Warm-up: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

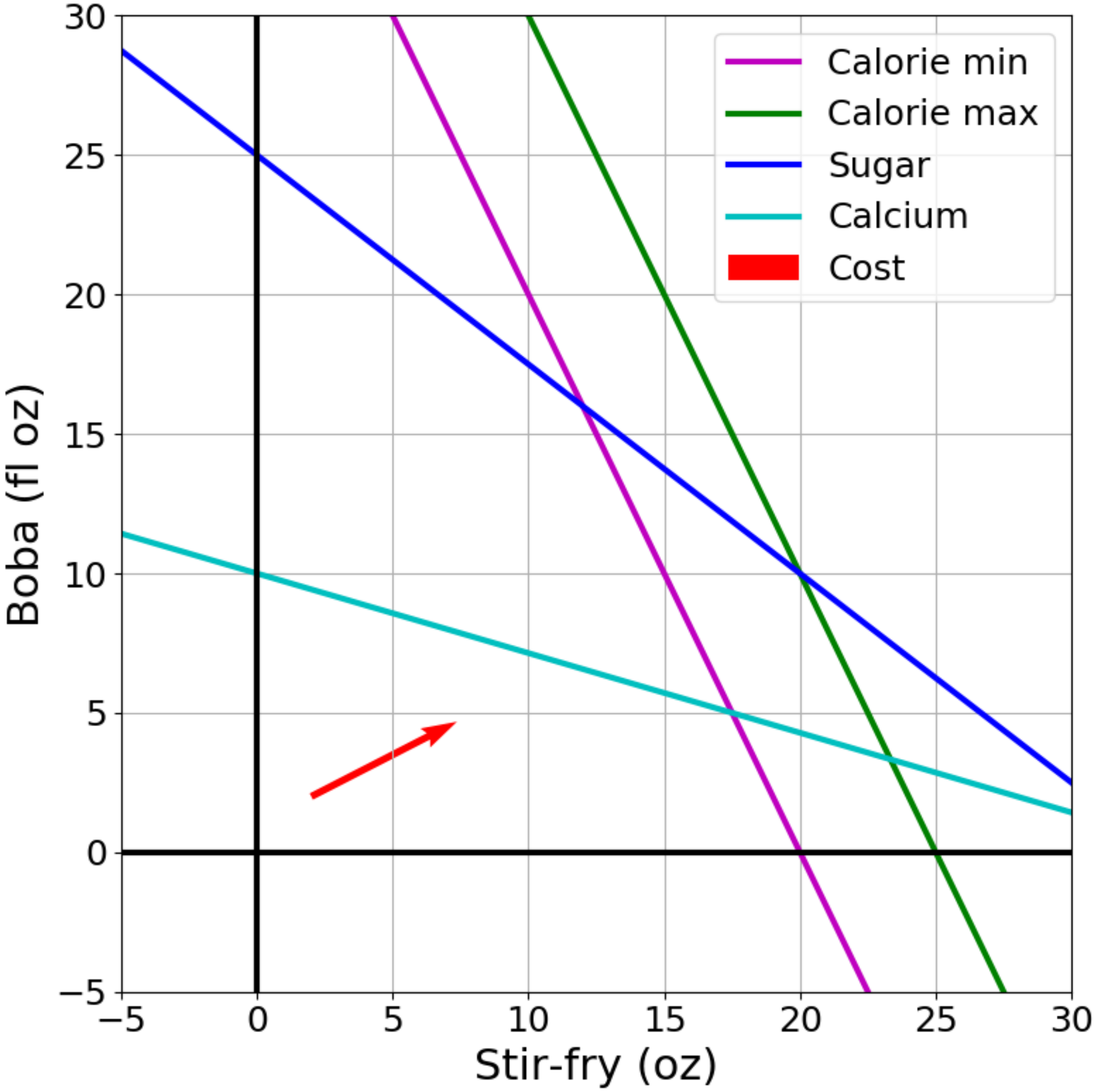
Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?


Calorie min
Calorie max
Sugar
Calcium
Cost
Boba (fl oz)
Stir-fry (oz)
30
25
20
15
10
5
0
−5
−5
0
5
10
15
20
25
30

Announcements

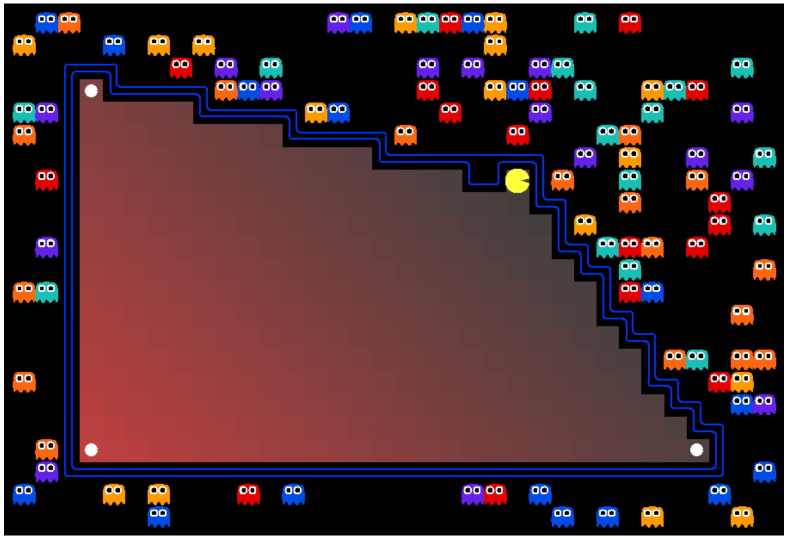
Assignments:

- HW3 (online)
  - Due Tonight, 10 pm
- HW4 (online)
  - Due 2/14, 10 pm
- P1: Search and Games due yesterday!!
- P2: Linear/Integer Programming
  - Due 2/23, 10pm (1 week after the exam)

- Exam 1 Feb 16!

AI: Representation and Problem Solving

Linear Programming



Instructor: Stephanie Rosenthal

Slide credits: CMU AI with drawings from http://ai.berkeley.edu

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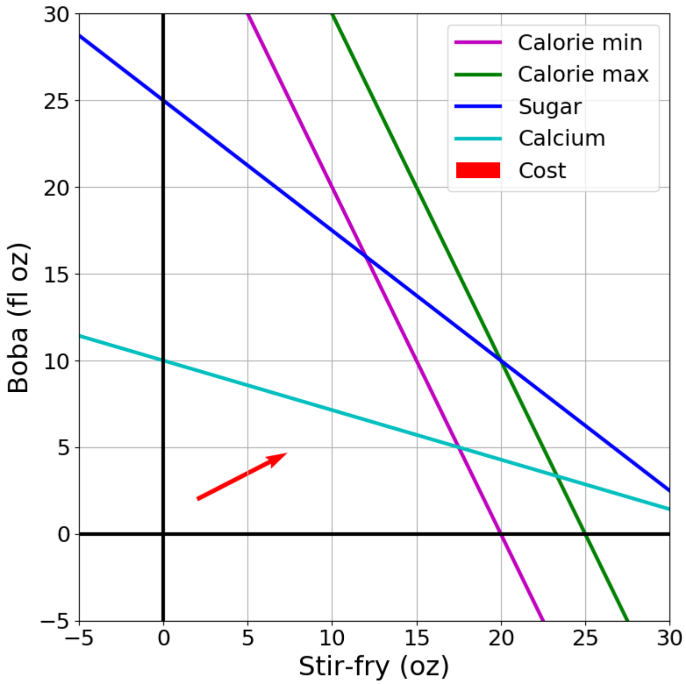
Optimization

Problem Description

Optimization Representation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

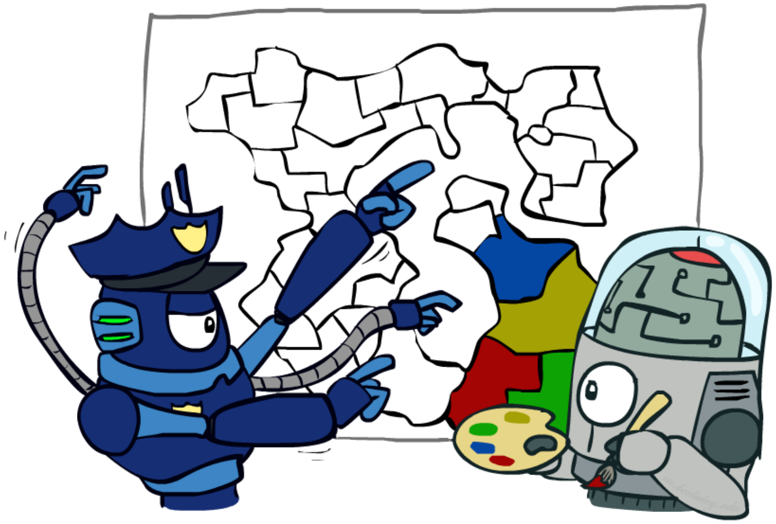
Graphical Representation



Constraint Satisfaction Problems

Map coloring

$$\begin{array}{ll} Any & \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x}\ satisfies\ \ constraints \end{array}$$



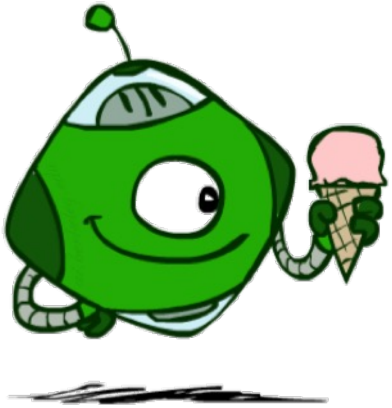
Notation Alert!

Optimization Formulation

Diet Problem

$$\mathit{Any} \quad \boldsymbol{x}$$

$$\text{s.t.} \quad \boldsymbol{x} \ \mathit{satisfies\ constraints}$$



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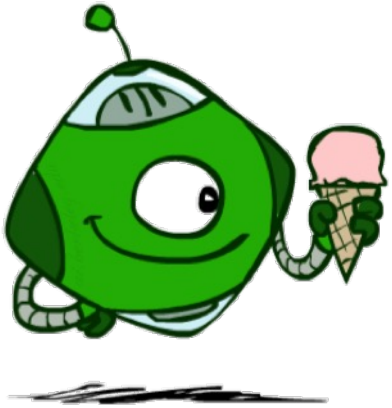
Notation Alert!

Optimization Formulation

Diet Problem

$$\min_{\boldsymbol{x}} \quad cost(\boldsymbol{x})$$ Objective

$$\text{s.t.} \quad \boldsymbol{x} \ satisfies \ constraints$$



Healthy Squad Goals

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Food	Cost	Calories	Sugar	Calcium
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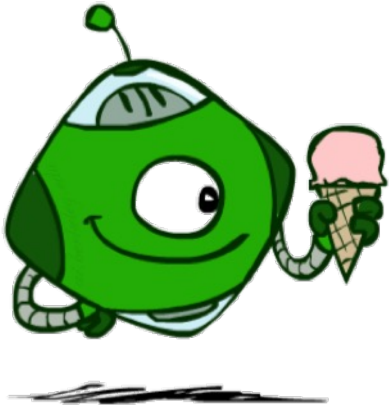
Boba (per fl oz)	0.5	50	4	70

Notation Alert!

Optimization Formulation

Diet Problem

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & cost(\boldsymbol{x}) \\ \text{s.t.} \quad & calories(\boldsymbol{x})\ contained \\ & sugar(\boldsymbol{x}) \leq limit \\ & \text{calcium}(\boldsymbol{x}) \geq limit \end{aligned}$$



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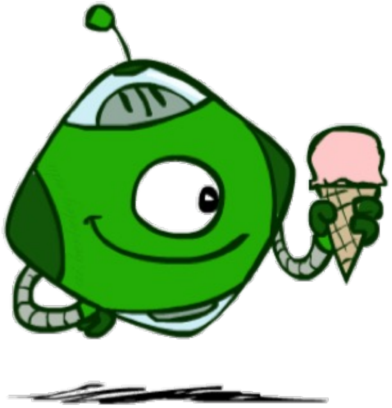
Optimization Formulation

Diet Problem

$$
\begin{aligned}
\min_{x_1, x_2} \quad & 1\,x_1 + 0.5\,x_2 \\
\text{s.t.} \quad & 100\,x_1 + 50\,x_2 \geq 2000 \\
& 100\,x_1 + 50\,x_2 \leq 2500 \\
& 3\,x_1 + \quad 4\,x_2 \leq 100 \\
& 20\,x_1 + 70\,x_2 \geq 700
\end{aligned}
$$



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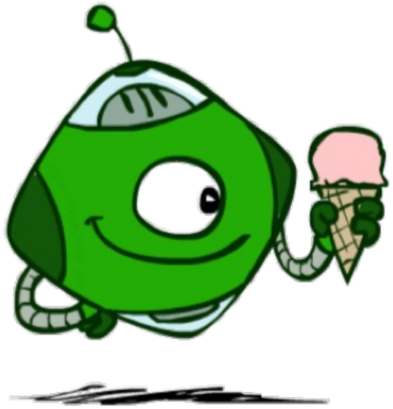
Optimization Formulation

Diet Problem



$$
\begin{aligned}
\min_{x_1, x_2} \quad & c_1\, x_1 + c_2\, x_2 \\
\text{s.t.} \quad & a_{1,1}\, x_1 + a_{1,2}\, x_2 \geq b_1 \\
& a_{2,1}\, x_1 + a_{2,2}\, x_2 \leq b_2 \\
& a_{3,1}\, x_1 + a_{3,2}\, x_2 \leq b_3 \\
& a_{4,1}\, x_1 + a_{4,2}\, x_2 \geq b_4
\end{aligned}
$$

Cost

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

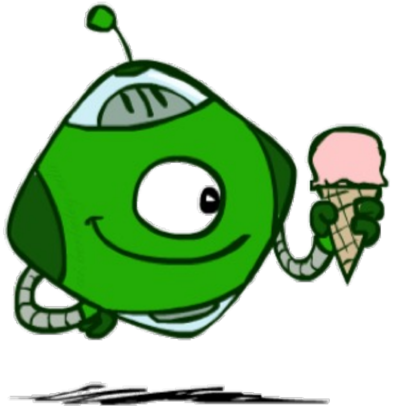
$$A = \begin{array}{cc} \text{Stir-fry} & \text{Boba} \\ \left[\begin{matrix} 100 \\ 100 \\ 3 \\ 20 \end{matrix}\right. & \left.\begin{matrix} 50 \\ 50 \\ 4 \\ 70 \end{matrix}\right] \end{array} \qquad \boldsymbol{b} = \begin{array}{cl} \text{Limit} & \\ \left[\begin{matrix} 2000 \\ 2500 \\ 100 \\ 700 \end{matrix}\right] & \begin{matrix} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{matrix} \end{array}$$

Notation Alert!

Optimization Formulation

Diet Problem

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & a_{1,1}\, x_1 + a_{1,2}\, x_2 \geq b_1 \\ & a_{2,1}\, x_1 + a_{2,2}\, x_2 \leq b_2 \\ & a_{3,1}\, x_1 + a_{3,2}\, x_2 \leq b_3 \\ & a_{4,1}\, x_1 + a_{4,2}\, x_2 \geq b_4 \end{aligned}$$



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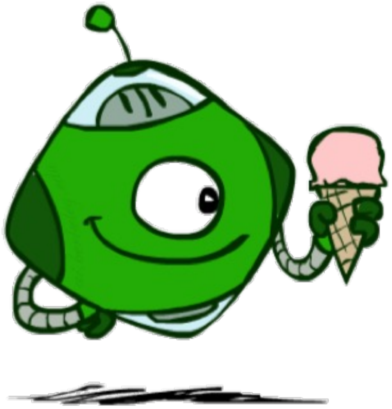
Notation Alert!

Optimization Formulation

Diet Problem



$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & -a_{1,1}\, x_1 - a_{1,2}\, x_2 \leq -b_1 \\ & a_{2,1}\, x_1 + a_{2,2}\, x_2 \leq b_2 \\ & a_{3,1}\, x_1 + a_{3,2}\, x_2 \leq b_3 \\ & -a_{4,1}\, x_1 - a_{4,2}\, x_2 \leq -b_4 \end{aligned}$$

Cost

$$c = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$A = \begin{array}{c} \begin{array}{cc} \text{Stir-fry} & \text{Boba} \end{array} \\ \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix} \end{array} \qquad b = \begin{array}{cl} \text{Limit} & \\ \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix} & \begin{array}{l} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{array} \end{array}$$

Optimization Formulation

Diet Problem

$$
\begin{array}{ll}
\min\limits_{\boldsymbol{x}} & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t.} & a_{1,1}\, x_1 + a_{1,2}\, x_2 \leq b_1 \\
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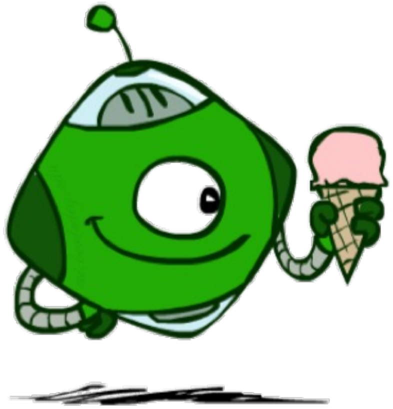
 & a_{3,1}\, x_1 + a_{3,2}\, x_2 \leq b_3 \\
 & a_{4,1}\, x_1 + a_{4,2}\, x_2 \leq b_4
\end{array}
$$



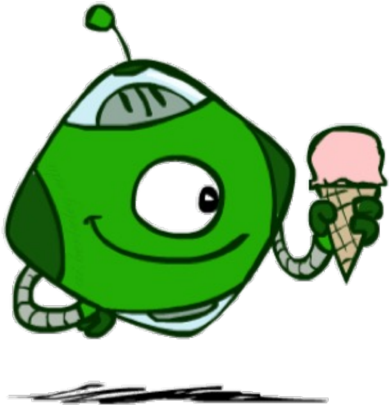
$$
\boldsymbol{c} = \begin{array}{c} \text{Cost} \\ \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \end{array}
$$

$$
A = \begin{array}{c} \begin{array}{cc} \text{Stir-fry} & \text{Boba} \end{array} \\ \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \end{array}
\quad
\boldsymbol{b} = \begin{array}{cl} \text{Limit} & \\ \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix} & \begin{array}{l} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{array} \end{array}
$$

Optimization Formulation

Diet Problem

$$\begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} \preceq \boldsymbol{b} \end{array}$$



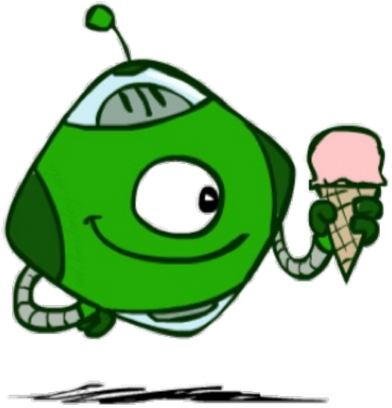
$$\begin{array}{cc} & \text{Cost} \\ \boldsymbol{c} = & \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} & \text{Stir-fry} & \text{Boba} \\ A = & \left[\begin{matrix} -100 \\ 100 \\ 3 \\ -20 \end{matrix}\right. & \left.\begin{matrix} -50 \\ 50 \\ 4 \\ -70 \end{matrix}\right] \end{array} \quad \begin{array}{ccc} & \text{Limit} & \\ \boldsymbol{b} = & \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix} & \begin{matrix} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{matrix} \end{array}$$

Notation Alert!

Poll 1

What has to increase to add more nutrition constraints?



$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

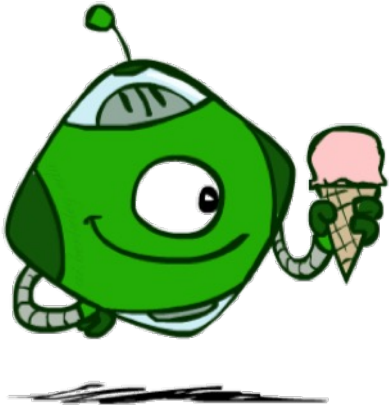
C) height A

D) width A

E) length $\boldsymbol{b}$

Poll 1

What has to increase to add more nutrition constraints?

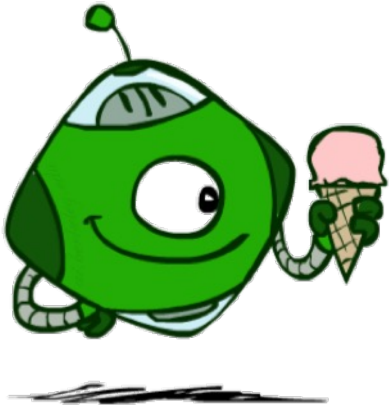


$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \qquad A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Poll 2

What has to increase to add more menu items?



$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

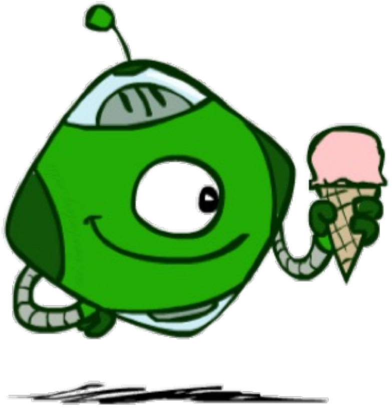
C) height A

D) width A

E) length $\boldsymbol{b}$

Poll 2

What has to increase to add more nutrition constraints?

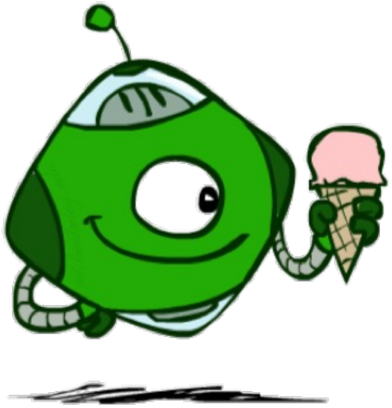


$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \qquad A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Poll 3

If $A \in \mathbb{R}^{M \times N}$, which of the following also equals N?



$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

C) length $\boldsymbol{b}$

Notation Alert!

Linear Programming

Linear objective with linear constraints

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{min.}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

As opposed to general optimization

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{min.}} \quad & f_0(\boldsymbol{x}) \\ \text{s.t.} \quad & f_i(\boldsymbol{x}) \leq 0, \quad i = 1 \ldots M \\ & \boldsymbol{a}_i^T \boldsymbol{x} = \boldsymbol{b}_i, \quad i = 1 \ldots P \end{aligned}$$

Linear Programming

Different formulations

Inequality form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

General form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d} \\ \text{s.t.} \quad & G\boldsymbol{x} \preceq \boldsymbol{h} \\ & A\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

Standard form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \mathrm{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \succeq 0 \end{aligned}$$

Important to pay attention to form!

Linear Programming

Different formulations

Inequality form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

General form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d} \\ \text{s.t.} \quad & G\boldsymbol{x} \preceq \boldsymbol{h} \\ & A\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

Standard form

$$\begin{aligned} \min_{\boldsymbol{x}}. \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \mathrm{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \succeq 0 \end{aligned}$$

Can switch between formulations!

Optimization

Problem Description

Optimization Representation

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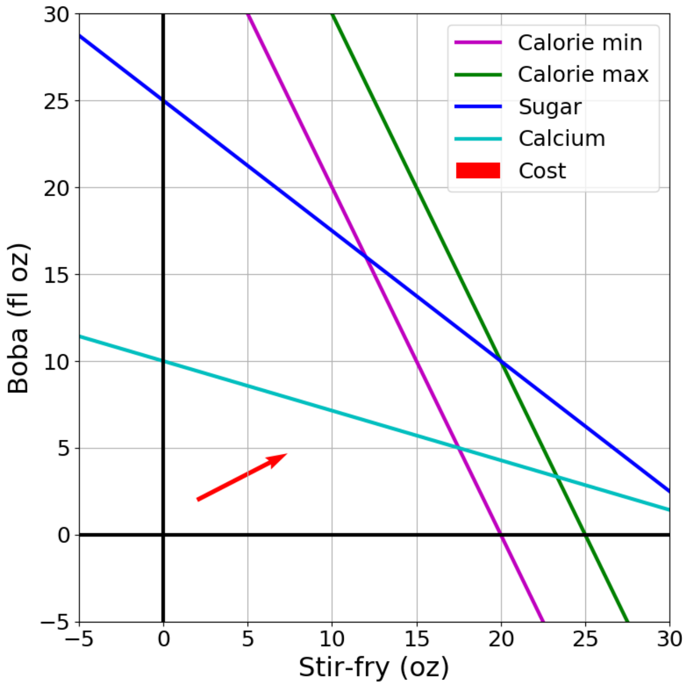
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Graphical Representation



Graphics Representation

Geometry / Algebra I Quiz

What shape does this inequality represent?

$$a_1 \, x_1 + a_2 \, x_2 \leq b_1$$

Graphics Representation

Geometry / Algebra I Quiz

What shape does this inequality represent?

$$a_1\,x_1 + a_2\,x_2 = b_1$$

$$a_1\,x_1 + a_2\,x_2 \leq b_1$$

$$\begin{aligned}
a_{1,1}\,x_1 + a_{1,2}\,x_2 &\leq b_1\\
a_{2,1}\,x_1 + a_{2,2}\,x_2 &\leq b_2\\
a_{3,1}\,x_1 + a_{3,2}\,x_2 &\leq b_3\\
a_{4,1}\,x_1 + a_{4,2}\,x_2 &\leq b_4
\end{aligned}$$

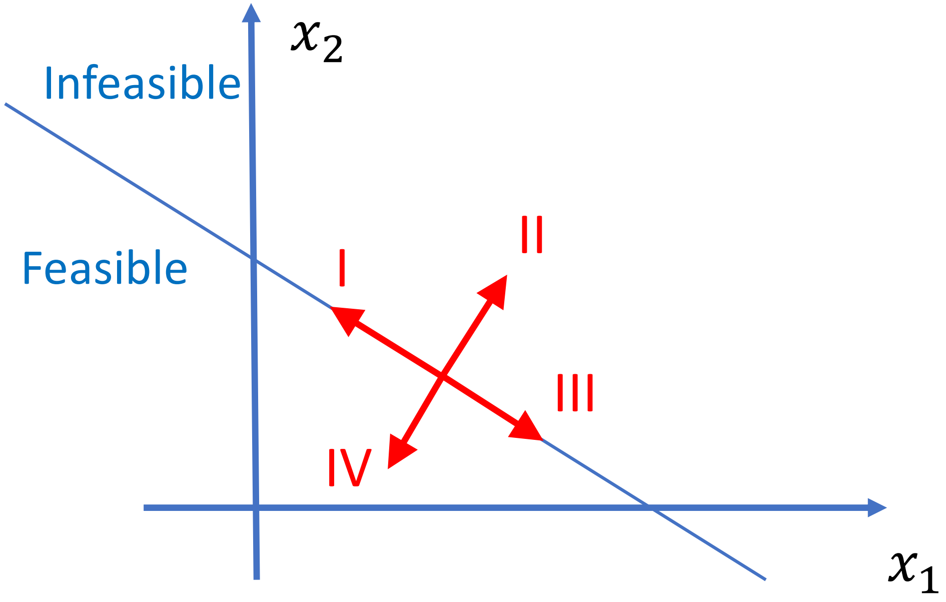
Poll 4

What is the relationship between the half plane:

$$a_1\ x_1 + a_2\ x_2 \leq b_1$$

and the vector:

$$[a_1, a_2]^T$$

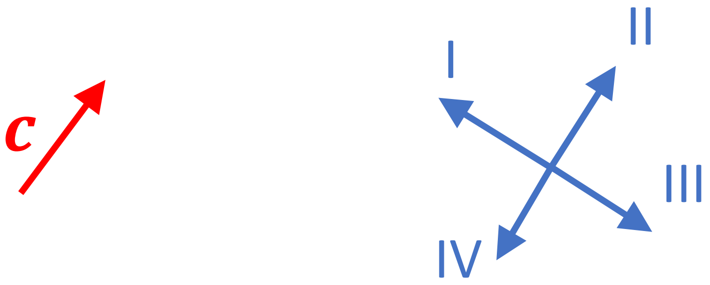


Poll 4

Given the cost vector $[c_1, c_2]^T$ and initial point $\boldsymbol{x}^{(0)}$,

Which unit vector step $\triangle\, \boldsymbol{x}$ will cause $\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} + \triangle\, \boldsymbol{x}$

to have the lowest cost $\boldsymbol{c}^T \boldsymbol{x}^{(1)}$?



Notation Alert!

Cost Contours

Given the cost vector $[c_1, c_2]^T$ where will

$\boldsymbol{c}^T\boldsymbol{x}$ = 0 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = 1 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = 2 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = -1 ?

$\boldsymbol{c}^T\boldsymbol{x}$ = -2 ?

Poll 5

As the magnitude of $\boldsymbol{c}$ increases, the distance between the contours lines of the objective $\boldsymbol{c}^T\boldsymbol{x}$:

A) Increases

B) Decreases