Warm-up: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

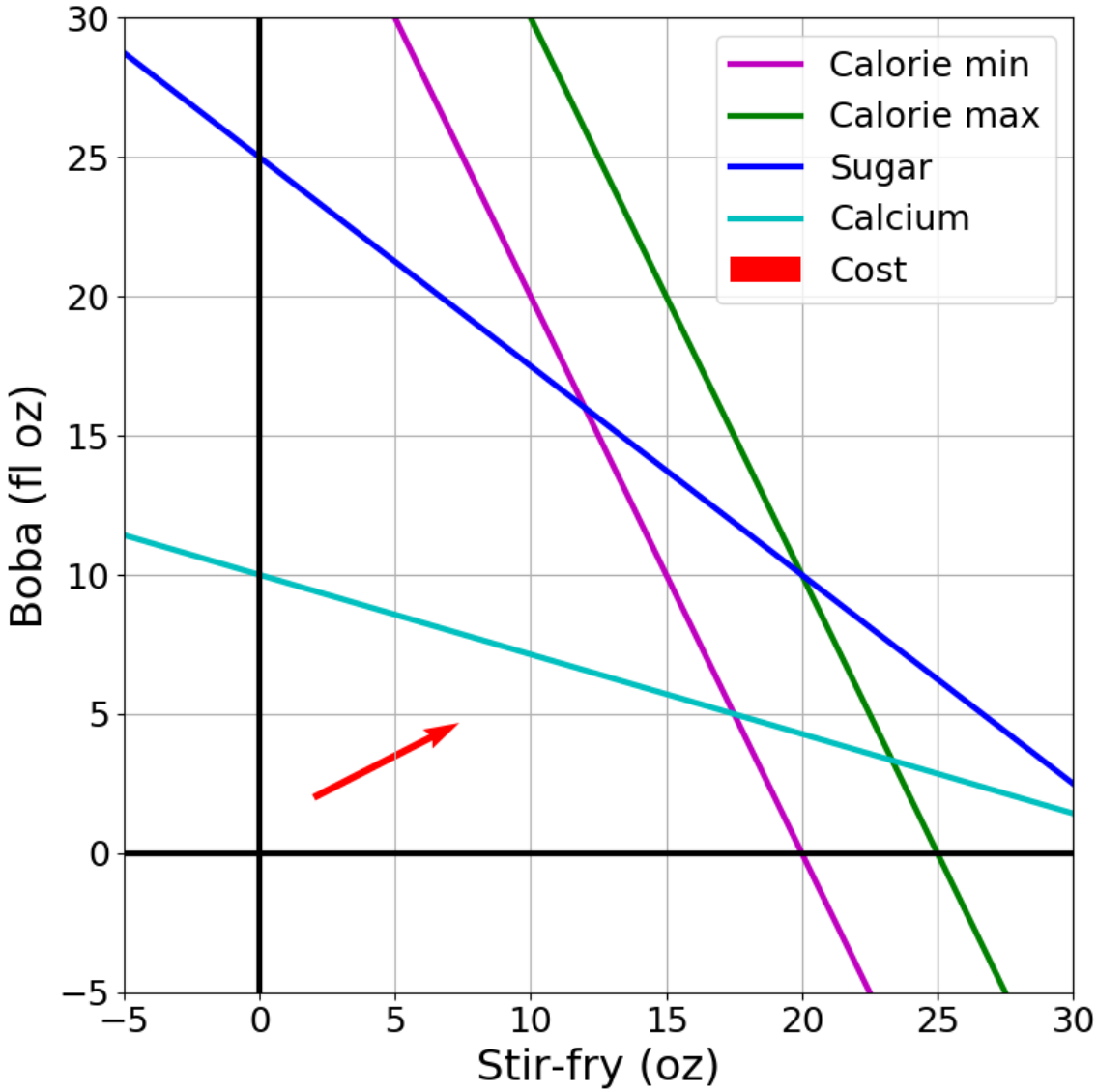
Healthy Squad Goals

- $2000 \leq \text{Calories} \leq 2500$
- $\text{Sugar} \leq 100$ g
- $\text{Calcium} \geq 700$ mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?


Calorie min
Calorie max
Sugar
Calcium
Cost
Boba (fl oz)
Stir-fry (oz)
30
25
20
15
10
5
0
−5
−5
0
5
10
15
20
25
30

Announcements

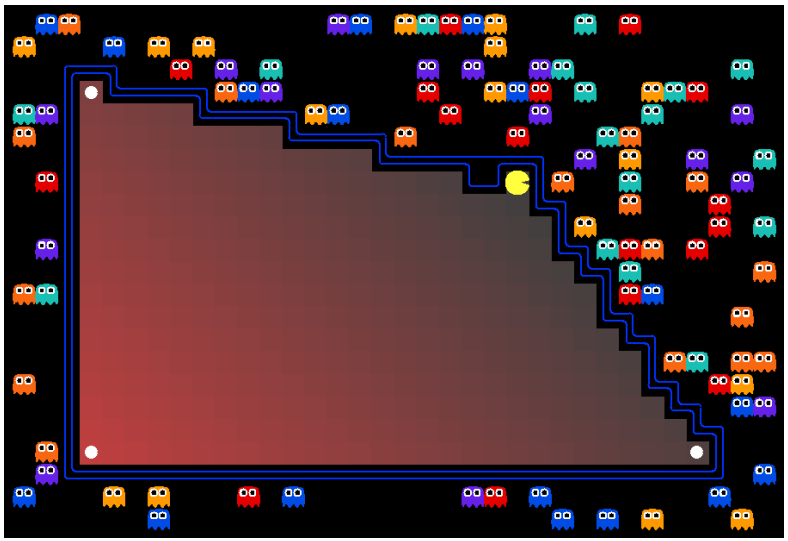
Assignments:

- HW3 (online)
  - Due Tonight, 10 pm
- HW4 (online)
  - Due 2/14, 10 pm
- P1: Search and Games due yesterday!! Wednesday
- P2: Linear/Integer Programming
  - Due 2/23, 10pm (1 week after the exam)

- Exam 1 Feb 16!

AI: Representation and Problem Solving

Linear Programming



Instructor: Stephanie Rosenthal

Slide credits: CMU AI with drawings from http://ai.berkeley.edu

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Optimization

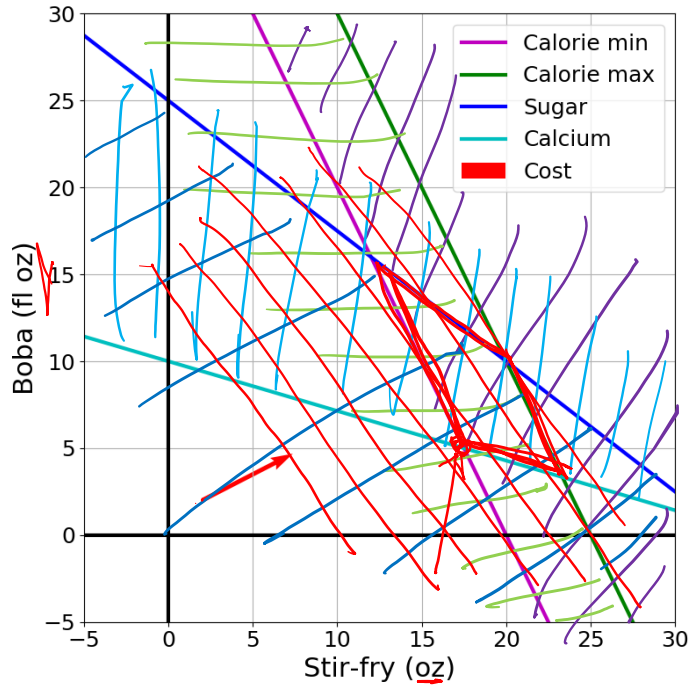
Problem Description

Optimization Representation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

Graphical Representation

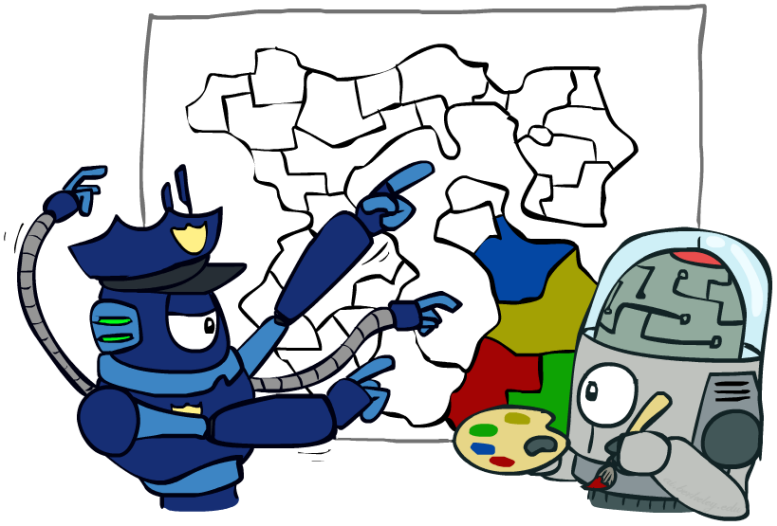


Constraint Satisfaction Problems

Map coloring

$Any \quad \vec{x}$ $[x_1, x_2 \in X_3 \ldots]$

s.t. $\boldsymbol{x}$ $satisfies$ $constraints$



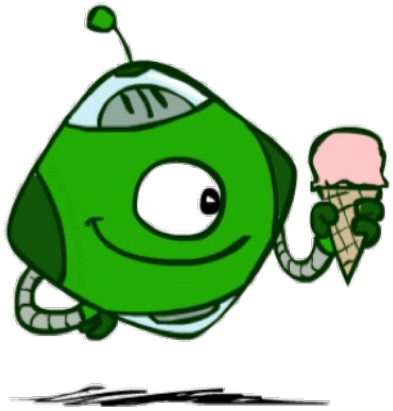
Notation Alert!

Optimization Formulation

Diet Problem

$Any \quad \boldsymbol{x}$

s.t. $\quad \boldsymbol{x}\ satisfies\ constraints$



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Notation Alert!

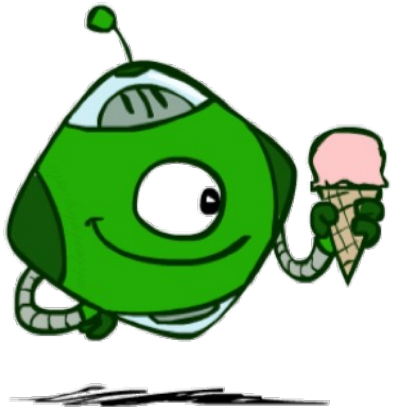
Optimization Formulation

Diet Problem

$$\min_{\boldsymbol{x}} \quad cost(\boldsymbol{x})$$

Objective

$$\text{s.t.} \quad \boldsymbol{x} \ satisfies \ constraints$$



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Boba (per fl oz)	0.5	50	4	70

Notation Alert!

Optimization Formulation

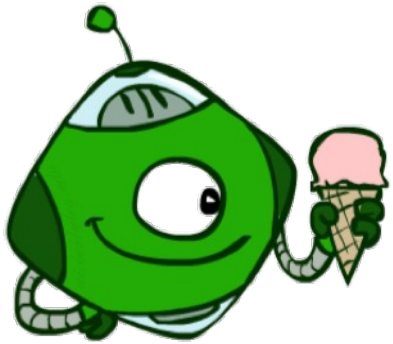
Diet Problem

$$\min_{\boldsymbol{x}} \quad cost(\boldsymbol{x})$$

$$\text{s.t.} \quad calories(\boldsymbol{x})\ contained$$

$$sugar(\boldsymbol{x}) \leq limit$$

$$\text{calcium}(\boldsymbol{x}) \geq limit$$



$cal(x) \geq 2000$

$cal(x) \leq 2500$

[Healthy Squad Goals]

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

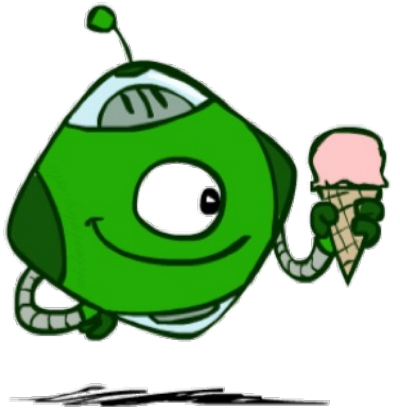
Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Optimization Formulation

Diet Problem

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad \begin{aligned} 100\,x_1 + 50\,x_2 &\geq 2000 \\ 100\,x_1 + 50\,x_2 &\leq 2500 \\ 3\,x_1 + 4\,x_2 &\leq 100 \\ 20\,x_1 + 70\,x_2 &\geq 700 \end{aligned}$$



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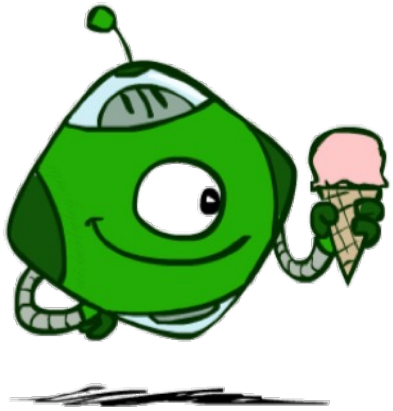
Notation Alert!

Optimization Formulation

Diet Problem



$$\min_{x_1, x_2} \quad c_1 x_1 + c_2 x_2$$

$$\text{s.t.} \quad \begin{bmatrix} a_{1,1} x_1 + a_{1,2} x_2 \geq b_1 \\ a_{2,1} x_1 + a_{2,2} x_2 \leq b_2 \\ a_{3,1} x_1 + a_{3,2} x_2 \leq b_3 \\ a_{4,1} x_1 + a_{4,2} x_2 \geq b_4 \end{bmatrix}$$

Cost

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Stir-fry, Boba

$$A = \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix}$$

Limit

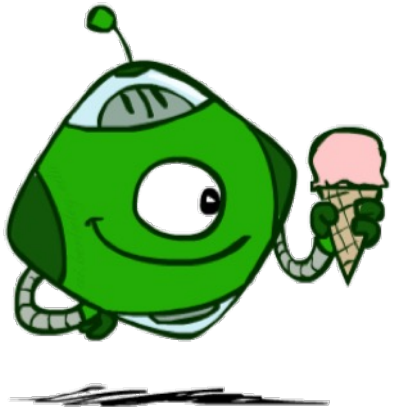
$$\boldsymbol{b} = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium

Notation Alert!

Optimization Formulation

Diet Problem



$$\min_{x} \quad c^T x$$

$$\text{s.t.} \quad \begin{aligned} a_{1,1}\, x_1 + a_{1,2}\, x_2 &\geq b_1 \\ a_{2,1}\, x_1 + a_{2,2}\, x_2 &\leq b_2 \\ a_{3,1}\, x_1 + a_{3,2}\, x_2 &\leq b_3 \\ a_{4,1}\, x_1 + a_{4,2}\, x_2 &\geq b_4 \end{aligned}$$

variables

Cost

$$c = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$ > variable

constraint

Stir-fry Boba

$$A = \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix}$$

Limit

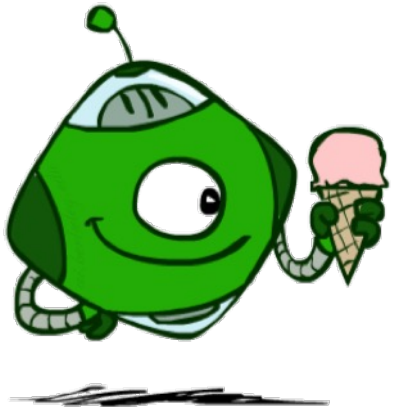
$$b = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium

Notation Alert!

Optimization Formulation

Diet Problem



$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\text{s.t.} \quad \begin{aligned} -a_{1,1}\, x_1 - a_{1,2}\, x_2 &\leq -b_1 \\ a_{2,1}\, x_1 + a_{2,2}\, x_2 &\leq b_2 \\ a_{3,1}\, x_1 + a_{3,2}\, x_2 &\leq b_3 \\ -a_{4,1}\, x_1 - a_{4,2}\, x_2 &\leq -b_4 \end{aligned}$$

Cost

$$c = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$A = \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix}$$

Stir-fry, Boba

Limit

$$b = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium

Optimization Formulation

Diet Problem

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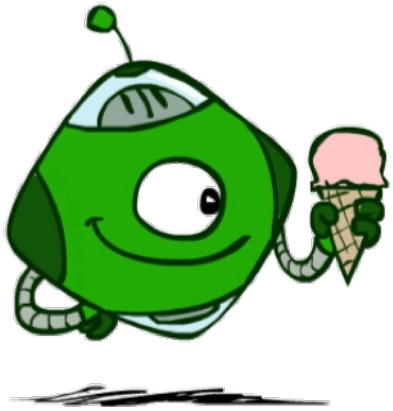


Cost

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Stir-fry Boba

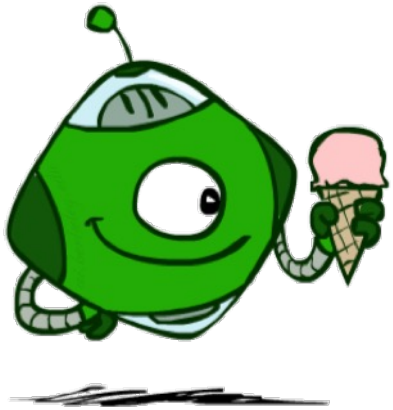
Limit

Calorie min
Calorie max
Sugar
Calcium

Optimization Formulation

Diet Problem

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x} \qquad \text{1 number}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$



Cost

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Stir-fry, Boba

Limit

Calorie min

Calorie max

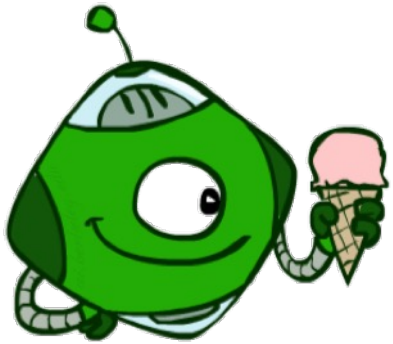
Sugar

Calcium

Notation Alert!

Poll 1

What has to increase to add more nutrition constraints?



$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

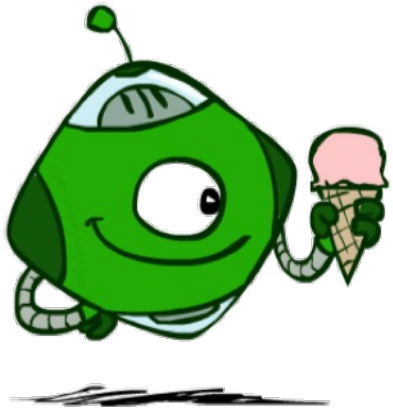
C) height A

D) width A

E) length $\boldsymbol{b}$

Poll 1

What has to increase to add more nutrition constraints?



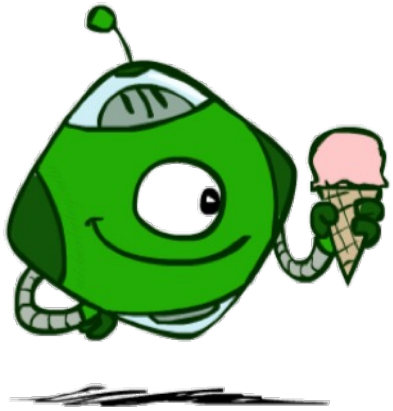
$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \qquad A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad \leq \qquad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Poll 2

What has to increase to add more menu items?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$



Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

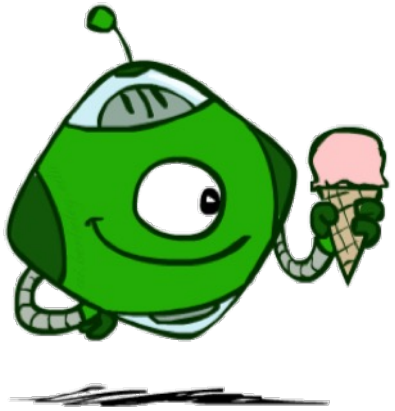
C) height A

D) width A

E) length $\boldsymbol{b}$

Poll 2

What has to increase to add more nutrition constraints?



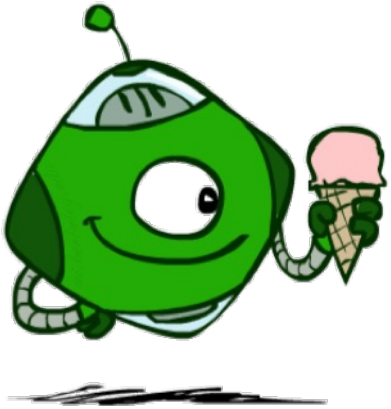
$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \qquad A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \qquad b = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Poll 3

If $A \in \mathbb{R}^{M \times N}$, which of the following also equals N?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$



Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

C) length $\boldsymbol{b}$

Notation Alert!

Linear Programming

Linear objective with linear constraints

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

As opposed to general optimization

$$\min_{\boldsymbol{x}} \quad f_0(\boldsymbol{x})$$
$$\text{s.t.} \quad f_i(\boldsymbol{x}) \leq 0, \quad i = 1 \ldots M$$
$$\boldsymbol{a}_i^T \boldsymbol{x} = \boldsymbol{b}_i, \quad i = 1 \ldots P$$

Linear Programming

Different formulations

Inequality form

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

General form

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{d}$$
$$\text{s.t.} \quad G\boldsymbol{x} \preceq \boldsymbol{h}$$
$$A\boldsymbol{x} = \boldsymbol{b}$$

Standard form

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad \mathrm{A}\boldsymbol{x} = \boldsymbol{b}$$
$$\boldsymbol{x} \succeq 0$$

Important to pay attention to form!

Linear Programming

Different formulations

Inequality form

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

General form

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{d} \\ \text{s.t.} \quad & G\boldsymbol{x} \preceq \boldsymbol{h} \\ & A\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

Standard form

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \mathrm{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \succeq 0 \end{aligned}$$

Can switch between formulations!

Optimization

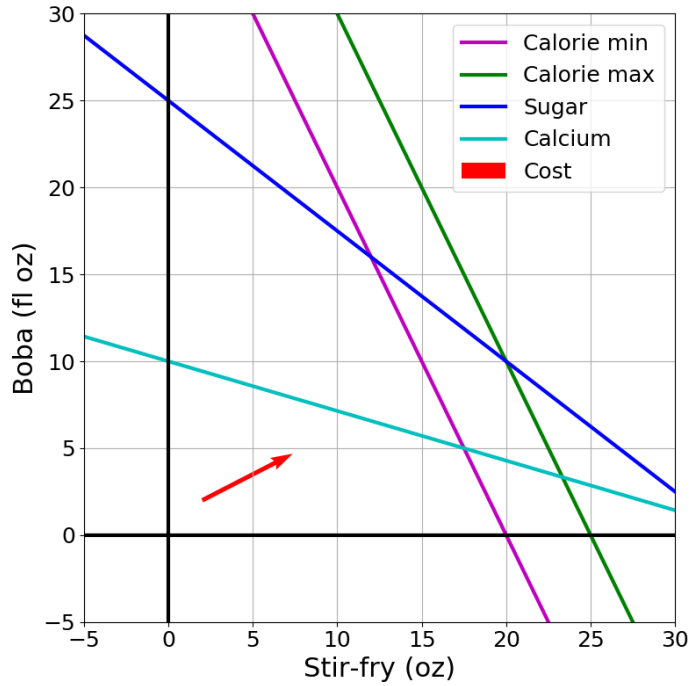
Problem Description

Optimization Representation

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \leq \boldsymbol{b}$$

Graphical Representation

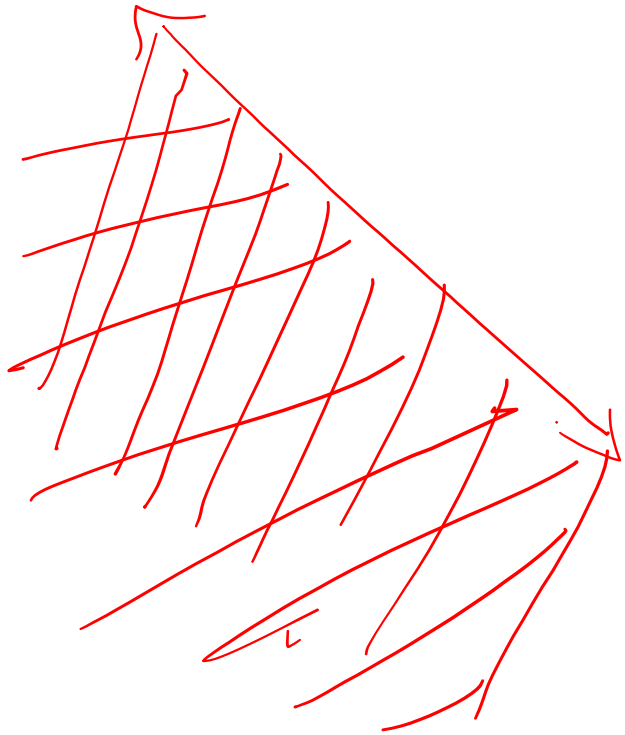


Graphics Representation

Geometry / Algebra I Quiz

What shape does this inequality represent?

$$a_1\, x_1 + a_2\, x_2 \leq b_1$$



Graphics Representation

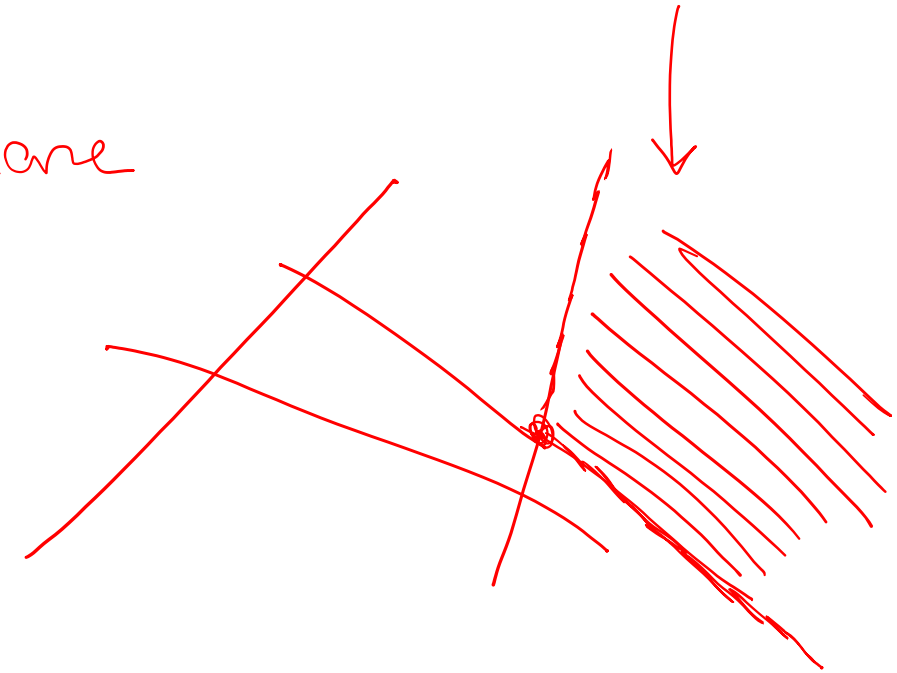
Geometry / Algebra I Quiz

What shape does this inequality represent?

$$a_1 x_1 + a_2 x_2 = b_1$$ line

$$a_1 x_1 + a_2 x_2 \leq b_1$$ half plane

$$\begin{bmatrix} a_{1,1} x_1 + a_{1,2} x_2 \leq b_1 \\ a_{2,1} x_1 + a_{2,2} x_2 \leq b_2 \\ a_{3,1} x_1 + a_{3,2} x_2 \leq b_3 \\ a_{4,1} x_1 + a_{4,2} x_2 \leq b_4 \end{bmatrix}$$



Poll 4

What is the relationship between the half plane:

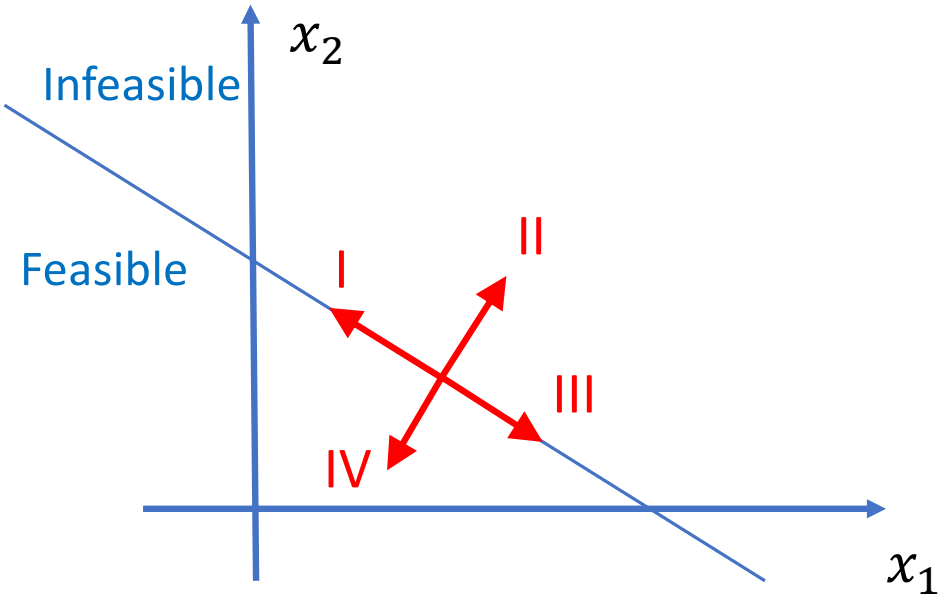
$$a_1\, x_1 + a_2\, x_2 \leq b_1$$

$1 x_1 + 1 x_2 \leq 5$

and the vector:

$$[a_1, a_2]^T$$

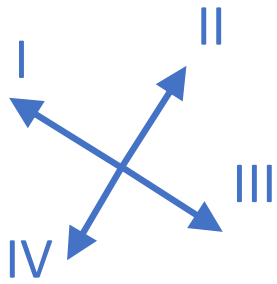
$[1 \;\; 1]$



Poll 5

STOP HERE

Given the cost vector $[c_1, c_2]^T$ and initial point $\boldsymbol{x}^{(0)}$,
Which unit vector step $\Delta\, \boldsymbol{x}$ will cause $\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} + \Delta\, \boldsymbol{x}$
to have the lowest cost $\boldsymbol{c}^T \boldsymbol{x}^{(1)}$?



Notation Alert!

Cost Contours

Given the cost vector $[c_1, c_2]^T$ where will

$\boldsymbol{c}^T \boldsymbol{x}$ = 0 ?

$\boldsymbol{c}^T \boldsymbol{x}$ = 1 ?

$\boldsymbol{c}^T \boldsymbol{x}$ = 2 ?

$\boldsymbol{c}^T \boldsymbol{x}$ = -1 ?

$\boldsymbol{c}^T \boldsymbol{x}$ = -2 ?

Poll 6

As the magnitude of $\boldsymbol{c}$ increases, the distance between the contours lines of the objective $\boldsymbol{c}^T \boldsymbol{x}$:

A) Increases

B) Decreases