, +c$
$+a, -b, +c$
$-a, +b, +c$

Counts

+a	+b	+c	0
+a	+b	-c	0
+a	-b	+c	3
+a	-b	-c	0
-a	+b	+c	4
-a	+b	-c	1
-a	-b	+c	2
-a	-b	-c	0

Announcements

Midterm 2

- Regrade requests due by tomorrow

Assignments

- P4
  - Due Tonight! (or up until Saturday) 10pm
- HW9
  - Due Tues 4/11 10pm
- P5
  - Out 4/11, Due 4/27
- HW10
  - Out 4/18, Due Tues 4/25 10pm

Announcements

Final Exam

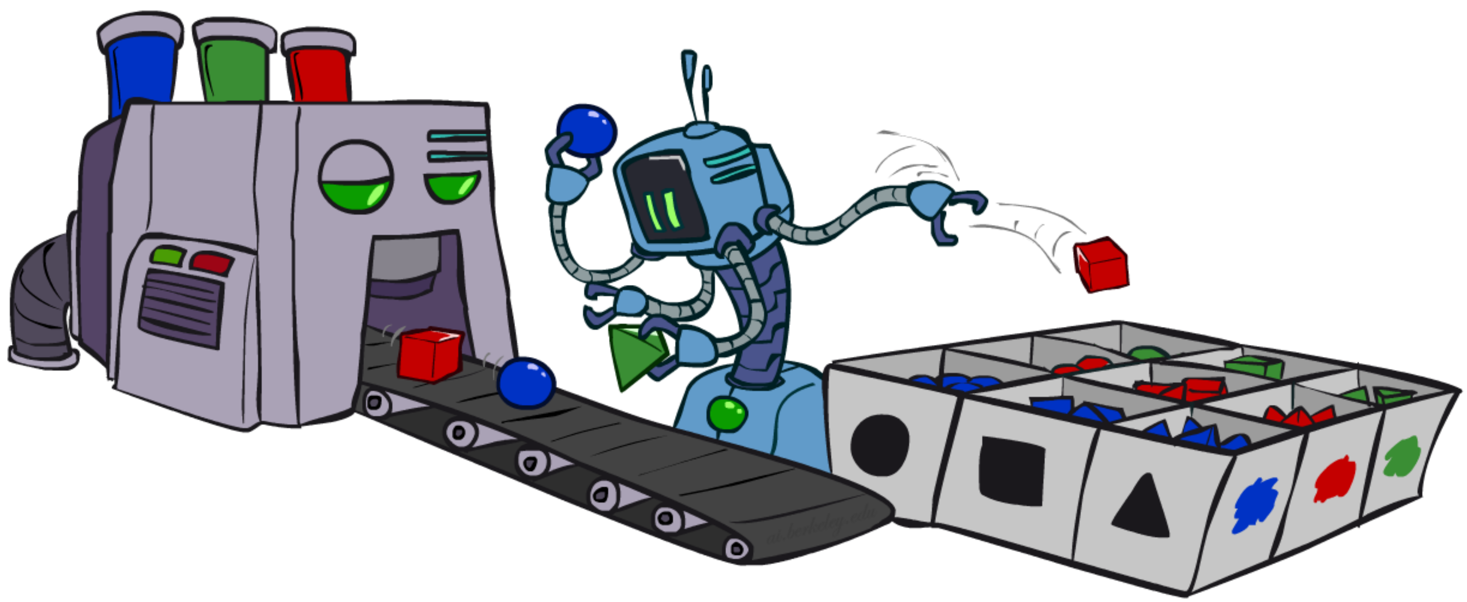
- Cumulative
  - Roughly half of final exam on final 1/3 of the course
  - The other half of the exam covering other topics
- 3 cheat sheets and calculator allowed

TA next semester!

- CSD application https://www.ugrad.cs.cmu.edu/ta/F23

AI: Representation and Problem Solving

Bayes Nets Sampling



Instructor: Stephanie Rosenthal

Slide credits: CMU AI and http://ai.berkeley.edu

Bayes Nets

✓ Part I: Representation

✓ Part II: Exact inference

✓ - Enumeration (always exponential complexity)

✓ - Variable elimination (worst-case exponential complexity, often better)

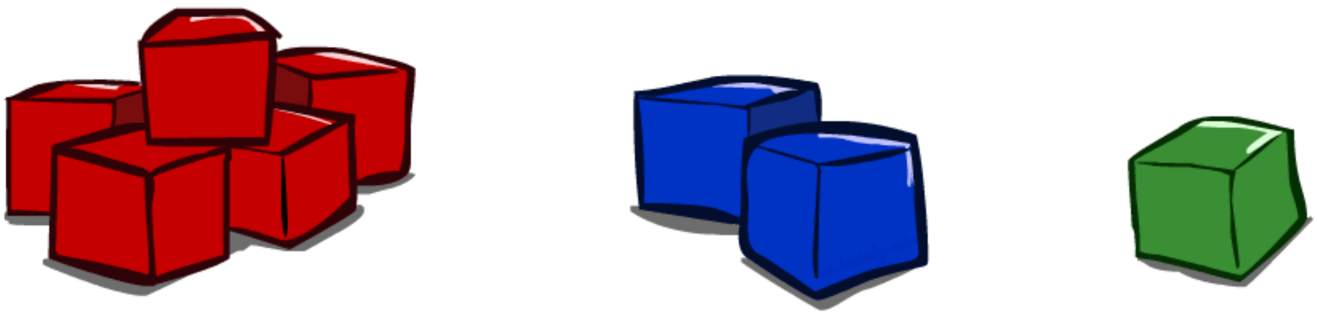
✓ - Inference is NP-hard in general

Part III: Approximate Inference

Inference vs Sampling

Motivation for Approximate Inference

Approximate Inference: Sampling



Warm-up as you walk in

Given these N=10 observations of the world:

What is the approximate value for
$P(-c|\ -a,+b)$?

A. 1/10
B. 5/10
C. 1/4
D. 1/5
E. I'm not sure

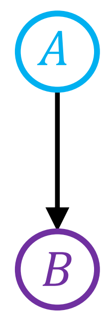
$-a,-b,+c$
$+a,-b,+c$
$-a,-b,+c$
$-a,+b,+c$
$+a,-b,+c$
$-a,+b,-c$
$-a,+b,+c$
$-a,+b,+c$
$+a,-b,+c$
$-a,+b,+c$

Counts

+a	+b	+c	0
+a	+b	-c	0
+a	-b	+c	3
+a	-b	-c	0
-a	+b	+c	4
-a	+b	-c	1
-a	-b	+c	2
-a	-b	-c	0

Sampling

How would you sample from a conditional distribution?



$P(A)$

+a	1/2
-a	1/2

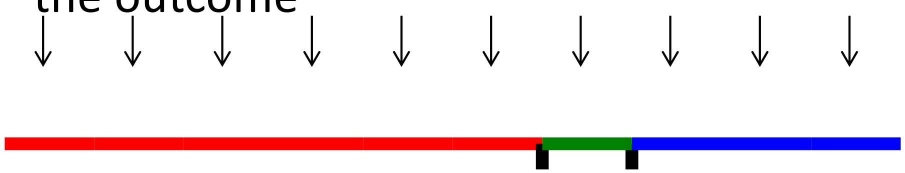
$P(B|A)$

+a	+b	1/10
	-b	9/10
-a	+b	1/2
	-b	1/2

Sampling

Sampling from given distribution

- Step 1: Get sample u from uniform distribution over [0, 1)
  - e.g. random() in python
- Step 2: Convert this sample u into an outcome for the given distribution by having each outcome associated with a sub-interval of [0,1) with sub-interval size equal to probability of the outcome

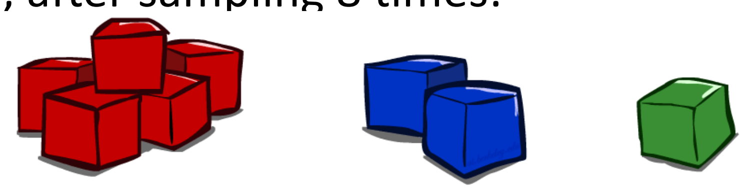


Example

C	P(C)
red	0.6
green	0.1
blue	0.3

$$0 \leq u < 0.6, \rightarrow C = red$$
$$0.6 \leq u < 0.7, \rightarrow C = green$$
$$0.7 \leq u < 1, \rightarrow C = blue$$

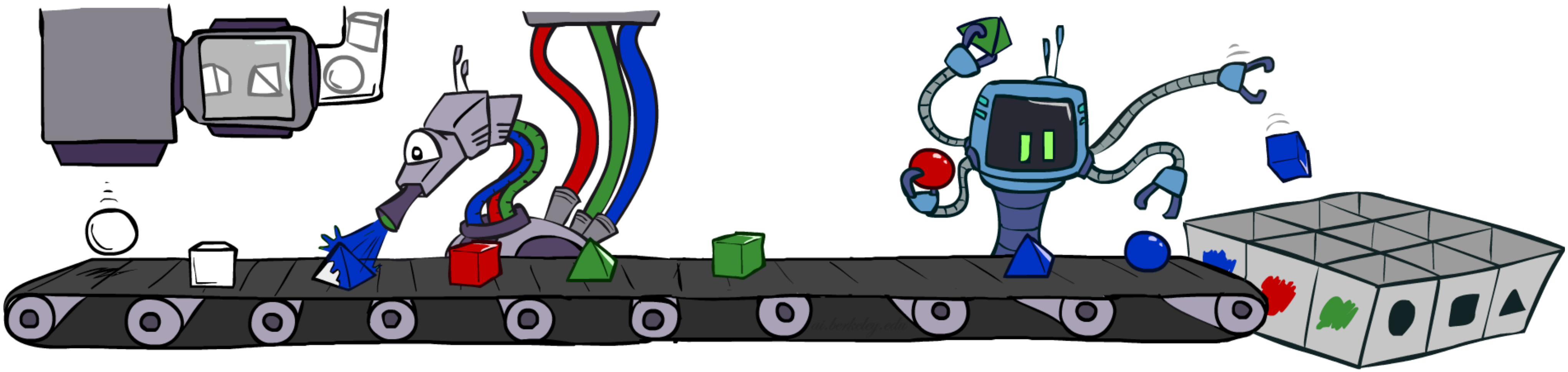
- If random() returns u = 0.83, then our sample is C = blue
- E.g, after sampling 8 times:



Sampling in Bayes' Nets

- Prior Sampling
- Rejection Sampling
- Likelihood Weighting
- Gibbs Sampling

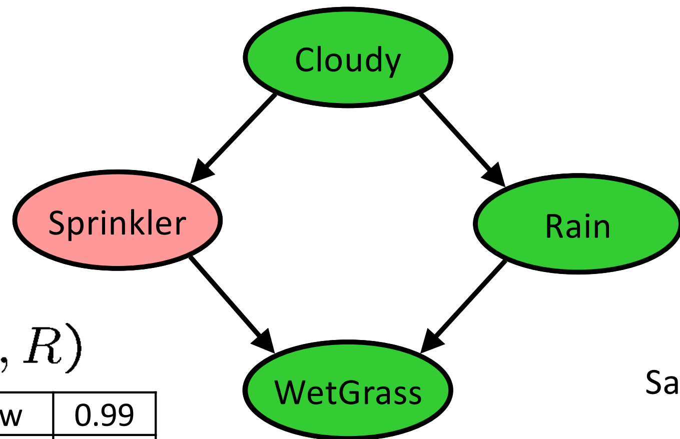
Prior Sampling



Prior Sampling

$P(C)$

+c	0.5
-c	0.5



$P(S|C)$

+c	+s	0.1
	-s	0.9
-c	+s	0.5
	-s	0.5

$P(R|C)$

+c	+r	0.8
	-r	0.2
-c	+r	0.2
	-r	0.8

$P(W|S,R)$

+s	+r	+w	0.99
		-w	0.01
	-r	+w	0.90
		-w	0.10
-s	+r	+w	0.90
		-w	0.10
	-r	+w	0.01
		-w	0.99

Samples:

+c, -s, +r, +w

-c, +s, -r, +w

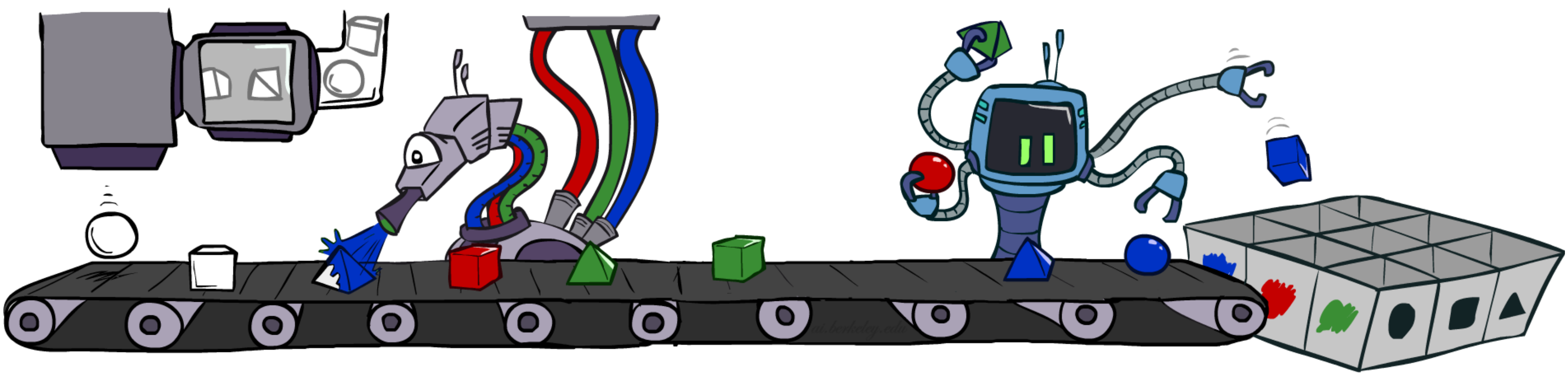
…

Prior Sampling

For i=1, 2, …, n

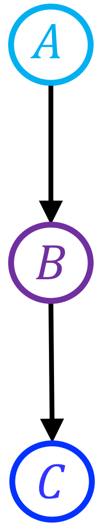
- Sample x_i from $P(X_i \mid Parents(X_i))$

Return $(x_1, x_2, \ldots, x_n)$



Poll 1

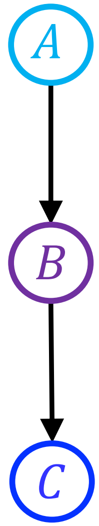
Prior Sampling: What does the value $\frac{N(+a,-b,+c)}{N}$ approximate?



A. $P(+a, -b, +c)$
B. $P(+c \mid +a, -b)$
C. $P(+c \mid -b)$
D. $P(+c)$
E. I don't know

Poll 1

Prior Sampling: What does the value $\frac{N(+a,-b,+c)}{N}$ approximate?

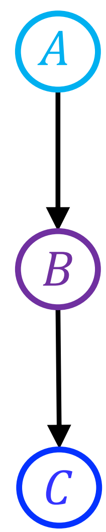


A. $P(+a,-b,+c)$
B. $P(+c \mid +a,-b)$
C. $P(+c \mid -b)$
D. $P(+c)$
E. I don't know

Poll 2

How many $\{-a, +b, -c\}$ samples out of N=1000 should we expect?

A. 1
B. 50
C. 125
D. 200
E. I have no idea



$P(A)$

+a	1/2
-a	1/2

$P(B|A)$

+a	+b	1/10
	-b	9/10
-a	+b	1/2
	-b	1/2

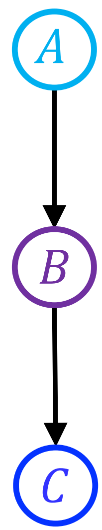
$P(C|B)$

+b	+c	4/5
	-c	1/5
-b	+c	1
	-c	0

Poll 2

How many $\{-a, +b, -c\}$ samples out of N=1000 should we expect?

A. 1
B. 50
C. 125
D. 200
E. I have no idea



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-a	+b	1/2
	-b	1/2

$P(C|B)$

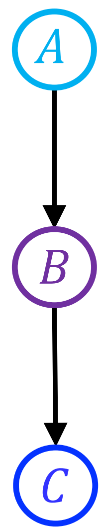
+b	+c	4/5
	-c	1/5
-b	+c	1
	-c	0

Probability of a sample

Given this Bayes Net & CPT,
what is $P(+a, +b, +c)$?

Algorithm: Multiply probability of each node given parents:

- w = 1.0
- for i=1, 2, …, n
  - Set w = w * $P(x_i \mid Parents(X_i))$
- return w



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-a	+b	1/2
	-b	1/2

$P(C|B)$

+b	+c	4/5
	-c	1/5
-b	+c	1
	-c	0

Prior Sampling

This process generates samples with probability:

$$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i|\text{Parents}(X_i)) = P(x_1 \ldots x_n)$$

…i.e. the BN's joint probability

Let the number of samples of an event be $N_{PS}(x_1 \ldots x_n)$

Then
$$\begin{aligned} \lim_{N\to\infty} \widehat{P}(x_1,\ldots,x_n) &= \lim_{N\to\infty} N_{PS}(x_1,\ldots,x_n)/N \\ &= S_{PS}(x_1,\ldots,x_n) \\ &= P(x_1 \ldots x_n) \end{aligned}$$

$S_{PS}(x_1 \ldots x_n)$ = the probability that a sample $(x_1 \ldots x_n)$ is drawn using Prior Sampling

Example

We'll get a bunch of samples from the BN:

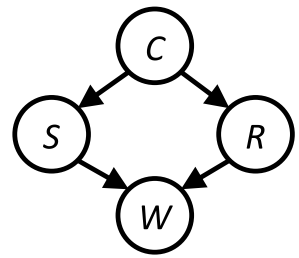
+c, -s, +r, +w

+c, +s, +r, +w

-c, +s, +r, -w

+c, -s, +r, +w

-c, -s, -r, +w



If we want to know P(W)

- We have counts <+w:4, -w:1>
- Normalize to get P(W) = <+w:0.8, -w:0.2>
- This will get closer to the true distribution with more samples
- Can estimate anything else, too
- What about P(C| +w)?   P(C| +r, +w)? P(C| -r, -w)?
- Fast: can use fewer samples if less time (what's the drawback?)

Practice Prior Sampling

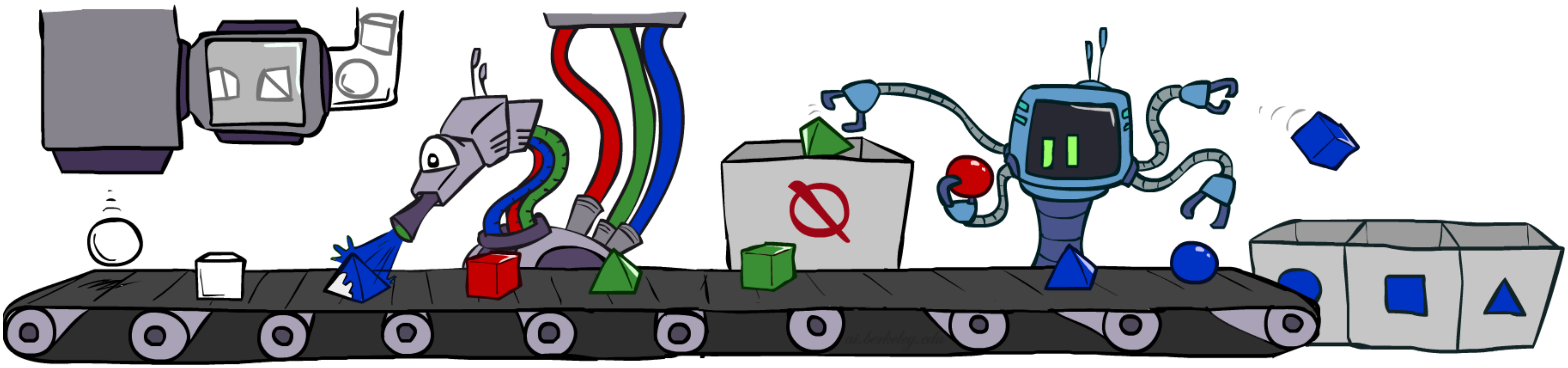
Use a random number generator to pick a number between 0 and 100, divide random number by 100 to get a probability

Check the relevant probability distribution to determine what value is selected.

Continue selecting values for each variable until you have a single sample

Record the sample, repeat the process for 9 more samples

Rejection Sampling



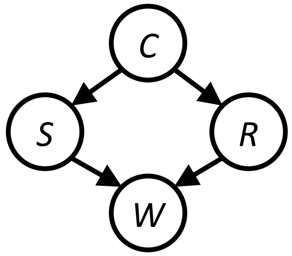
Rejection Sampling

Let’s say we want P(C)

- No point keeping all samples around
- Just tally counts of C as we go

Let’s say we want P(C| +s)

- Same thing: tally C outcomes, but ignore (reject) samples which don’t have S=+s
- This is called rejection sampling
- It is also consistent for conditional probabilities (i.e., correct in the limit)



+c, -s, +r, +w
+c, +s, +r, +w
-c, +s, +r, -w
+c, -s, +r, +w
-c, -s, -r, +w

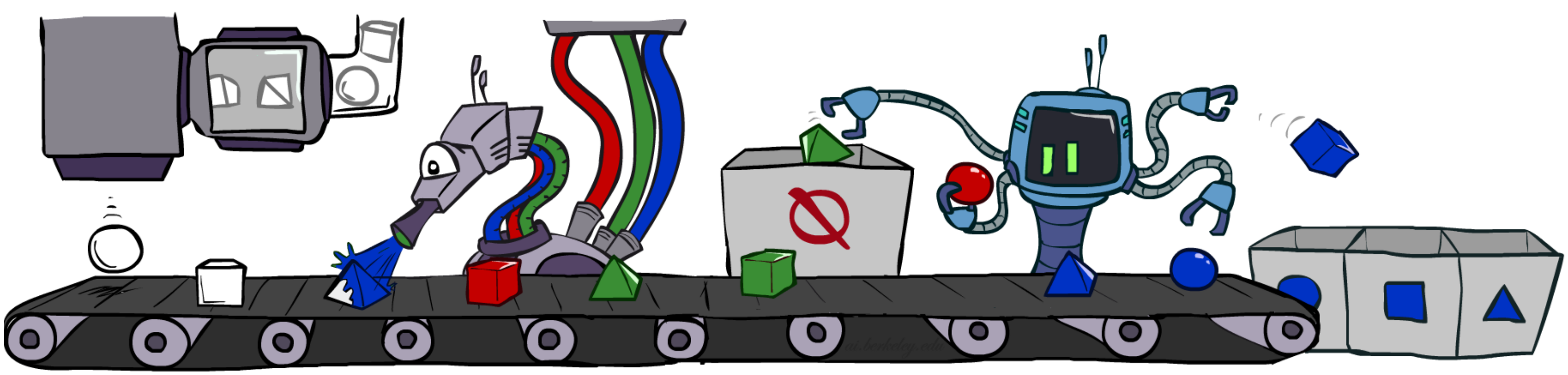
Rejection Sampling

IN: evidence instantiation

For i=1, 2, …, n

- Sample x_i from $P(X_i \mid Parents(X_i))$
- If x_i not consistent with evidence
  - Reject: Return, and no sample is generated in this cycle

Return $(x_1, x_2, \ldots, x_n)$



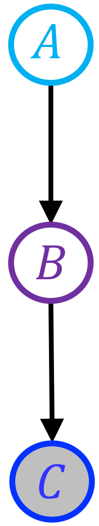
Poll 3

What queries can we (approximately) answer with rejection sampling samples (evidence: $+c$)?

A. $P(+a, -b, +c)$
B. $P(+a, -b \mid +c)$
C. Both
D. Neither
E. I have no idea

Counts $N(A, B, C)$

+a	+b	+c	4
+a	+b	-c	
+a	-b	+c	3
+a	-b	-c	
-a	+b	+c	2
-a	+b	-c	
-a	-b	+c	1
-a	-b	-c	



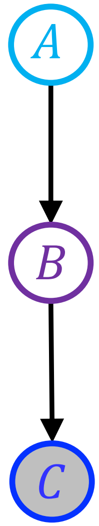
Poll 3

What queries can we (approximately) answer with rejection sampling samples (evidence: $+c$)?

A. $P(+a, -b, +c)$
B. $P(+a, -b \mid +c)$
C. Both ← If we also have total number of attempts
D. Neither
E. I have no idea

Counts $N(A, B, C)$

+a	+b	+c	4
+a	+b	-c	
+a	-b	+c	3
+a	-b	-c	
-a	+b	+c	2
-a	+b	-c	
-a	-b	+c	1
-a	-b	-c	



Practice Rejection Sampling

Use a random number generator to pick a number between 0 and 100, divide random number by 100 to get a probability

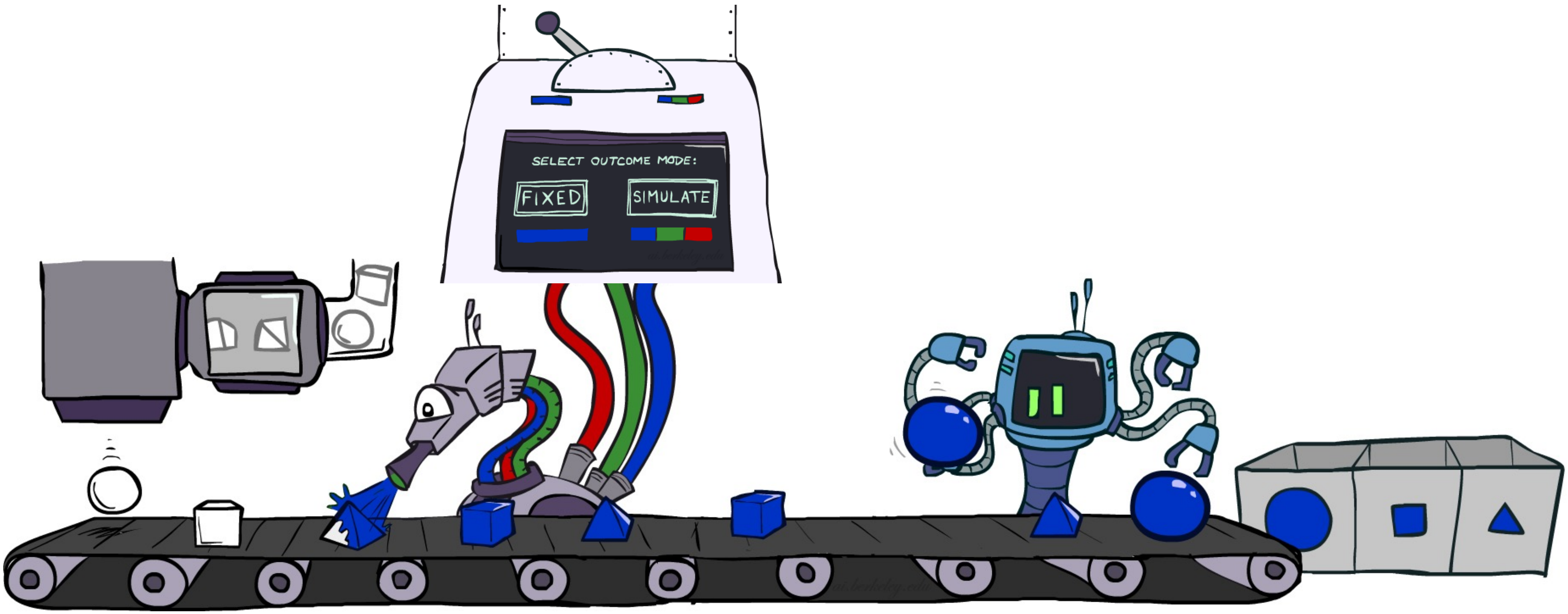
Check the relevant probability distribution to determine what value is selected.

REJECT ENTIRE SAMPLE IF VALUE IS INCONSISTENT WITH EVIDENCE

Continue selecting values for each variable until you have a single sample

Record the sample, repeat the process for 9 more samples

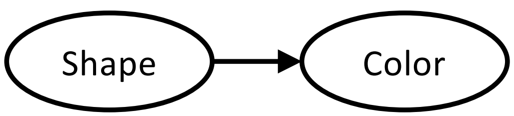
Likelihood Weighting



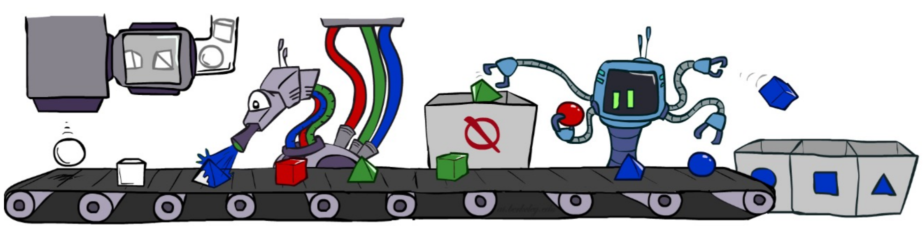
Likelihood Weighting

Problem with rejection sampling:

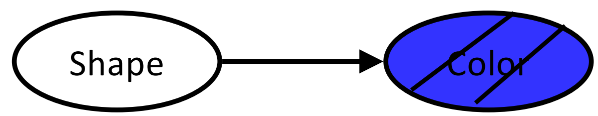
- If evidence is unlikely, rejects lots of samples
- Evidence not exploited as you sample
- Consider P(Shape|blue)



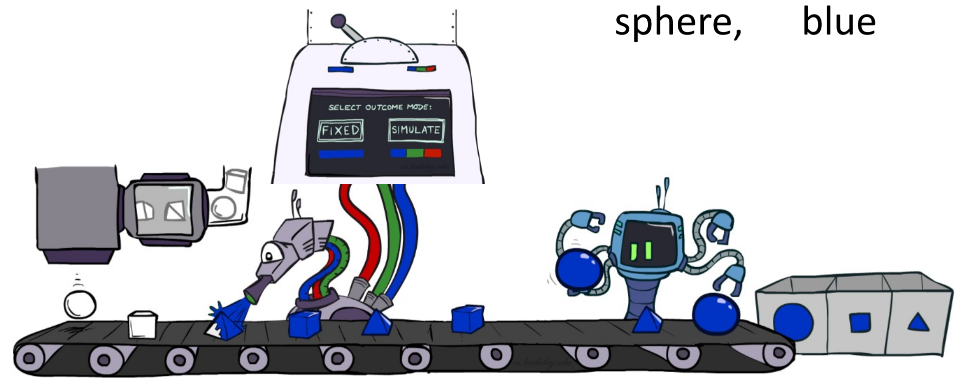
~~pyramid, green~~
~~pyramid, red~~
sphere, blue
~~cube, red~~
~~sphere, green~~



- Idea: fix evidence variables and sample the rest
  - Problem: sample distribution not consistent!
  - Solution: weight by probability of evidence given parents



pyramid, blue
pyramid, blue
sphere, blue
cube, blue
sphere, blue



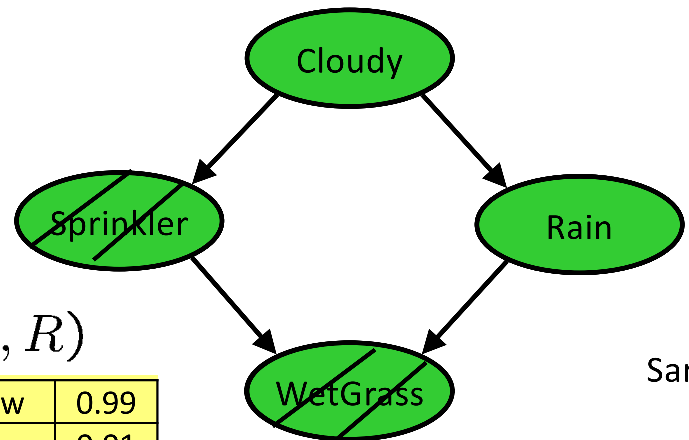
Likelihood Weighting

$P(C)$

+c	0.5
-c	0.5

$P(S|C)$

+c	+s	0.1
	-s	0.9
-c	+s	0.5
	-s	0.5



$P(R|C)$

+c	+r	0.8
	-r	0.2
-c	+r	0.2
	-r	0.8

$P(W|S,R)$

+s	+r	+w	0.99
		-w	0.01
	-r	+w	0.90
		-w	0.10
-s	+r	+w	0.90
		-w	0.10
	-r	+w	0.01
		-w	0.99

Samples:

+c, +s, +r, +w

...

$$w = 1.0 \times 0.1 \times 0.99$$

Likelihood Weighting

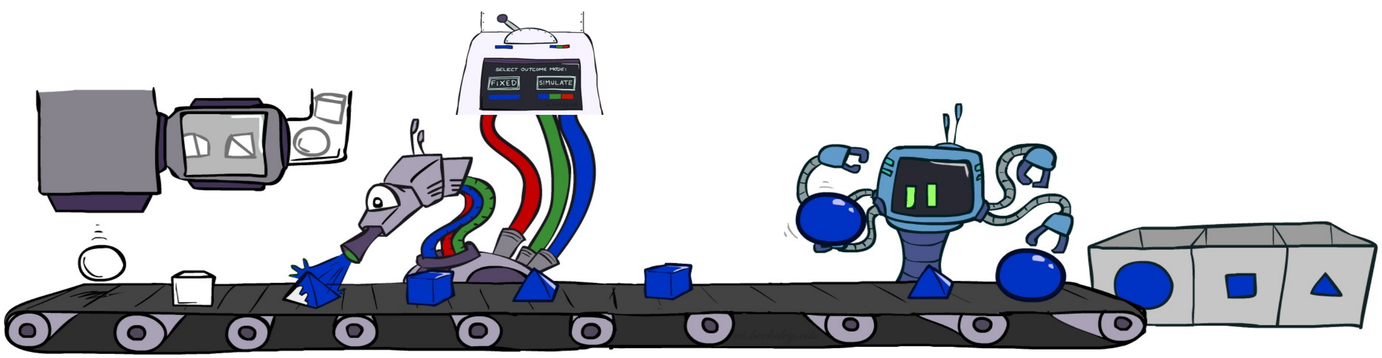
IN: evidence instantiation

w = 1.0

for i=1, 2, …, n

- if X_i is an evidence variable
  - X_i = observation x_i for X_i
  - Set w = w * $P(x_i \mid Parents(X_i))$
- else
  - Sample x_i from $P(X_i \mid Parents(X_i))$

return $(x_1, x_2, …, x_n)$, w



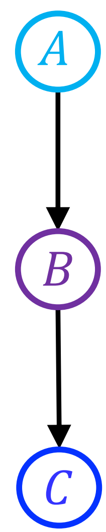
Likelihood Weighting

No evidence: Prior Sampling	Some evidence: Likelihood Weighted Sampling	All evidence: Likelihood Weighted
Input: no evidence for i=1, 2, …, n - Sample x_i from $P(X_i \mid Parents(X_i))$ return $(x_1, x_2, \ldots, x_n)$	Input: evidence instantiation w = 1.0 for i=1, 2, …, n if X_i is an evidence variable - X_i = observation x_i for X_i - Set $w = w * P(x_i \mid Parents(X_i))$ else - Sample x_i from $P(X_i \mid Parents(X_i))$ return $(x_1, x_2, \ldots, x_n)$, w	Input: evidence instantiation w = 1.0 for i=1, 2, …, n - Set $w = w * P(x_i \mid Parents(X_i))$ return w

Remember Poll 2

How many $\{-a, +b, -c\}$ samples out of N=1000 should we expect?

A. 1
B. 50
C. 125
D. 200
E. I have no idea



$P(A)$

+a	1/2
-a	1/2

$P(B|A)$

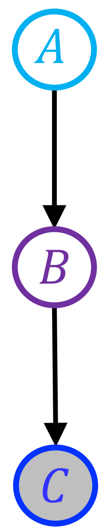
+a	+b	1/10
	-b	9/10
-a	+b	1/2
	-b	1/2

$P(C|B)$

+b	+c	4/5
	-c	1/5
-b	+c	1
	-c	0

Likelihood Weighting

How many $\{-a, +b, -c\}$ samples out of N=1000 should we expect?



$P(A)$

+a	1/2
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$P(B|A)$

+a	+b	1/10
	-b	9/10
-a	+b	1/2
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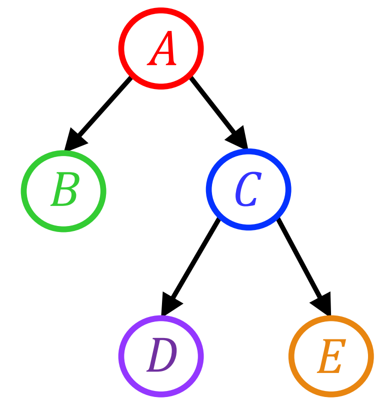
$P(C|B)$

+b	+c	4/5
	-c	1/5
-b	+c	1
	-c	0

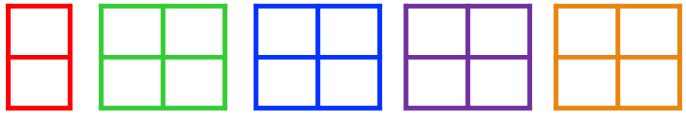
Likelihood Weighting Distribution



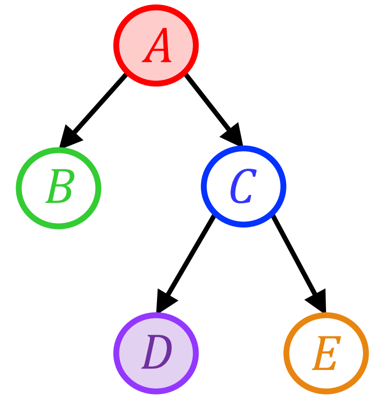
Consistency of likelihood weighted sampling distribution

Joint from Bayes nets

$$P(A, B, C, D, E) = P(A)\, P(B|A)\, P(C|A)\, P(D|C)\, P(E|C)$$



Likelihood Weighting Distribution

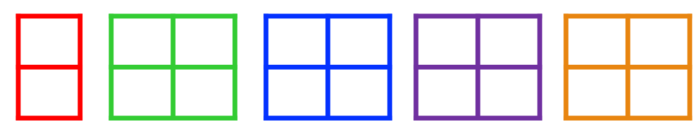


Consistency of likelihood weighted sampling distribution

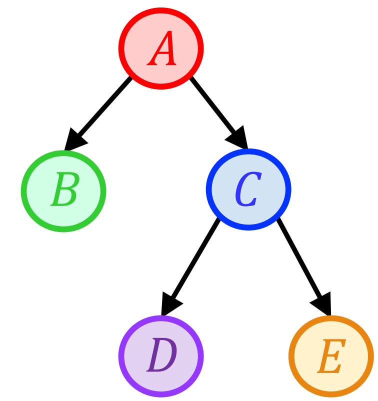
Evidence: $+a$, $-d$

Joint from Bayes nets

$$P(A, B, C, D, E) = P(+a)\ P(B|+a)\ P(C|+a)\ P(-d|C)\ P(E|C)$$



Likelihood Weighting Distribution

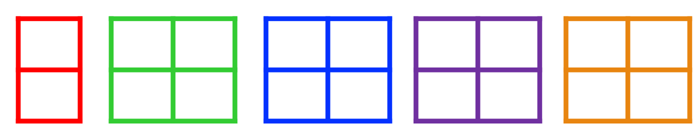


Consistency of likelihood weighted sampling distribution

Evidence: $+a, +b, -c, -d, +e$

Joint from Bayes nets

$$P(A, B, C, D, E) = P(+a)\ P(+b|+a)\ P(-c|+a)\ P(-d|-c)\ P(+e|-c)$$



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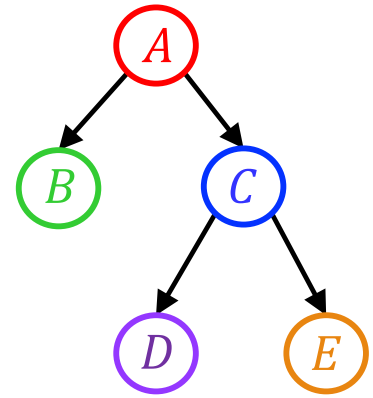
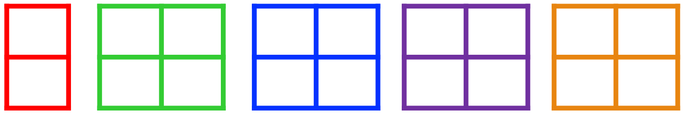


Consistency of likelihood weighted sampling distribution

Evidence: None

Joint from Bayes nets

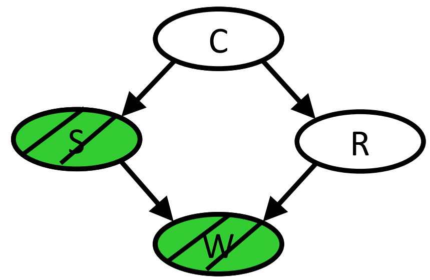
$$P(A, B, C, D, E) = P(A)\, P(B|A)\, P(C|A)\, P(D|C)\, P(E|C)$$



Likelihood Weighting

Sampling distribution if z sampled and e fixed evidence

$$S_{WS}(\mathbf{z},\mathbf{e}) = \prod_{i=1}^{l} P(z_i|\text{Parents}(Z_i))$$



Now, samples have weights

$$w(\mathbf{z},\mathbf{e}) = \prod_{i=1}^{m} P(e_i|\text{Parents}(E_i))$$

Together, weighted sampling distribution is consistent

$$S_{\text{WS}}(z,e) \cdot w(z,e) = \prod_{i=1}^{l} P(z_i|\text{Parents}(z_i)) \prod_{i=1}^{m} P(e_i|\text{Parents}(e_i))$$

$$= P(\mathbf{z},\mathbf{e})$$

Poll 4

Given a fixed query, two identical samples from likelihood weighted sampling will have the same exact weights.

A. True
B. False
C. It depends
D. I don’t know

Poll 5

What does the following likelihood weighted value approximate?

$$\text{weight}_{(+a,-b,+c)} \cdot \frac{N(+a,-b,+c)}{N}$$

A. $P(+a, -b, +c)$
B. $P(+a, -b \mid +c)$
C. I'm not sure

Practice Likelihood Weighted Sampling

For each variable:

If evidence variable: you know the value, and multiply the weight of the sample by P(evidence|parents)

Else: Use a random number generator to pick a number between 0 and 100, divide random number by 100 to get a probability. Check the relevant probability distribution to determine what value is selected.

Continue selecting values for each variable until you have a single sample

Record the sample, repeat the process for 9 more samples

The probability for a given set of variables equals

(COUNT OF SAMPLES * WEIGHT)/(TOTAL COUNT OF SAMPLES)

Likelihood Weighting

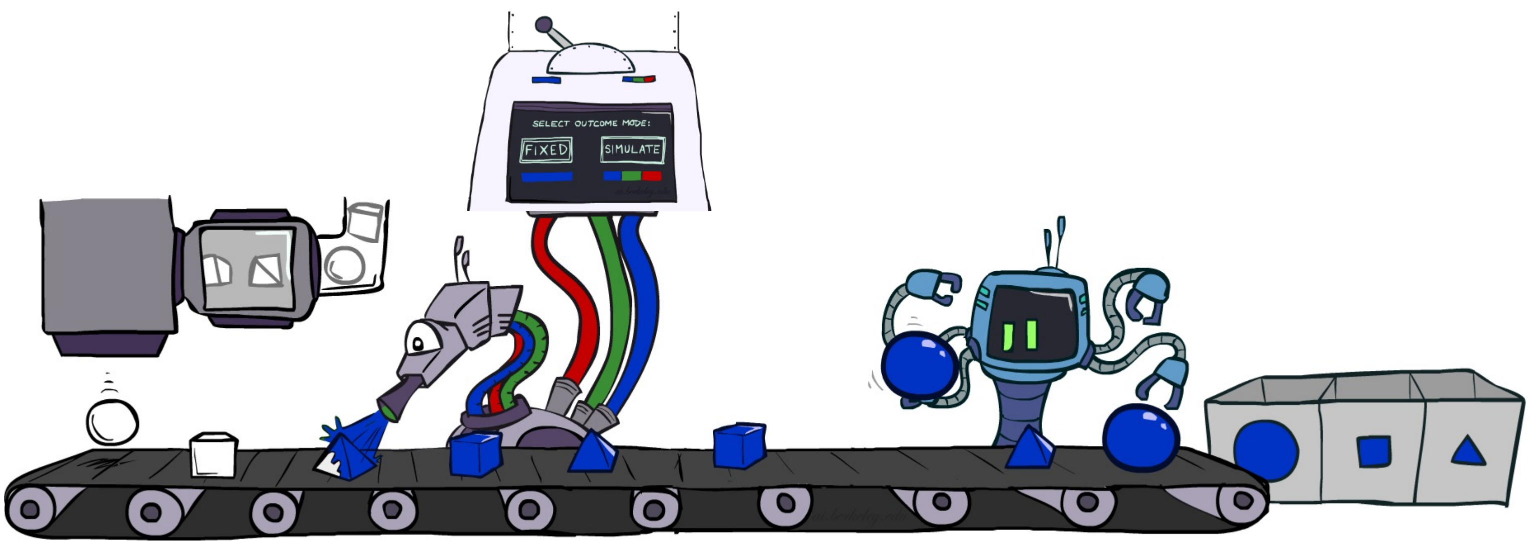
Likelihood weighting is good

- We have taken evidence into account as we generate the sample
- E.g. here, W's value will get picked based on the evidence values of S, R
- More of our samples will reflect the state of the world suggested by the evidence

Likelihood weighting doesn't solve all our problems

- Evidence influences the choice of downstream variables, but not upstream ones (C isn't more likely to get a value matching the evidence)

We would like to consider evidence when we sample every variable



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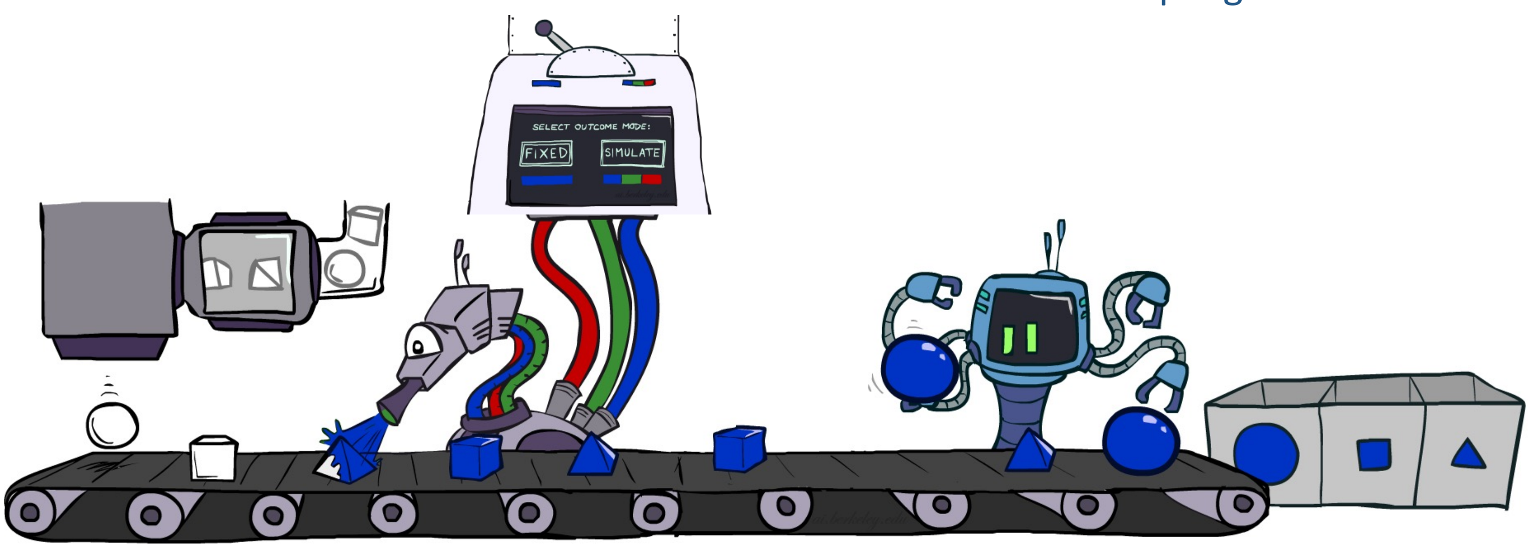
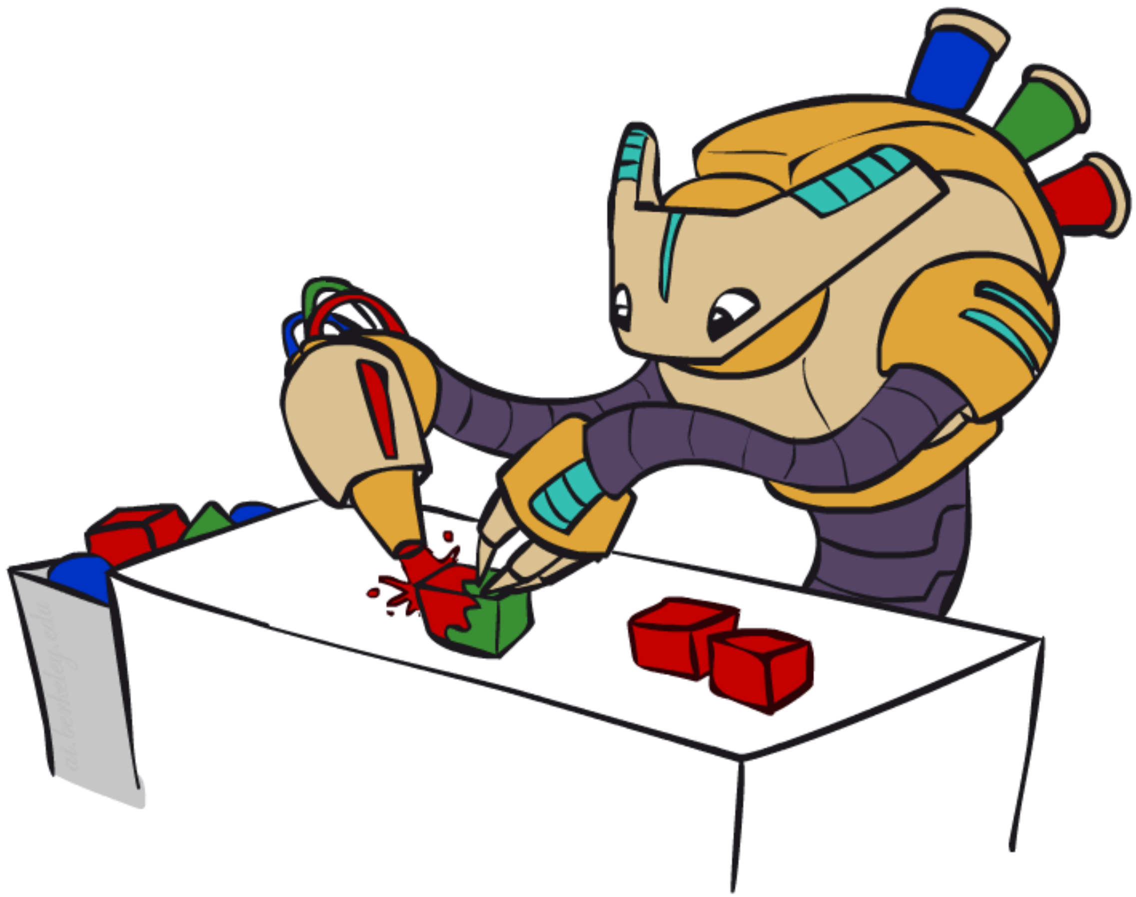
→ Gibbs sampling



Gibbs Sampling



Gibbs Sampling

Iteration Procedure: keep track of a full instantiation $x_1, x_2, \ldots, x_n$.

1. Start with an arbitrary instantiation consistent with the evidence.
2. Sample one variable at a time, conditioned on all the rest, but keep evidence fixed.
3. Keep repeating this for a long time.

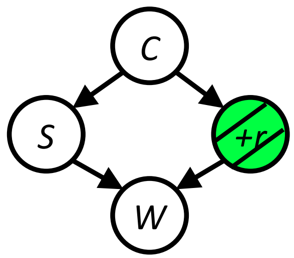
When done, keep last values of variables as 1 sample.

Repeat iteration for each sample.

Gibbs Sampling Example: P(S | +r)

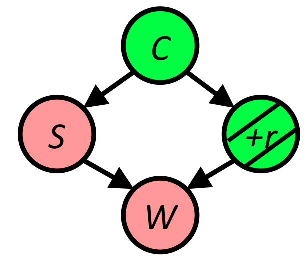
Step 1: Fix evidence

- R = +r



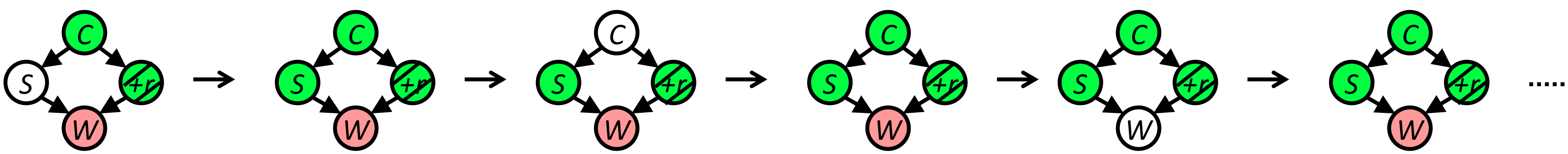
Step 2: Initialize other variables

- Randomly



Steps 3: Repeat

- Choose a non-evidence variable X
- Resample X from P(X | all other variables)



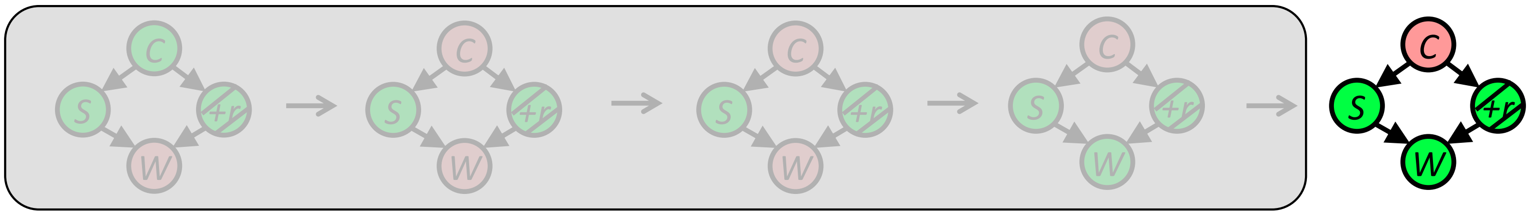
Sample from $P(S|+c, -w, +r)$ Sample from $P(C|+s, -w, +r)$ Sample from $P(W|+s, +c, +r)$

Gibbs Sampling Example: P(S | +r)

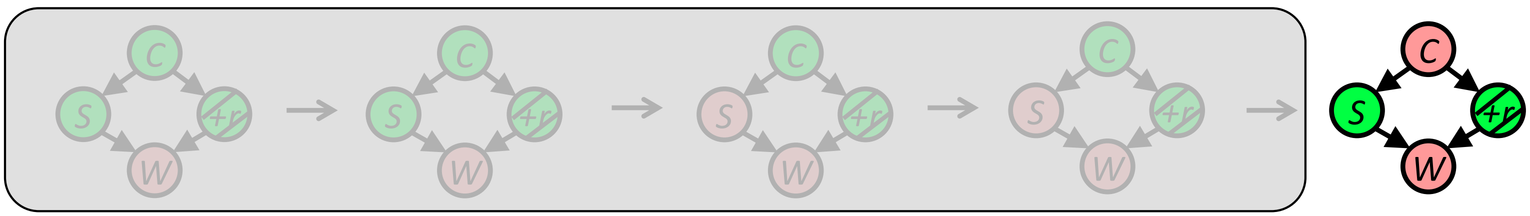
Each iteration here is repeated 5 times.

Keep only the last sample from each iteration:

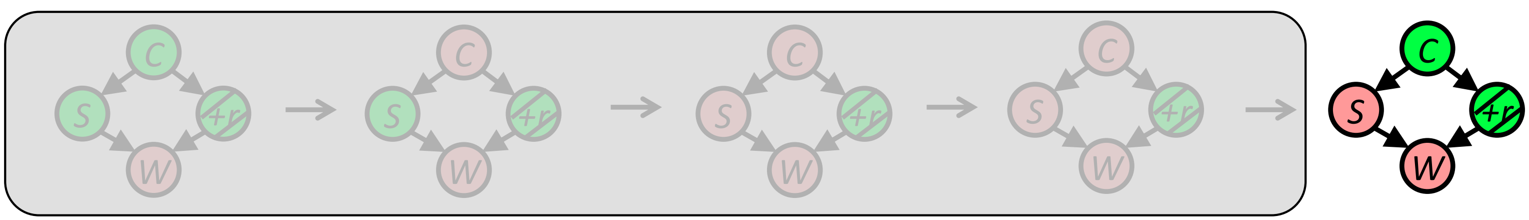
1.



2.

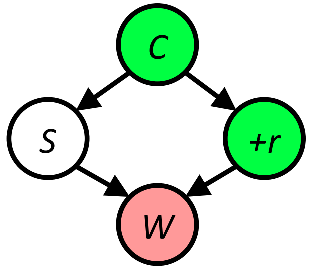


3.



Efficient Resampling of One Variable

Sample from P(S | +c, +r, -w)



$$
\begin{aligned}
P(S|+c,+r,-w) &= \frac{P(S,+c,+r,-w)}{P(+c,+r,-w)} \\
&= \frac{P(S,+c,+r,-w)}{\sum_s P(s,+c,+r,-w)} \\
&= \frac{P(+c)P(S|+c)P(+r|+c)P(-w|S,+r)}{\sum_s P(+c)P(s|+c)P(+r|+c)P(-w|s,+r)} \\
&= \frac{P(+c)P(S|+c)P(+r|+c)P(-w|S,+r)}{P(+c)P(+r|+c)\sum_s P(s|+c)P(-w|s,+r)} \\
&= \frac{P(S|+c)P(-w|S,+r)}{\sum_s P(s|+c)P(-w|s,+r)}
\end{aligned}
$$

Many things cancel out – only CPTs with S remain!

More generally: only CPTs that have resampled variable need to be considered, and joined together

Practice Gibbs Sampling

Procedure: keep track of a full instantiation $x_1, x_2, \ldots, x_n$.

1. Start with an arbitrary instantiation consistent with the evidence.
2. Sample one variable at a time, conditioned on all the rest, but keep evidence fixed.
3. Iterate to repeat this for a long time.
4. Keep only the last set of variables as ONE sample
5. Repeat with new sample

Property: in the limit of repeating this infinitely many times the resulting sample is coming from the correct distribution

Gibbs Sampling

Property: in the limit of repeating this infinitely many times the resulting sample is coming from the correct distribution

Rationale: both upstream and downstream variables condition on evidence.

In contrast: likelihood weighting only conditions on upstream evidence, and hence weights obtained in likelihood weighting can sometimes be very small. Sum of weights over all samples is indicative of how many "effective" samples were obtained, so want high weight.

Further Reading on Gibbs Sampling

Gibbs sampling produces sample from the query distribution P(Q | e) in limit of re-sampling infinitely often

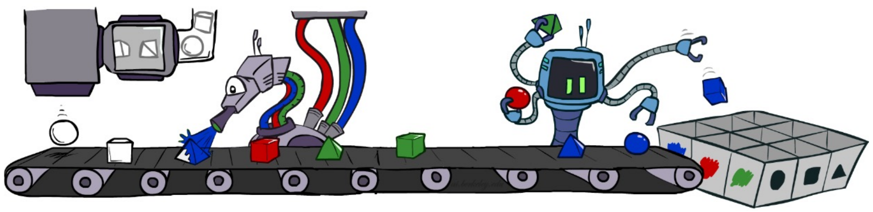
Gibbs sampling is a special case of more general methods called Markov chain Monte Carlo (MCMC) methods

- Metropolis-Hastings is one of the more famous MCMC methods (in fact, Gibbs sampling is a special case of Metropolis-Hastings)

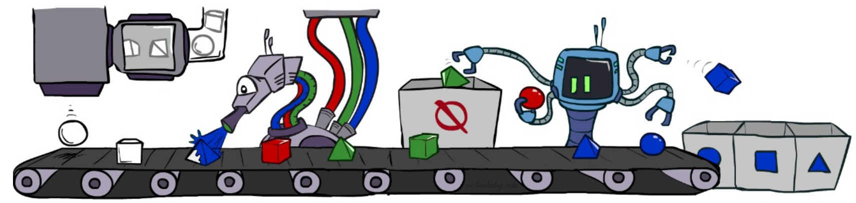
You may read about Monte Carlo methods – they're just sampling

Bayes' Net Sampling Summary

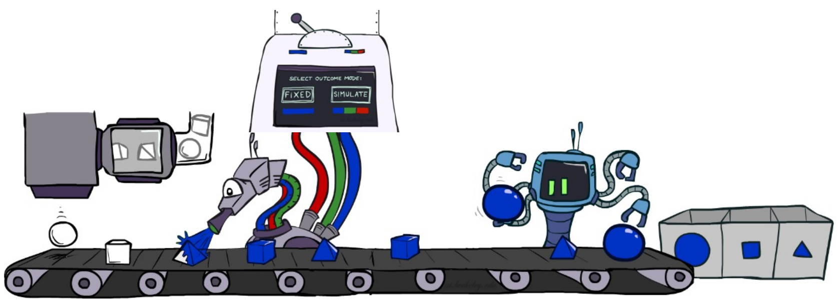
Prior Sampling P(Q, E)



Rejection Sampling P(Q | e)



Likelihood Weighting P(Q , e)



Gibbs Sampling P(Q | e)