Announcements

Feedback (take time now to fill it out):

- www.cmu.edu/hub/fce
- https://ugrad.cs.cmu.edu/ta/S23/feedback

Assignments:

- HW10 due tonight
- P5 due Thursday night

Final Exam:

- More heavily weighted towards the last 1/3 of the material
- All material in the class (lectures, activities, recitations, homeworks) are fair game for the final
- Look at post on Piazza with instructions

Homework 10 BONUS

We created a 10pt bonus HW10 problem so you can practice voting strategies on Gradescope

Due 5/4 at 4:30pm, no late days

Strong preference that you complete earlier!

Warm-up

Design an algorithm to determine the winner of three candidates a, b, c given the ranking provided by n individual voters, described by a $3 \times n$ matrix M

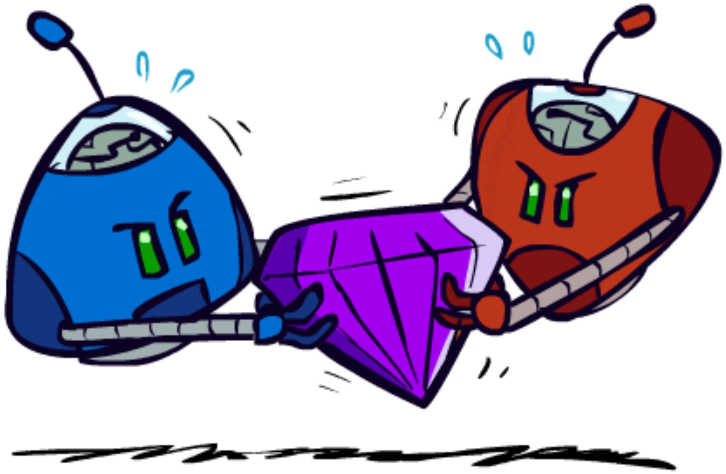
```
function voting($M$)
Input: $M$ where $M_{ij} \in$ {a, b, c} is the candidate at rank $j$ for voter $i$
Output: $x \in$ {a, b, c} describes the winner

Return $x$
```

Example Matrix M

	Voter 1	Voter 2	Voter 3	Voter 4
Rank 1	a	c	b	a
Rank 2	b	b	c	b
Rank 3	c	a	a	c

AI: Representation and Problem Solving

Game Theory: Equilibrium (cont) & Social Choice



Instructors: Stephanie Rosenthal

Slide credits: CMU AI, Fei Fang

Image credit: ai.berkeley.edu

Normal-Form Games

A game in normal form consists of the following elements

- Set of players
- Set of actions for each player
- Payoffs / Utility functions
    - Determines the utility for each player given the actions chosen by all players (referred to as action profile)
- Bimatrix game is special case: two players, finite action sets

Players move simultaneously and the game ends immediately afterwards

Find Mixed Strategy Nash Equilibrium

How to find mixed strategy NE (after iterative removal)?

Berry

		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

If $s_A = (p, 1-p)$ and $s_B = (q, 1-q)$ with $0 < p, q < 1$ is a NE, what are the necessary conditions for p and q?

$$u_A(F, s_B) = u_A(C, s_B) \qquad u_B(s_A, F) = u_B(s_A, C)$$

Why? Remember Theorem 1: A mixed strategy is BR iff all actions in the support are BR.

So…if $s_A \in BR(s_B)$, then $F \in BR(s_B)$and $C \in BR(s_B)$

Poll 1 (graded for accuracy)

If $s_A = (p, 1-p)$ and $s_B = (q, 1-q)$ with $0 < p, q < 1$ is a NE of the game, which equations should p and q satisfy?

A. $2q = 3(1-q)$

B. $2p = 3(1-p)$

C. $q = 2(1-q)$

D. $p = 2(1-p)$

E. $p = q$

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	3,2

Poll 2 (graded for accuracy)

If $s_A = (p, 1 - p)$ and $s_B = (q, 1 - q)$ with $0 < p, q < 1$ is a NE of the game, which equations should p and q satisfy?

A. $1p + 2(1 - p) = 2p + 1(1 - p)$

B. $1q + 2(1 - q) = 2q + 1(1 - q)$

C. $4q + 1(1 - q) = 3q + 3(1 - q)$

D. $4p + 1(1 - p) = 3p + 3(1 - p)$

E. $p = q$

Berry

Alex

	Football	Concert
Football	4,1	1,2
Concert	3,2	3,1

Power of Commitment

What is the PSNEs in this game and the players’ utilities?

What action should player 2 choose if player 1 commits to playing b? What is player 1’s utility?

What action should player 2 choose if player 1 commits to playing a and b uniformly randomly? What is player 1’s expected utility?

Player 2

Player 1

	c	d
a	2,1	4,0
b	1,0	3,2

Stackelberg Equilibrium

Stackelberg Game

- Leader commits to a strategy first
- Follower responds after observing the leader’s strategy

Stackelberg Equilibrium

- Follower best responds to leader’s strategy
- Leader commits to a strategy that maximize her utility assuming follower best responds

Player 2

Player 1

	c	d
a	2,1	4,0
b	1,0	3,2

Stackelberg Equilibrium

If the leader can only commit to a pure strategy, or you know that the leader's strategy in equilibrium is a pure strategy, the Stackelberg equilibrium can be found by enumerating leader's pure strategy

If ties for the follower are broken by the follower such that the leader benefits, the leader can exploit this. This is the strong Stackelberg equilibrium (SSE)

In general, the leader can commit to a mixed strategy

$u^{SSE} \geq u^{NE}$ (first-mover advantage)!

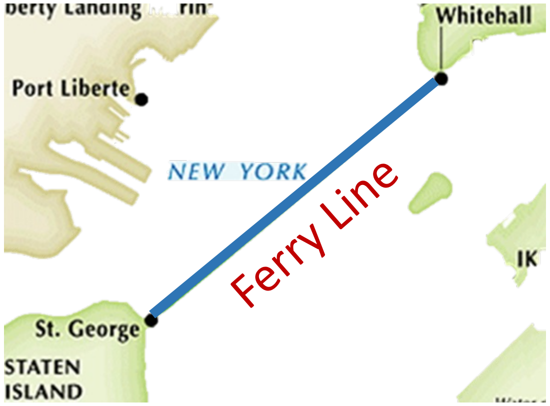
Berry

		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

Player 2

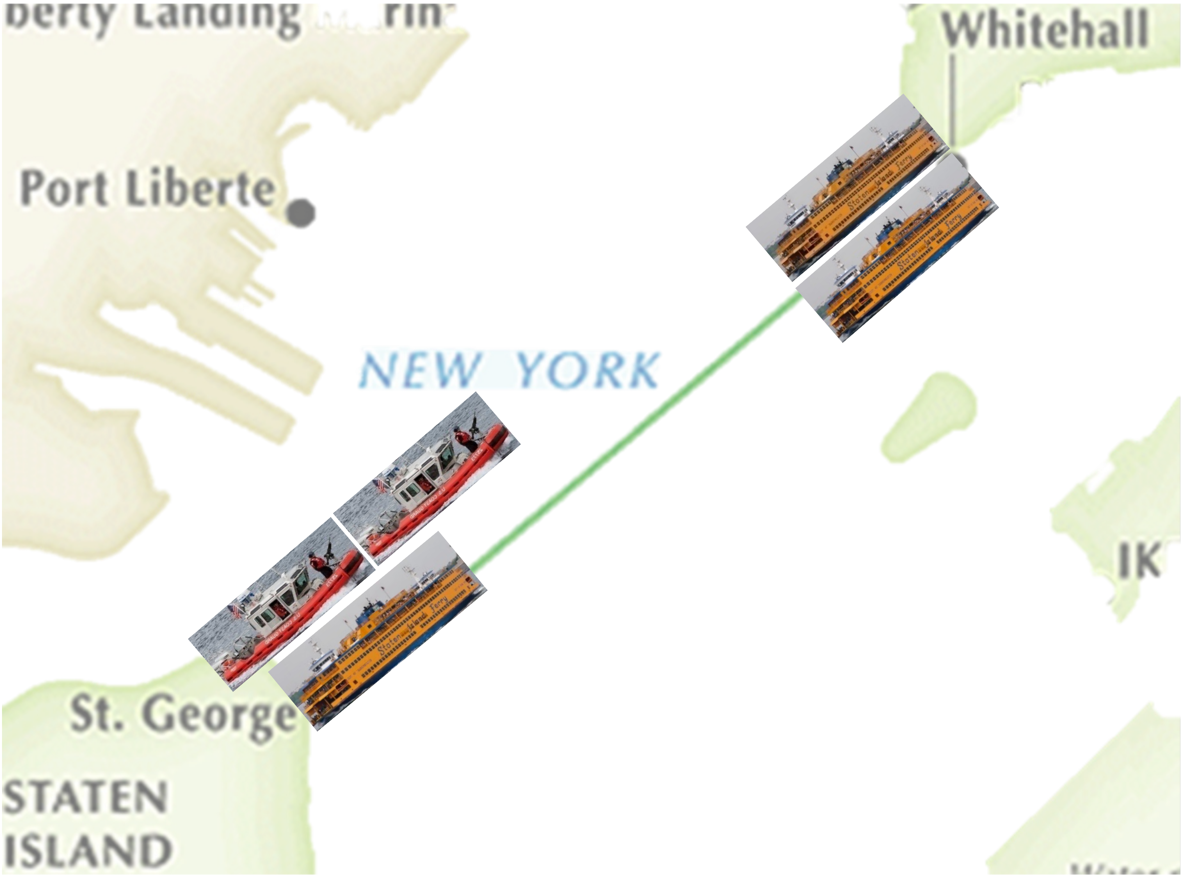
		c	d
Player 1	a	2,1	4,0
	b	1,0	3,2

Protecting Staten Island Ferry



Protecting Staten Island Ferry

Previous USCG Approach



Problem

Optimal Patrol Strategy for Protecting Moving Targets with Multiple Mobile Resources
Fei Fang, Albert Xin Jiang, Milind Tambe
In AAMAS-13: The Twelfth International Conference on Autonomous Agents and Multiagent Systems, May 2013

Game Model and Linear Programming-based Solution

Stackelberg game: Leader – Defender, Follower – Attacker

Attacker’s payoff: $u_i(t)$ if not protected, 0 otherwise

Zero-sum → Strong Stackelberg Equilibrium=Nash Equilibrium =Minimax (Minimize Attacker’s Maximum Expected Utility)

$$\min_{p_r,v} v$$

s.t. $v \geq \mathbb{E}[U^{att}(i,t)] = u_i(t)\times\mathbb{P}[unprotected(i,t)], \forall i,t$

Adversary

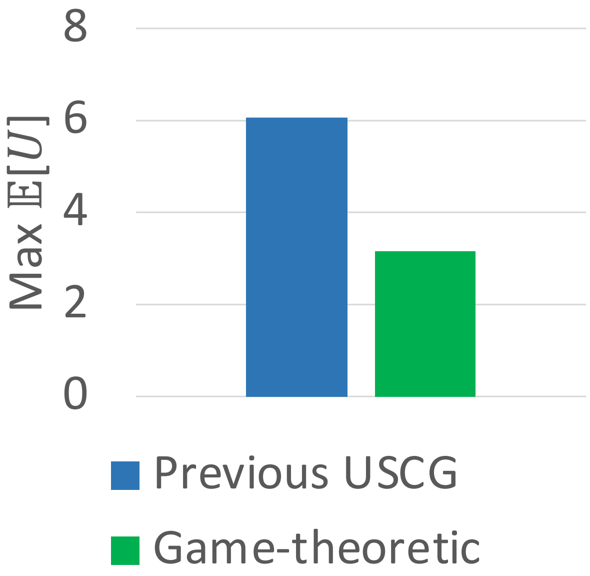
Defender

p_r		**10:00:00 AM** **Target 1**	**10:00:01 AM** **Target 1**	**…**	**10:30:00 AM** **Target 3**	**…**
30%	Purple Route	-5,5	-4,4		0,0	
40%	Orange Route					
20%	Blue Route					
	……					

$$\sum_r p_r \leq 1$$

Evaluation: Simulation & Real-World Feedback

Reduce potential risk by 50%



Deployed by US Coast Guard

USCG evaluation

- Point defense to zone defense
- Increased randomness

Professional mariners:

- Apparent increase in Coast Guard patrols

Game Theory: Social Choice

Warm-up

Design an algorithm to determine the winner of three candidates a, b, c given the ranking provided by n individual voters, described by a $3 \times n$ matrix M

```
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Input: $M$ where $M_{ij} \in$ {a, b, c} is the candidate at rank $j$ for voter $i$
Output: $x \in$ {a, b, c} describes the winner



Return $x$
```

Example Matrix M

	Voter 1	Voter 2	Voter 3	Voter 4
Rank 1	a	c	b	a
Rank 2	b	b	c	b
Rank 3	c	a	a	c

Social Choice Theory

A mathematical theory that deal with aggregation of individual preferences

Wide applications in economics, public policy, etc.

Origins in Ancient Greece

18th century

- Formal foundations by Condorcet and Borda

19th Century

- Charles Dodgson

20th Century

- Nobel Prize in Economics

20th Century – Winners of Nobel Memorial Prize in Economic Sciences

Kenneth Arrow

Amartya Kumar Sen

Voting Model

Model

- Set of voters $N = \{1..n\}$
- Set of alternatives A ($|A| = m$)
- Each voter has a ranking over the alternatives
- Preference profile: collection of all voters' rankings

Voter ID	**1**	**2**	**3**	**4**
Ranking	a	c	b	a
	b	b	c	b
	c	a	a	c

Voting Rules

Voting rule: function that maps preference profiles to alternatives that specifies the winner of the election

```
function voting($M$)
Input: $M$ where $M_{ij}$ ∈ {a, b, c} is the candidate at rank $j$ for voter $i$
Output: $x$ ∈ {a, b, c} describes the winner

Return $x$
```

Example Matrix M

a	c	b	a
b	b	c	b
c	a	a	c

Voting Rules

Plurality (used in many political elections)

- Each voter give one point to top alternative
- Alternative with most points win

Who's the winner?   a

Voter ID	1	2	3	4
Ranking	a	c	b	a
	b	b	c	b
	c	a	a	c

Voting Rules

Borda count (used for national election in Slovenia)

- Each voter awards $m - k$ points to alternative ranked k^{th}
- Alternative with most points win

Who's the winner? b

Voter ID	1	2	3	4
Ranking	a	c	b	a
	b	b	c	b
	c	a	a	c

Voting Rules

Borda count (used for national election in Slovenia)

- Each voter awards $m - k$ points to alternative ranked k^{th}
- Alternative with most points win

Who’s the winner? b

Voter ID	1	2	3	4
Ranking	a	c	b	a
	b	b	c	b
	c	a	a	c

$m - k$
2
1
0

a: 2+0+0+2=4; b: 1+1+2+1=5; c: 0+2+1+0=3

Pairwise Election

Alternative x beats y in pairwise election if majority of voters prefer x to y

Who beats who in pairwise election?

b beats c

Voter ID	1	2	3	4
Ranking	a	c	b	a
	b	b	c	b
	c	a	a	c

Voting Rules

Plurality with runoff

- First round: two alternatives with highest plurality scores survive
- Second round: pairwise election between the two

x beats y if majority of voters prefer x to y

Who’s the winner?

Voter ID	**1**	**2**	**3**	**4**	**5**
Ranking	a	c	b	a	c
	b	b	c	b	b
	c	a	a	c	a

Voting Rules

Plurality with runoff

- First round: two alternatives with highest plurality scores survive
- Second round: pairwise election between the two

x beats y if majority of voters prefer x to y

Who's the winner? a and c survive, and then c beats a

Voter ID	1	2	3	4	5
Ranking	a	c	b	a	c
	b	b	c	b	b
	c	a	a	c	a

Voting Rules

Single Transferable Vote (STV)

- (used in Ireland, Australia, New Zealand, Maine, San Francisco, Cambridge)
- $m - 1$ rounds: In each round, alternative with least plurality votes is eliminated
- Alternative left is the winner

Who's the winner?

Voter ID	1	2	3	4	5
Ranking	a	d	b	a	b
	b	b	c	b	d
	d	c	a	d	a
	c	a	d	c	c

Voting Rules

Single Transferable Vote (STV)

- (used in Ireland, Australia, New Zealand, Maine, San Francisco, Cambridge)
- $m - 1$ rounds: In each round, alternative with least plurality votes is eliminated
- Alternative left is the winner

Who's the winner? c is eliminated, then d, then a, leaving b as the winner.

Voter ID	1	2	3	4	5
Ranking	a	d	b	a	b
	b	b	c	b	d
	d	c	a	d	a
	c	a	d	c	c

Note: When d is eliminated, the vote from voter 2 is effectively transferred to b

Representation of Preference Profile

Identity of voters does not matter

Only record how many voters has a preference

33 voters	16 voters	3 voters	8 voters	18 voters	22 voters
a	b	c	c	d	e
b	d	d	e	e	c
c	c	b	b	c	b
d	e	a	d	b	d
e	a	e	a	a	a

Tie Breaking

Commonly used tie breaking rules include

- Borda count
- Having the most votes in the first round
- …

Social Choice Axioms

How do we choose among different voting rules? What are the desirable properties?

Majority consistency

Majority consistency: Given a voting rule that satisfies Majority Consistency, if a majority of voters (> 50% of voters) rank alternative x first, then x should be the final winner.

Poll 3

Which rules are NOT majority consistent?

A. Plurality: Each voter give one point to top alternative

B. Borda count: Each voter awards $m - k$ points to alternative ranked k^{th}

C. Plurality with runoff: Pairwise election between two alternatives with highest plurality scores

D. STV: In each round, alternative with least plurality votes is eliminated

E. None

Condorcet Consistency

Recall: x beats y in a pairwise election if majority of voters prefer x to y

Condorcet winner is the alternative that beats every other alternative in pairwise election

Does a Condorcet winner always exist?

Condorcet paradox = cycle in majority preferences

Voter ID	**1**	**2**	**3**
Ranking over alternatives (first row is the most preferred)	a	c	b
	b	a	c
	c	b	a

Condorcet Consistency

Condorcet consistency: A voting rule that satisfies majority consistency should select a Condorcet Winner as the final winner if one exists.

Which of the introduced voting rules (Plurality, Borda count, Plurality with runoff, STV) are Condorcet consistent?

Poll 4

Which rules ARE Condorcet consistent?

A. Plurality: Each voter give one point to top alternative

B. Borda count: Each voter awards $m - k$ points to alternative ranked k^{th}

C. Plurality with runoff: Pairwise election between two alternatives with highest plurality scores

D. STV: In each round, alternative with least plurality votes is eliminated

E. None

Condorcet Consistency

Winner under different voting rules in this example

- Plurality:
- Borda:
- Plurality with runoff:
- STV:
- Condorcet winner:

33 voters	16 voters	3 voter	8 voters	18 voters	22 voters
a	b	c	c	d	e
b	d	d	e	e	c
c	c	b	b	c	b
d	e	a	d	b	d
e	a	e	a	a	a

Strategy-Proofness

Using Borda Count

Who is the winner?

Voter ID	1	2	3
Ranking over alternatives (first row is the most preferred)	b	b	a
	a	a	b
	c	c	c
	d	d	d

$m-k$
3
2
1
0

Who is the winner now?

Voter ID	1	2	3
Ranking over alternatives (first row is the most preferred)	b	b	a
	a	a	c
	c	c	d
	d	d	b

$m-k$
3
2
1
0

Strategy-Proofness

A single voter can manipulate the outcome!

Voter ID	1	2	3
Ranking over alternatives (first row is the most preferred)	b	b	a
	a	a	b
	c	c	c
	d	d	d

$m-k$
3
2
1
0

b: 2*3+1*2=8
a: 2*2+1*3=7

b is the winner

Voter ID	1	2	3
Ranking over alternatives (first row is the most preferred)	b	b	a
	a	a	c
	c	c	d
	d	d	b

$m-k$
3
2
1
0

b: 2*3+1*0=6
a: 2*2+1*3=7

a is the winner

Strategy-Proofness

A voting rule is strategyproof (SP) if a voter can never benefit from lying about his preferences (regardless of what other voters do)

- Benefit: a more preferred alternative is selected as winner

Do not lie: b is the winner

Voter ID	1	2	3
Ranking	b	b	a
	a	a	b
	c	c	c
	d	d	d

Lie: a is the winner

Voter ID	1	2	3
Ranking	b	b	a
	a	a	c
	c	c	d
	d	d	b

If a voter’s preference is a>b>c, c will be selected w/o lying, and b will be selected w/ lying, then the voter still benefits

Poll 5

Which of the introduced voting rules are strategyproof?

A. Plurality: Each voter give one point to top alternative

🚫 B. Borda count: Each voter awards $m - k$ points to alternative ranked k^{th}

C. Plurality with runoff: Pairwise election between two alternatives with highest plurality scores

D. STV: In each round, alternative with least plurality votes is eliminated

E. None

Greedy Algorithm for f −Manipulation

Given voting rule f and preference profile of $n-1$ voters, how can the last voter report preference to let a specific alternative y *uniquely* win (no tie breaking)?

Greedy algorithm for f −Manipulation

```
Rank $y$ in the first place
While there are unranked alternatives
    If $\exists x$ that can be placed in the next spot without preventing $y$ from winning
        place this alternative in the next spot
    else
        return false
return true (with final ranking)
```

Correctness proved (Bartholdi et al., 1989)

Greedy Algorithm for f −Manipulation

Example with Borda count voting rule

Voter ID	**1**	**2**	**3**
Ranking over alternatives (first row is the most preferred)	b	b	a
	a	a	
	c	c	
	d	d	

Other Properties

A voting rule is dictatorial if there is a voter who always gets their most preferred alternative

A voting rule is constant if the same alternative is always chosen (regardless of the stated preferences)

A voting rule is onto if any alternative can win, for some set of stated preferences

Which of the introduced voting rules (Plurality, Borda count, Plurality with runoff, STV) are dictatorial, constant or onto?

Results in Social Choice Theory

Constant functions and dictatorships are SP   Why?

Theorem (Gibbard-Satterthwaite): If $m \geq 3$, then any voting rule that is SP and onto is dictatorial

- Any voting rule that is onto and nondictatorial is manipulable
- It is impossible to have a voting rule that is strategyproof, onto, and nondictatorial

Activity: Favorites of 15281

Plurality Vote

Borda Count

Plurality with Runoff

Single Transferrable Vote

Learning Objectives

Understand the voting model

Find the winner under the following voting rules

- Plurality, Borda count, Plurality with runoff, Single Transferable Vote

Describe the following concepts, axioms, and properties of voting rules

- Pairwise election, Condorcet winner
- Majority consistency, Condorcet consistency, Strategyproof
- Dictatorial, constant, onto

Understand the possibility of satisfying multiple properties

Describe the greedy algorithm for voting rule manipulation