15-281	Artificial Intelligence	HW10

INSTRUCTIONS

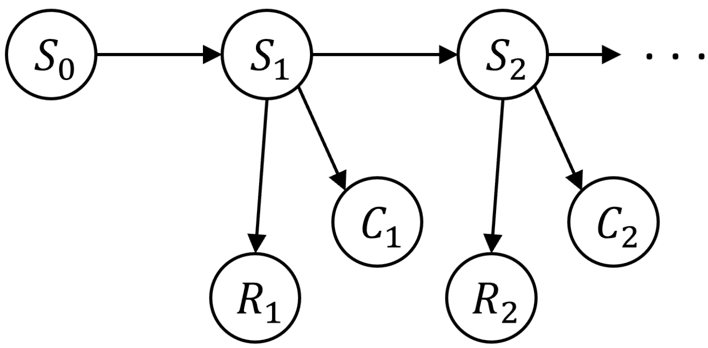
- **Due: Monday, November 25th, 2024 at 10:00 PM EST.** Remember that you may use up to 2 slip days for the Written making the last day to submit **Wednesday, November 27th, 2024 at 10:00 PM EST**.
- **Format:** Write your answers in the `yoursolution.tex` file and compile a pdf (preferred) or you can type directly on the blank pdf. Make sure that your answers are within the dedicated regions for each question/part. If you do not follow this format, we may deduct points. Also feel free to annotate directly on the pdf, but we will not accept printed+scanned solutions.
- **Images:** To insert pictures, we recommend drawing it on PowerPoint or Google Drawings, saving it as an image and including it in your latex source.
- **How to submit:** Submit a pdf with your answers on Gradescope. Log in and click on our class 15-281 and click on the submission titled HW10 and upload your pdf containing your answers.
- **Policy:** See the course website for homework policies and Academic Integrity.
- **Credit:** Please show your work to receive partial credit! We cannot award credit for partially correct solutions if there is no work.

Name	
Andrew ID	
Hours to complete?	◯ (0, 2] hours ◯ (2, 3] hours ◯ (3, 4] hours ◯ (4, 5] hours ◯ (5, 6] hours ◯ (6, 7] hours ◯ (7, 8] hours ◯ $>$ 8 hours

Q1. [62 pts] Dynamic Bayes Net and Hidden Markov Model

A professor wants to know if students are getting enough sleep. Each day, the professor observes whether the students sleep in class, and whether they have red eyes. Let S_t be the random variable of the student having enough sleep, R_t be the random variable for the student having red eyes, and C_t be the random variable of the student sleeping in class on day t. The professor has the following theory:

- The prior probability of getting enough sleep at time t, with no observations, is 0.6
- The probability of getting enough sleep on night t is 0.9 given that the student got enough sleep the previous night, and 0.2 if not
- The probability of having red eyes is 0.1 if the student got enough sleep, and 0.7 if not
- The probability of sleeping in class is 0.2 if the student got enough sleep, and 0.4 if not



S_0	$P(S_0)$
$+s$	0.6
$-s$	0.4

S_{t+1}	S_t	$P(S_{t+1} \mid S_t)$
$+s_{t+1}$	$+s_t$	0.9
$-s_{t+1}$	$+s_t$	0.1
$+s_{t+1}$	$-s_t$	0.2
$-s_{t+1}$	$-s_t$	0.8

R_t	S_t	$P(R_t \mid S_t)$
$+r$	$+s$	0.1
$-r$	$+s$	0.9
$+r$	$-s$	0.7
$-r$	$-s$	0.3

C_t	S_t	$P(C_t \mid S_t)$
$+c$	$+s$	0.2
$-c$	$+s$	0.8
$+c$	$-s$	0.4
$-c$	$-s$	0.6

Using the DBN above and these evidence values

- $[-r_1, -c_1]$ = not red eyes, not sleeping in class
- $[+r_2, -c_2]$ = red eyes, not sleeping in class
- $[+r_3, +c_3]$ = red eyes, sleeping in class

we want to compute $P(S_t \mid r_{1:t}, c_{1:t})$ for each of $t = 1, 2, 3$ as well as perform smoothing to get $P(S_2 \mid r_{1:3}, c_{1:3})$.

In order to do so, we will compute intermediate values which will correspond to the predict and update steps of our forward algorithm as well as finding the value of α **(the normalization constant)** in each case.

Note: Please round your answers to 3 decimal places at the *end* of each calculation. That is, if you need to compute multiple intermediate values to get your answer, do not round until you get your final answer. Please also note we will only be able to award partial credit if work is shown.

Also note: You can (and should) check all your solutions via the Gradescope 'Online 10' assignment. This can serve as a kind of autograder. :)

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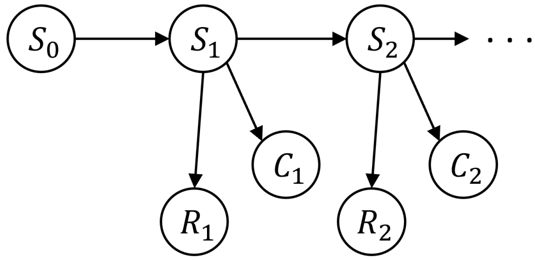

S_0	$P(S_0)$
$+s$	0.6
$-s$	0.4

S_{t+1}	S_t	$P(S_{t+1} \mid S_t)$
$+s_{t+1}$	$+s_t$	0.9
$-s_{t+1}$	$+s_t$	0.1
$+s_{t+1}$	$-s_t$	0.2
$-s_{t+1}$	$-s_t$	0.8

R_t	S_t	$P(R_t \mid S_t)$
$+r$	$+s$	0.1
$-r$	$+s$	0.9
$+r$	$-s$	0.7
$-r$	$-s$	0.3

C_t	S_t	$P(C_t \mid S_t)$
$+c$	$+s$	0.2
$-c$	$+s$	0.8
$+c$	$-s$	0.4
$-c$	$-s$	0.6

Round all numerical answers to 3 decimal places. Please also note we will only be able to award partial credit if work is shown. Evidence values: $[-r_1, -c_1]$, $[+r_2, -c_2]$, $[+r_3, +c_3]$

(a) [16 pts] State Estimation: $t = 1$

(i) [6 pts] Predict:

Use the Bayes' Net's CPTs to write an equivalent expression for $P(+s_1)$:

$P(+s_1) =$

$P(-s_1) =$

(ii) [6 pts] Update:

Use the Bayes' Net's CPTs and α **to find an equivalent expression for** $P(+s_1 | -r_1, -c_1)$:

When solving for probabilities, **don't** leave α in your answer.

$P(+s_1 | -r_1, -c_1) =$

$P(-s_1 | -r_1, -c_1) =$

(iii) [4 pts] What was the value for α?

$\alpha =$

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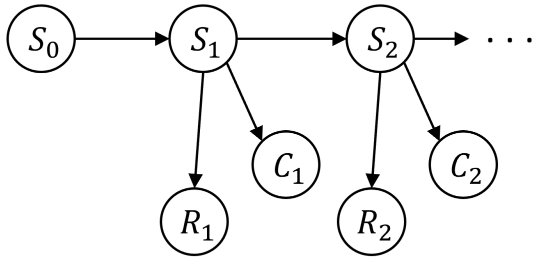

S_0	$P(S_0)$
$+s$	0.6
$-s$	0.4

S_{t+1}	S_t	$P(S_{t+1} \mid S_t)$
$+s_{t+1}$	$+s_t$	0.9
$-s_{t+1}$	$+s_t$	0.1
$+s_{t+1}$	$-s_t$	0.2
$-s_{t+1}$	$-s_t$	0.8

R_t	S_t	$P(R_t \mid S_t)$
$+r$	$+s$	0.1
$-r$	$+s$	0.9
$+r$	$-s$	0.7
$-r$	$-s$	0.3

C_t	S_t	$P(C_t \mid S_t)$
$+c$	$+s$	0.2
$-c$	$+s$	0.8
$+c$	$-s$	0.4
$-c$	$-s$	0.6

Round all numerical answers to 3 decimal places. Please also note we will only be able to award partial credit if work is shown. Evidence values: $[-r_1, -c_1]$, $[+r_2, -c_2]$, $[+r_3, +c_3]$

(b) [16 pts] State Estimation: $t = 2$

(i) [6 pts] Predict:

$P(+s_2 \mid -r_1, -c_1) =$

$P(-s_2 \mid -r_1, -c_1) =$

(ii) [6 pts] Update:

Use the Bayes' Net's CPTs, previous probabilities, and α to find an equivalent expression for $P(+s_2 \mid r_{1:2}, c_{1:2})$:

When solving for probabilities, **don't** leave α in your answer.

$P(+s_2 \mid r_{1:2}, c_{1:2}) =$

$P(-s_2 \mid r_{1:2}, c_{1:2}) =$

(iii) [4 pts] What was the value for α?

$\alpha =$

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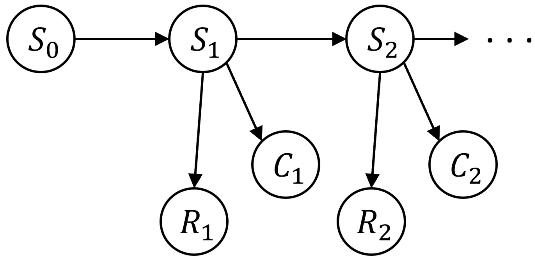

S_0	$P(S_0)$
$+s$	0.6
$-s$	0.4

S_{t+1}	S_t	$P(S_{t+1} \mid S_t)$
$+s_{t+1}$	$+s_t$	0.9
$-s_{t+1}$	$+s_t$	0.1
$+s_{t+1}$	$-s_t$	0.2
$-s_{t+1}$	$-s_t$	0.8

R_t	S_t	$P(R_t \mid S_t)$
$+r$	$+s$	0.1
$-r$	$+s$	0.9
$+r$	$-s$	0.7
$-r$	$-s$	0.3

C_t	S_t	$P(C_t \mid S_t)$
$+c$	$+s$	0.2
$-c$	$+s$	0.8
$+c$	$-s$	0.4
$-c$	$-s$	0.6

Round all numerical answers to 3 decimal places. Please also note we will only be able to award partial credit if work is shown. Evidence values: $[-r_1, -c_1]$, $[+r_2, -c_2]$, $[+r_3, +c_3]$

(c) [20 pts] State Estimation: $t = 3$

(i) [8 pts] Predict:

$P(+s_3 \mid r_{1:2}, c_{1:2}) =$

$P(-s_3 \mid r_{1:2}, c_{1:2}) =$

(ii) [8 pts] Update:

Use the Bayes' Net's CPTs, previous probabilities, and α to find an equivalent expression for $P(+s_3 \mid r_{1:3}, c_{1:3})$:

When solving for probabilities, **don't** leave α in your answer.

$P(+s_3 \mid r_{1:3}, c_{1:3}) =$

$P(-s_3 \mid r_{1:3}, c_{1:3}) =$

(iii) [4 pts] What was the value for α?

$\alpha =$

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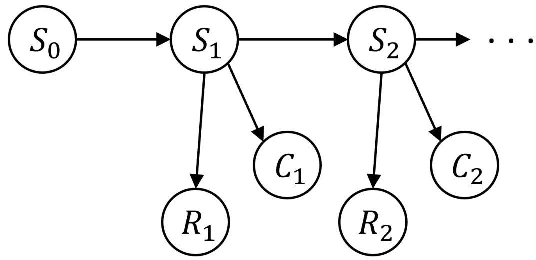


S_0	$P(S_0)$
$+s$	0.6
$-s$	0.4

S_{t+1}	S_t	$P(S_{t+1} \mid S_t)$
$+s_{t+1}$	$+s_t$	0.9
$-s_{t+1}$	$+s_t$	0.1
$+s_{t+1}$	$-s_t$	0.2
$-s_{t+1}$	$-s_t$	0.8

R_t	S_t	$P(R_t \mid S_t)$
$+r$	$+s$	0.1
$-r$	$+s$	0.9
$+r$	$-s$	0.7
$-r$	$-s$	0.3

C_t	S_t	$P(C_t \mid S_t)$
$+c$	$+s$	0.2
$-c$	$+s$	0.8
$+c$	$-s$	0.4
$-c$	$-s$	0.6

Round all numerical answers to 3 decimal places. Please also note we will only be able to award partial credit if work is shown. Evidence values: $[-r_1, -c_1]$, $[+r_2, -c_2]$, $[+r_3, +c_3]$

(d) [10 pts] We can build upon the previous three parts and use smoothing to compute $P(S_2 \mid r_{1:3}, c_{1:3})$.

(i) [4 pts] **Backward message:** $P(+r_3, +c_3 \mid S_2) = \sum_{s_3} P(+r_3, +c_3 \mid s_3) P(s_3 \mid S_2)$

$P(+r_3, +c_3 \mid +s_2) =$

$P(+r_3, +c_3 \mid -s_2) =$

(ii) [4 pts] **Smoothing**: $P(S_2 \mid r_{1:3}, c_{1:3}) = \alpha P(S_2 \mid r_{1:2}, c_{1:2}) P(+r_3, +c_3 \mid S_2)$
When solving for probabilities, **don't** leave α in your answer.

$P(+s_2 \mid r_{1:3}, c_{1:3}) =$

$P(-s_2 \mid r_{1:3}, c_{1:3}) =$

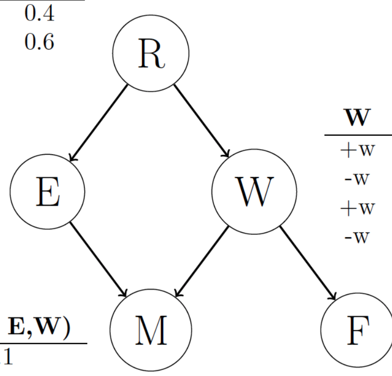
(iii) [2 pts] What was the value for α?

$\alpha =$

Q2. [38 pts] Sampling

Consider the following Bayes Net and corresponding probability tables.

R	P(R)
+r	0.4
-r	0.6



E	R	P(E | R)
+e	+r	0.25
-e	+r	0.75
+e	-r	0.6
-e	-r	0.4

W	R	P(W | R)
+w	+r	0.9
-w	+r	0.1
+w	-r	0.2
-w	-r	0.8

M	E	W	P(M | E,W)
+m	+e	+w	0.1
-m	+e	+w	0.9
+m	+e	-w	0.45
-m	+e	-w	0.55
+m	-e	+w	0.35
-m	-e	+w	0.65
+m	-e	-w	0.9
-m	-e	-w	0.1

F	W	P(F | W)
+f	+w	0.15
-f	+w	0.85
+f	-w	0.75
-f	-w	0.25

Consider the case where we are sampling to approximate the query $P(R \mid +f, +m)$.

(a) [24 pts] Fill in the following table with the probabilities of *drawing* each respective sample given that we are using each of the following sampling techniques. *Hint*: $P(+f, +m) = 0.2682$.

Method	$< +r, +e, -w, +m, +f >$	$< +r, -e, +w, -m, +f >$
Prior sampling		
Rejection sampling		
Likelihood weighting		

(b) [14 pts] We are going to use Gibbs sampling to estimate the probability of getting the sample $< +r, +e, -w, +m, +f >$. We will start from the sample $< -r, -e, -w, +m, +f >$ and resample E first then R. What is the probability of drawing sample $< +r, +e, -w, +m, +f >$?

Answer: