Warm-up as you walk in

Given these N=10 observations of the world:

What is the approximate value for
$P(-c \mid -a, +b)$?

A. 1/10
B. 5/10
C. 1/4
D. 1/5
E. I'm not sure

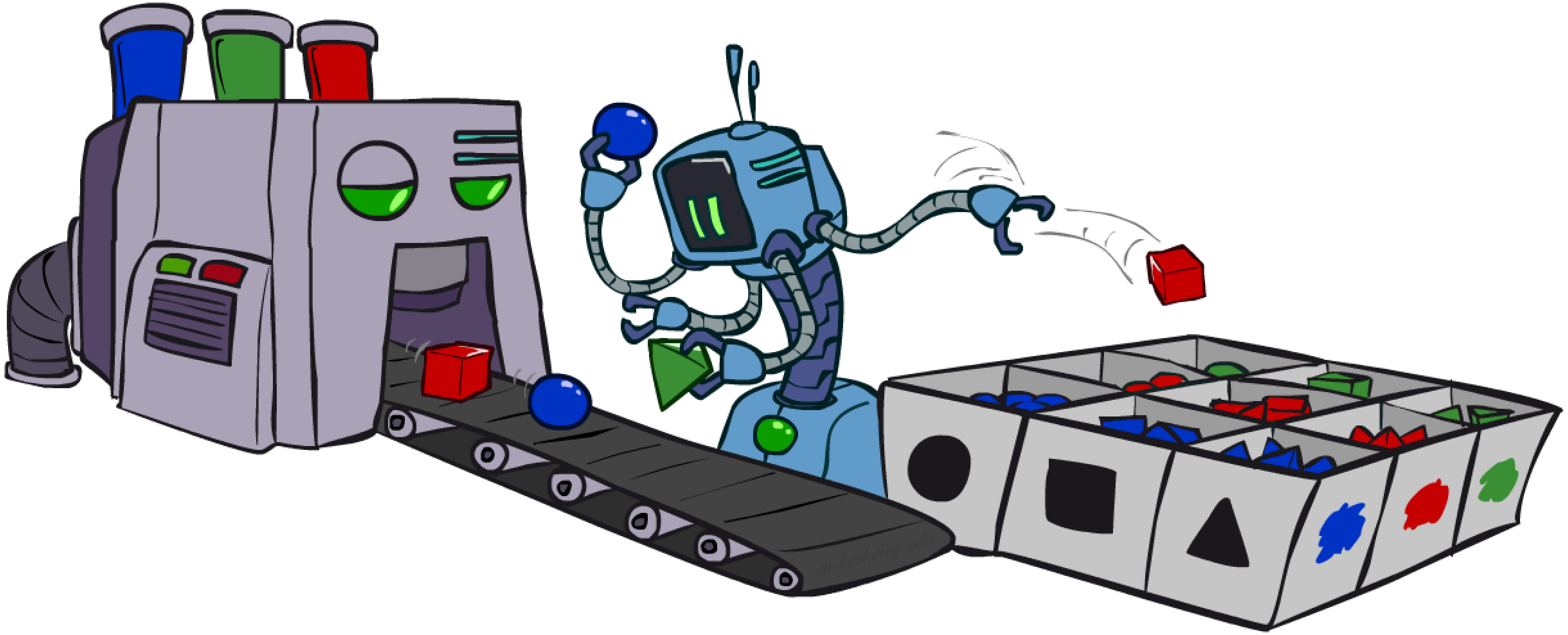
$-a, -b, +c$
$+a, -b, +c$
$-a, -b, +c$
$-a, +b, +c$
$+a, -b, +c$
$-a, +b, -c$
$-a, +b, +c$
$-a, +b, +c$
$+a, -b, +c$
$-a, +b, +c$

Counts

+a	+b	+c	0
+a	+b	-c	0
+a	-b	+c	3
+a	-b	-c	0
-a	+b	+c	4
-a	+b	-c	1
-a	-b	+c	2
-a	-b	-c	0

AI: Representation and Problem Solving

Bayes Nets Sampling



Instructor: Pat Virtue

Slide credits: CMU AI and http://ai.berkeley.edu

Review: Bayes Nets

Joint distributions → answer any query

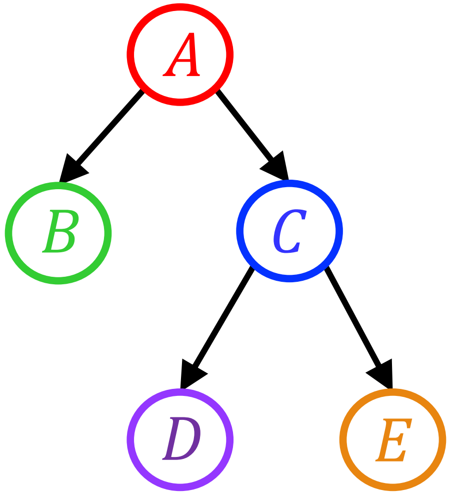
$$P(a \mid e) = \frac{1}{Z} P(a, e) = \frac{1}{Z} \sum_b \sum_c \sum_d P(a, b, c, d, e)$$

Break down joint using chain rule

$$P(A, B, C, D, E) = P(A)\, P(B|A)\, P(C|A, B)\, P(D|A, B, C)\, P(E|A, B, C, D)$$

With Bayes nets

$$P(A, B, C, D, E) = P(A)\, P(B|A)\, P(C|A)\, P(D|C)\, P(E|C)$$



Bayes Nets

✓ Part I: Representation

✓ Part II: Exact inference

✓ ▪ Enumeration (always exponential complexity)

✓ ▪ Variable elimination (worst-case exponential complexity, often better)

✓ ▪ Inference is NP-hard in general

Part III: Approximate Inference

Warm-up as you walk in

Given these N=10 observations of the world:

What is the approximate value for
$P(-c|\underbrace{-a,+b})$?

A. 1/10
B. 5/10
C. 1/4
D. 1/5
E. I'm not sure

Sample

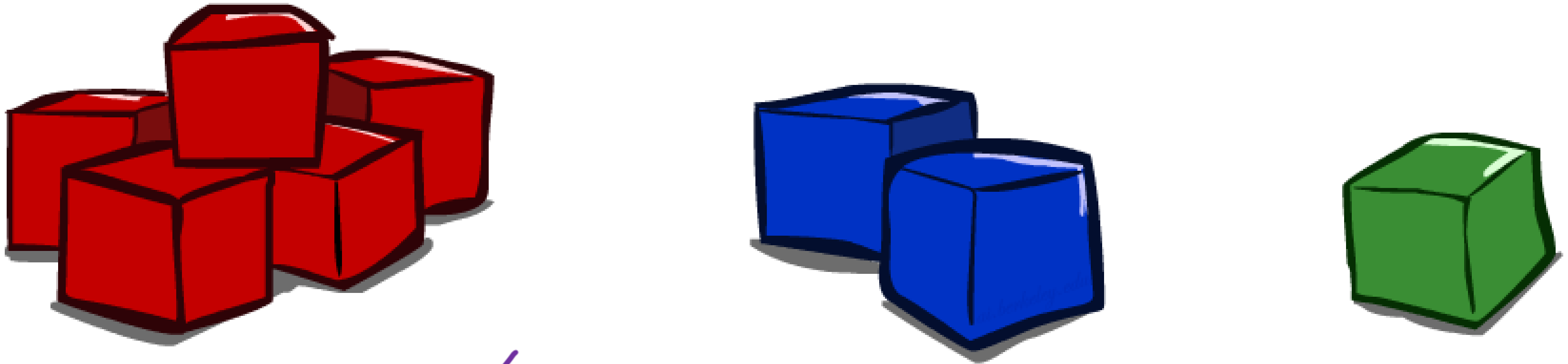
$-a,-b,+c$
$+a,-b,+c$
$-a,-b,+c$
$-a,+b,+c$
$+a,-b,+c$
$-a,+b,-c$
$-a,+b,+c$
$-a,+b,+c$
$+a,-b,+c$
$-a,+b,+c$

10

Counts

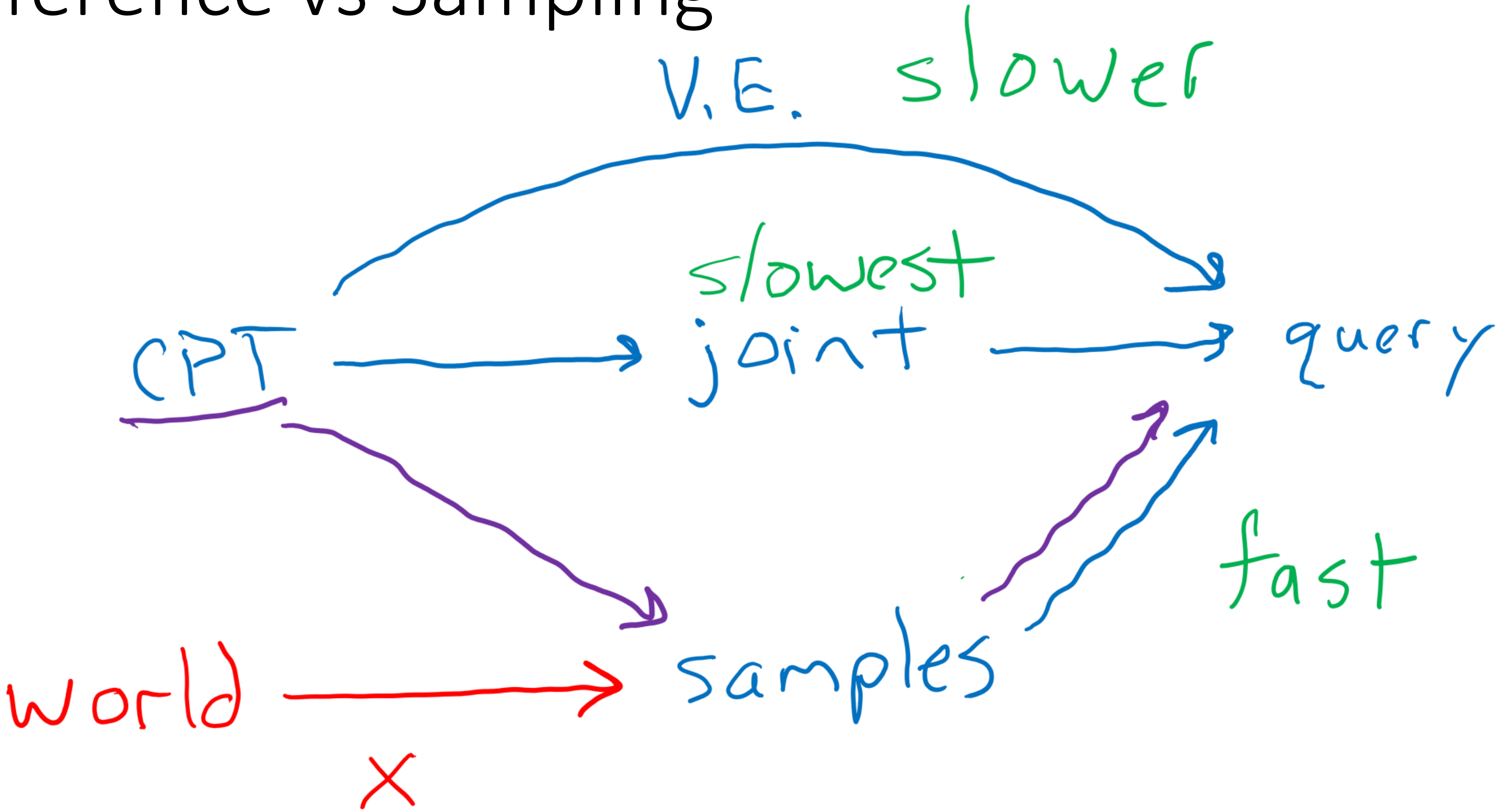
+a	+b	+c	0 /10
+a	+b	-c	0 /10
+a	-b	+c	3 /10
+a	-b	-c	0
-a	+b	+c	4
-a	+b	-c	1
-a	-b	+c	2
-a	-b	-c	0

Approximate Inference: Sampling



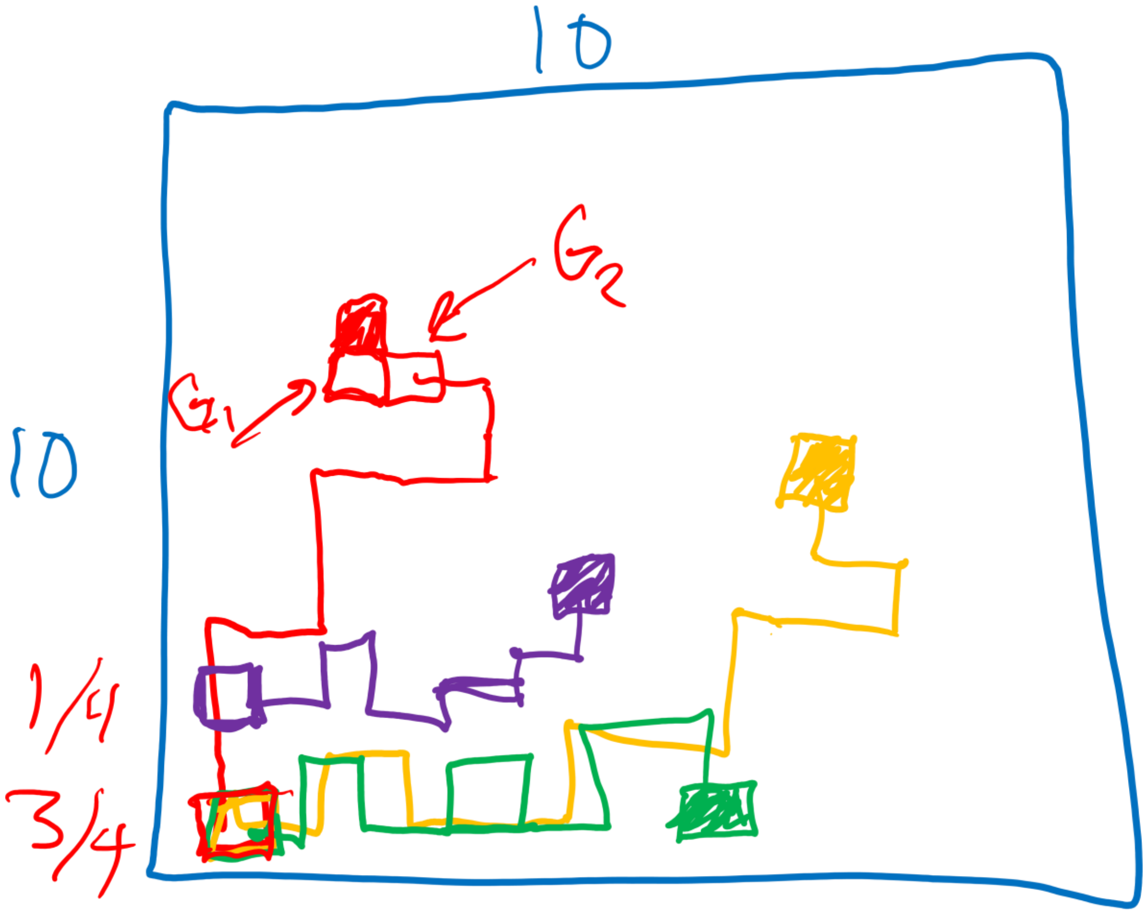
5/8 2/8 1/8

Inference vs Sampling



Motivation for Approximate Inference

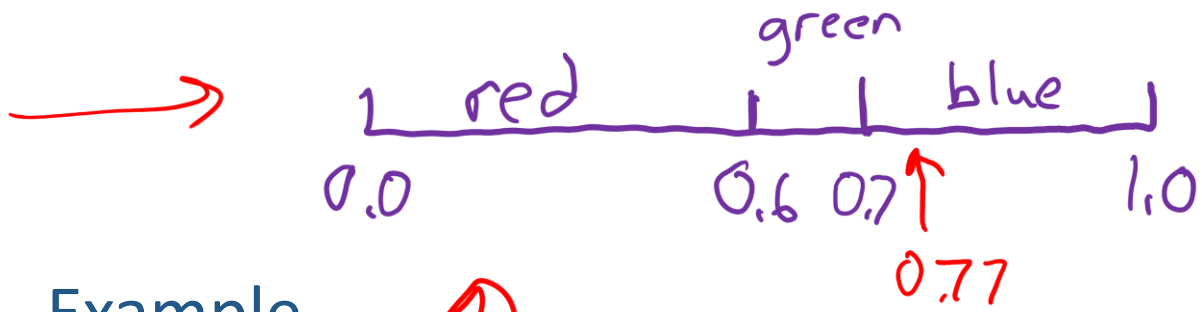
$$\frac{P(G_{t+1} \mid G_t)}{P(G_0)}$$



$P(G_{20})$

$$G_0 \to G_1 \to G_2 \to$$

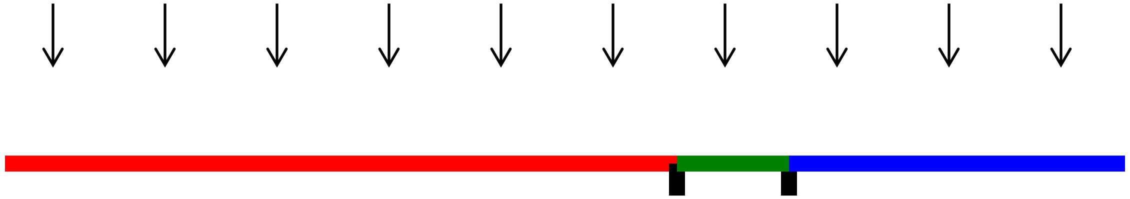
Sampling



Sampling from given distribution

- Step 1: Get sample *u* from uniform distribution over [0, 1)
  - e.g. random() in python

- Step 2: Convert this sample *u* into an outcome for the given distribution by having each outcome associated with a sub-interval of [0,1) with sub-interval size equal to probability of the outcome

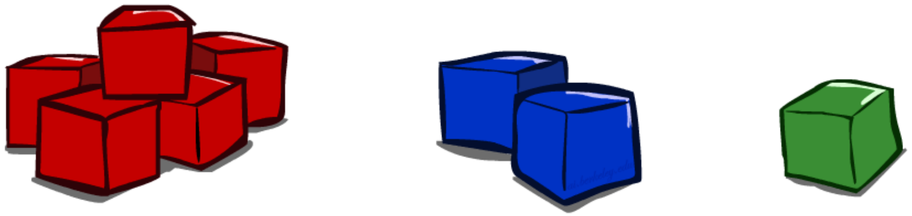


Example

C	P(C)
red	0.6
green	0.1
blue	0.3

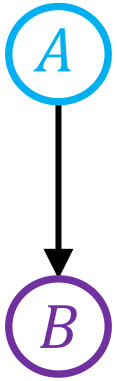
$$0 \leq u < 0.6, \rightarrow C = red$$
$$0.6 \leq u < 0.7, \rightarrow C = green$$
$$0.7 \leq u < 1, \rightarrow C = blue$$

- If random() returns *u* = 0.83, then our sample is *C* = blue
- E.g, after sampling 8 times:



Sampling

How would you sample from a conditional distribution?



$P(A)$

+a	1/2
-a	1/2

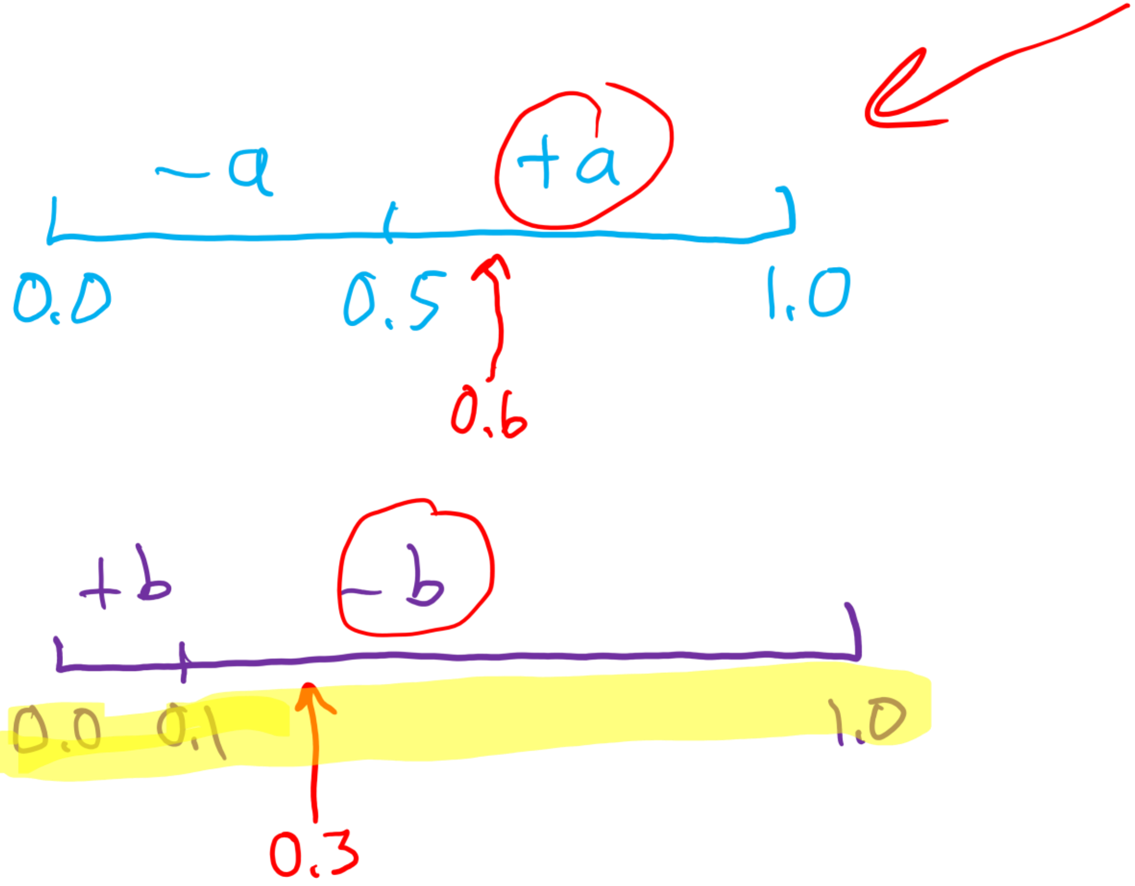
$P(B|A)$

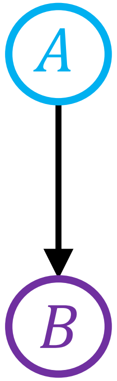
+a	+b	1/10
	-b	9/10
-a	+b	1/2
	-b	1/2

Sampling

$P(A, B)$

How would you sample from a conditional distribution?





$P(A)$

+a	1/2
-a	1/2

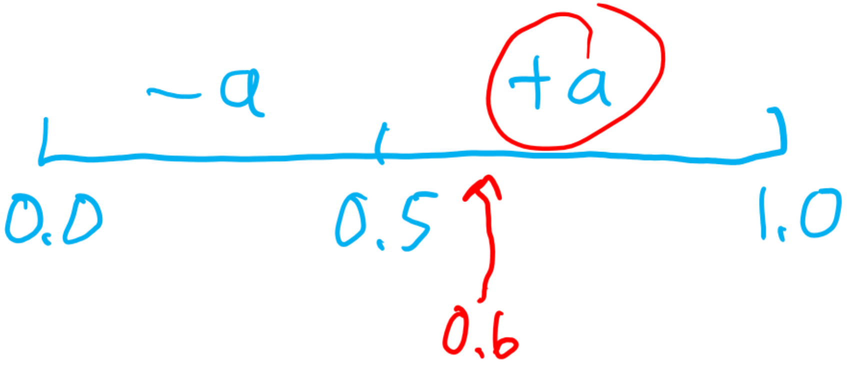
$P(B|A)$

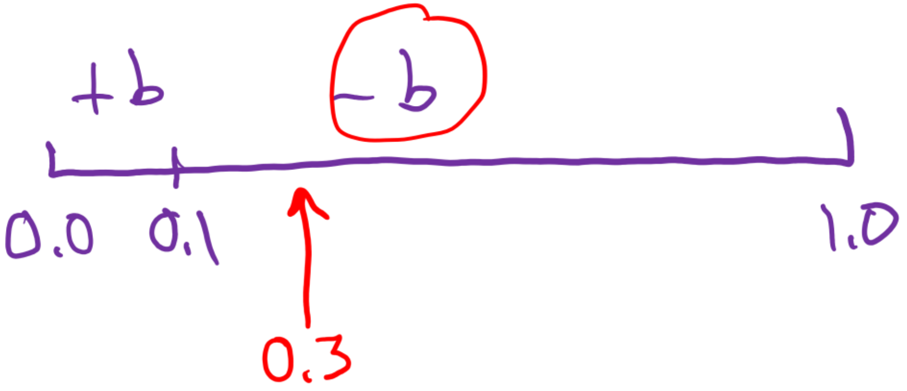
+a	+b	1/10
	-b	9/10
-a	+b	1/2
	-b	1/2

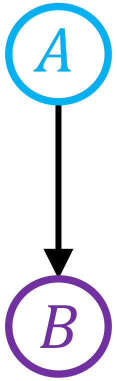
Sampling

$P(A, B)$

How would you sample from a conditional distribution?





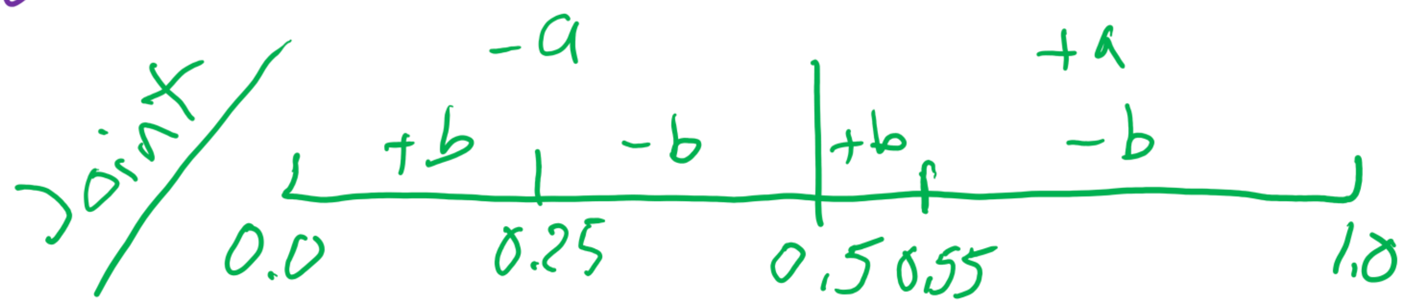


$P(A)$

+a	1/2
-a	1/2

$P(B|A)$

+a	+b	1/10
	-b	9/10
-a	+b	1/2
	-b	1/2



Sampling in Bayes' Nets

Prior Sampling

Rejection Sampling

Likelihood Weighting

Gibbs Sampling

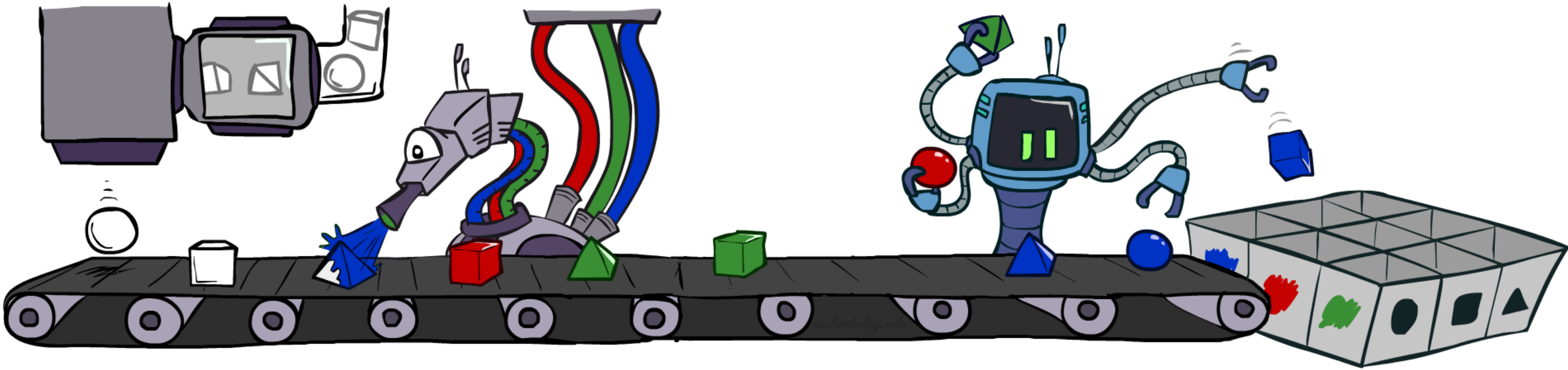
Handout

No evidence: Prior Sampling	Some evidence: Likelihood Weighted Sampling	All evidence: Likelihood Weighted
Input: no evidence for i=1, 2, …, m ▪ Sample x_i from $P(X_i \mid Parents(X_i))$ return $(x_1, x_2, …, x_m)$	Input: evidence instantiation w = 1.0 for i=1, 2, …, m if X_i is an evidence variable ▪ X_i = observation x_i for X_i ▪ Set $w = w * P(x_i \mid Parents(X_i))$ else ▪ Sample x_i from $P(X_i \mid Parents(X_i))$ return $(x_1, x_2, …, x_m)$, w	Input: evidence instantiation w = 1.0 for i=1, 2, …, m ▪ Set $w = w * P(x_i \mid Parents(X_i))$ return w

Prior Sampling

$S \rightarrow C$

S C



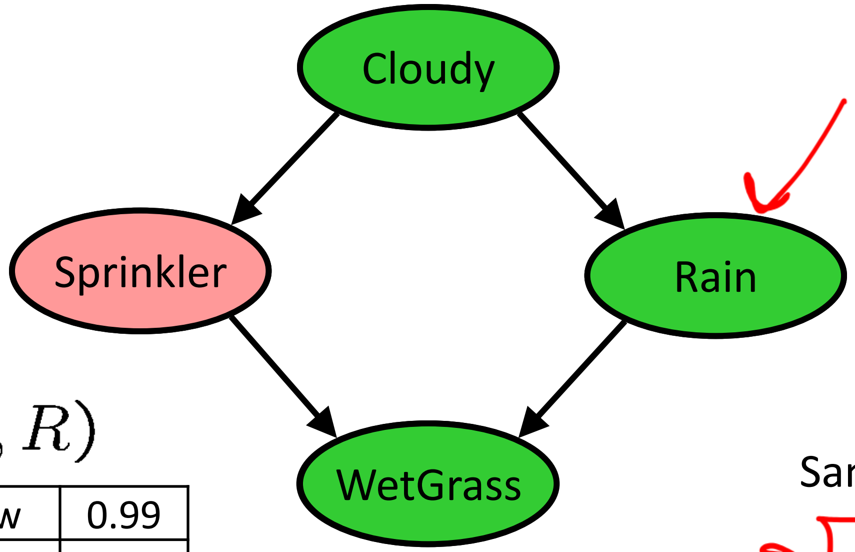
Prior Sampling

$P(C)$

+c	0.5
-c	0.5

$P(S|C)$

+c	+s	0.1
	-s	0.9
-c	+s	0.5
	-s	0.5



$P(R|C)$

+c	+r	0.8
	-r	0.2
-c	+r	0.2
	-r	0.8

$P(W|S,R)$

+s	+r	+w	0.99
		-w	0.01
	-r	+w	0.90
		-w	0.10
-s	+r	+w	0.90
		-w	0.10
	-r	+w	0.01
		-w	0.99

Samples:

+c, -s, +r, +w

-c, +s, -r, +w

…

Prior Sampling

for j in 1,..., N

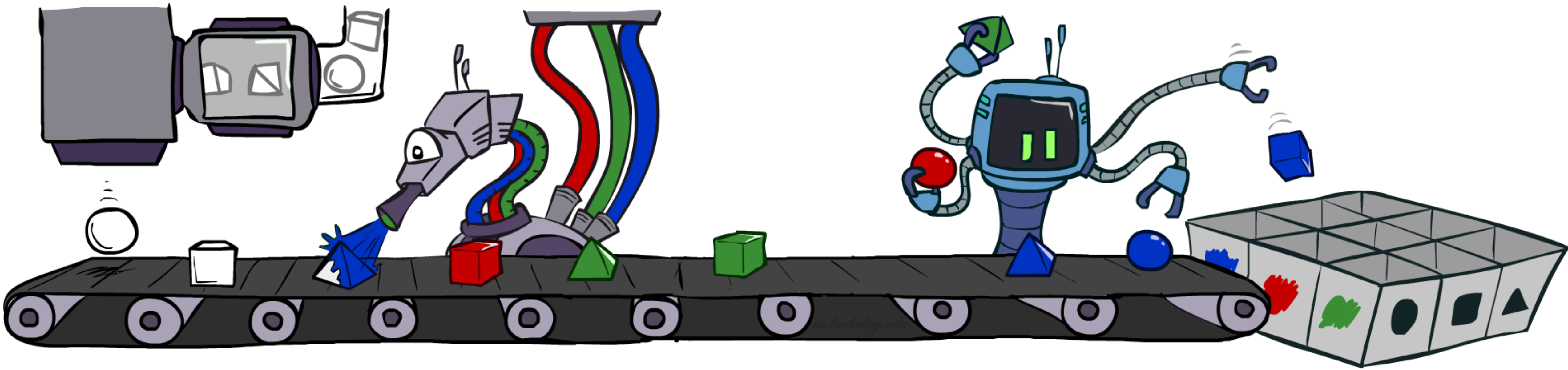
m: # of R.V. nodes in Bayes Net

N: num_samples

For i=1, 2, …, m

- Sample x_i from $P(X_i \mid Parents(X_i))$

Return $(x_1, x_2, \ldots, x_m)$

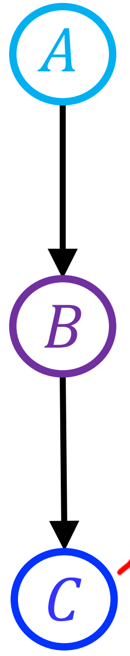


Poll 1

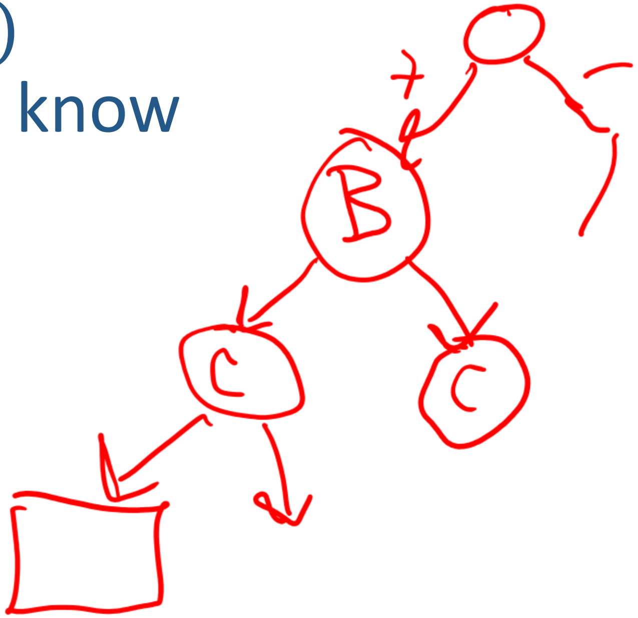
Prior Sampling: What does the value $\frac{N(+a,-b,+c)}{N}$ approximate?

1000

A. $P(+a,-b,+c)$
B. $P(+c \mid +a,-b)$
C. $P(+c \mid -b)$
D. $P(+c)$
E. I don't know



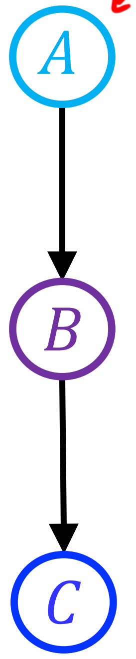
$P(C|B)$



Samples

+a	+b	+c
−a	+b	+c
+a	+b	+c
...		

Poll 1



Prior Sampling: What does the value $\frac{N(+a,-b,+c)}{N}$ approximate?

A. $P(+a, -b, +c)$
B. $P(+c \mid +a, -b)$
C. $P(+c \mid -b)$
D. $P(+c)$
E. I don't know

Let's assume for simplicity that all probabilities are 0.5:

$\mathrm{P}(a) = 0.5\ \forall a$
$\mathrm{P}(b \mid a) = 0.5\ \forall a, b$
$\mathrm{P}(c \mid b) = 0.5\ \forall b, c$

Let's take 1000 samples:

$+a$ $\frac{498}{1000}$ $-b$ $\frac{250}{498}$ $+c$ $\frac{126}{250}$

Poll 2

How many $\{-a, +b, -c\}$ samples out of N=1000 should we expect?

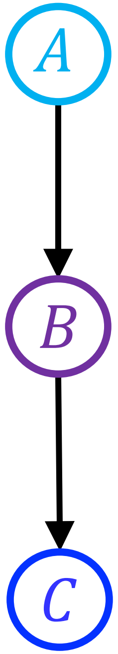
A. 1
B. 50
C. 125
D. 200
E. I have no idea

N

1000

P

$\frac{1}{2}\ \frac{1}{2}\ \frac{1}{5}$



$P(A)$

+a	1/2
-a	1/2

$P(B|A)$

+a	+b	1/10
	-b	9/10
-a	+b	1/2
	-b	1/2

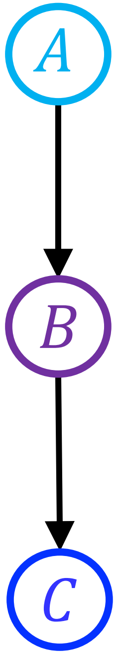
$P(C|B)$

+b	+c	4/5
	-c	1/5
-b	+c	1
	-c	0

Poll 2

How many $\{-a, +b, -c\}$ samples out of N=1000 should we expect?

A. 1
B. 50
C. 125
D. 200
E. I have no idea



$P(A)$

+a	1/2
-a	1/2

$P(B|A)$

+a	+b	1/10
	-b	9/10
-a	+b	1/2
	-b	1/2

$P(C|B)$

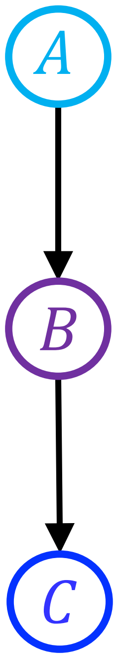
+b	+c	4/5
	-c	1/5
-b	+c	1
	-c	0

Probability of a sample

Given this Bayes Net & CPT,
what is $P(+a, +b, +c)$?

Algorithm: Multiply probability of each node given parents:

- w = 1.0
- for i=1, 2, …, m
  - Set w = w * $\boxed{P(x_i \mid \text{Parents}(X_i))}$
- return w



$P(A)$

+a	1/2
-a	1/2

$P(B|A)$

+a	+b	1/10
	-b	9/10
-a	+b	1/2
	-b	1/2

$P(C|B)$

+b	+c	4/5
	-c	1/5
-b	+c	1
	-c	0

Example

We'll get a bunch of samples from the BN:

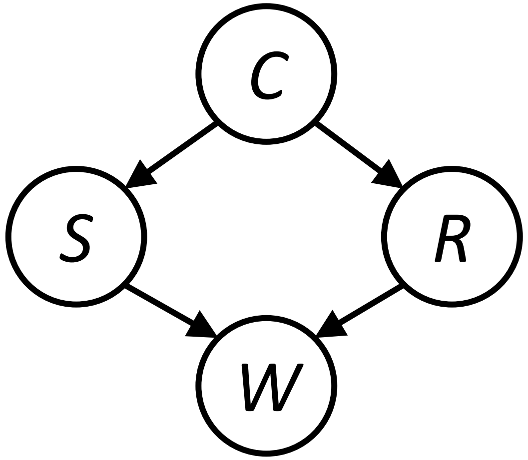
+c, -s, +r, +w

+c, +s, +r, +w

-c, +s, +r, -w   +c   0

+c, -s, +r, +w

-c, -s, -r, +w   -c   0

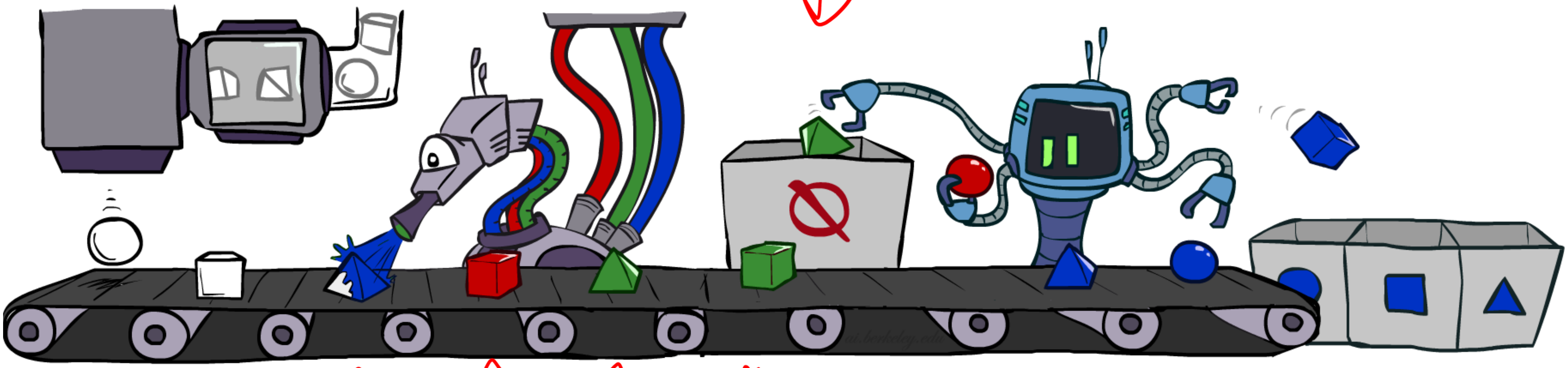


If we want to know P(W)

- We have counts <+w:4, -w:1>
- Normalize to get P(W) = <+w:0.8, -w:0.2>
- This will get closer to the true distribution with more samples
- Can estimate anything else, too
- What about P(C| +w)? P(C| +r, +w)? P(C| -r, -w)?
- Fast: can use fewer samples if less time (what's the drawback?)

Rejection Sampling

$P(\text{Shape} \mid \text{blue})$



Rejection Sampling

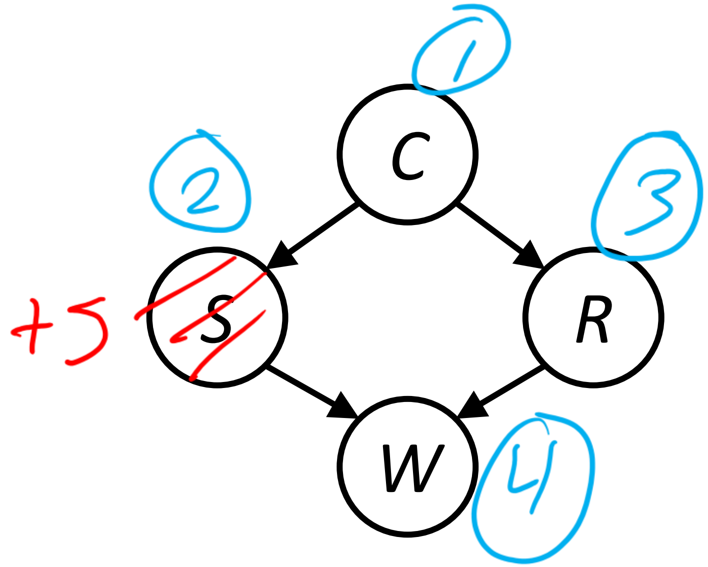
$P(C \mid +s)$

Let's say we want P(C)

- No point keeping all samples around
- Just tally counts of C as we go

Let's say we want P(C| +s)

- Same thing: tally C outcomes, but ignore (reject) samples which don't have S=+s
- This is called rejection sampling
- It is also consistent for conditional probabilities (i.e., correct in the limit)



+c, -s, +r, +w
+c, +s, +r, +w
-c, +s, +r, -w
+c, -s, +r, +w
-c, -s, -r, +w

Rejection Sampling

one sample

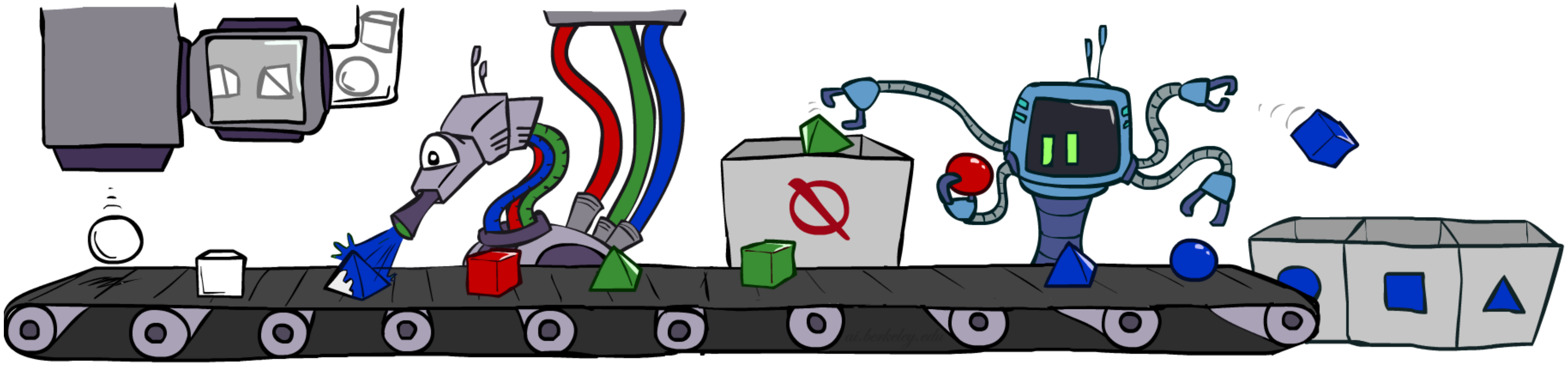
do until N=1000

IN: evidence instantiation

For i=1, 2, …, m

- Sample x_i from $P(X_i \mid Parents(X_i))$
- If x_i not consistent with evidence
  - Reject: Return, and no sample is generated in this cycle

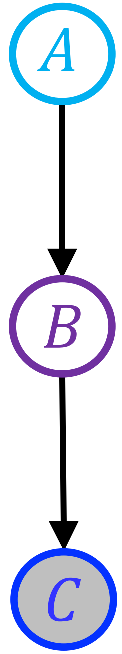
Return $(x_1, x_2, …, x_m)$



Poll 3

What queries can we (approximately) answer with rejection sampling samples (evidence: $+c$)?
Select all that apply

N_{total}



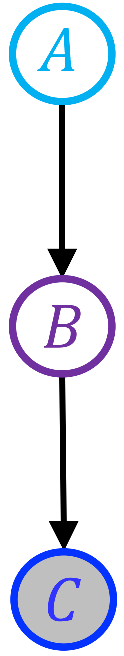
A. $P(+a, -b, +c)$ 37%
B. $\boxed{P(+a, -b \mid +c)}$ 97%
C. $P(+a, -b \mid -c)$ 11%
D. None of the above

Counts $N(A, B, C)$

+a	+b	+c	4
+a	+b	-c	
+a	-b	+c	3
+a	-b	-c	
-a	+b	+c	2
-a	+b	-c	
-a	-b	+c	1
-a	-b	-c	

/10

Poll 3



What queries can we (approximately) answer with rejection sampling samples (evidence: $+c$)?
Select all that apply

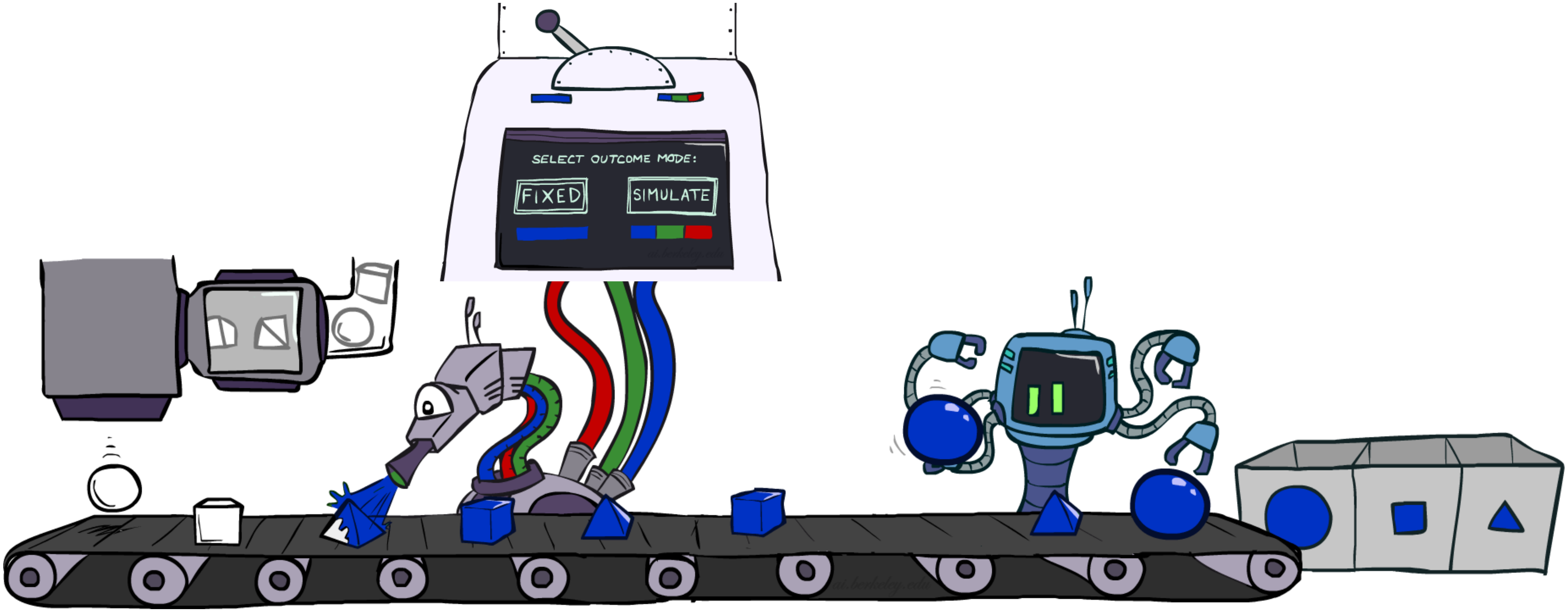
A. $P(+a, -b, +c)$
B. $P(+a, -b \mid +c)$
C. $P(+a, -b \mid -c)$
D. None of the above

A ~~and C~~: only if we also have total number of attempts

Counts $N(A, B, C)$

+a	+b	+c	4
+a	+b	-c	
+a	-b	+c	3
+a	-b	-c	
-a	+b	+c	2
-a	+b	-c	
-a	-b	+c	1
-a	-b	-c	

Likelihood Weighting

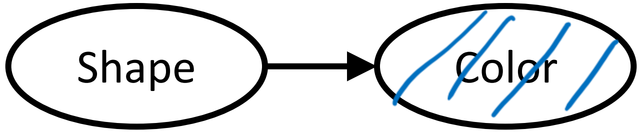


Likelihood Weighting

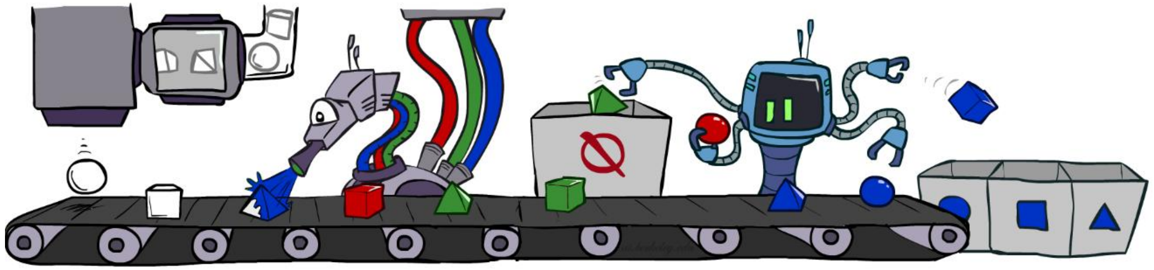
Problem with rejection sampling:

- If evidence is unlikely, rejects lots of samples
- Evidence not exploited as you sample
- Consider P(Shape|blue)

$P(C|S)$



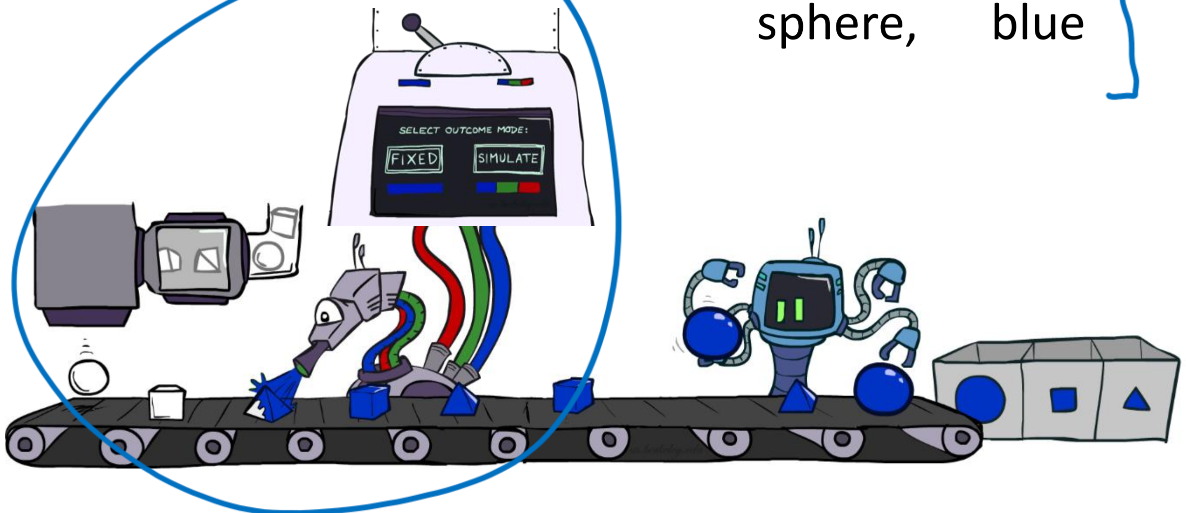
~~pyramid, green~~
~~pyramid, red~~
sphere, blue
~~cube, red~~
~~sphere, green~~



- Idea: fix evidence variables and sample the rest
  - Problem: sample distribution not consistent!
  - Solution: weight by probability of evidence given parents

$P(Color=pyramid) \approx 2/5$

pyramid, blue
pyramid, blue
sphere, blue
cube, blue
sphere, blue



Likelihood Weighting

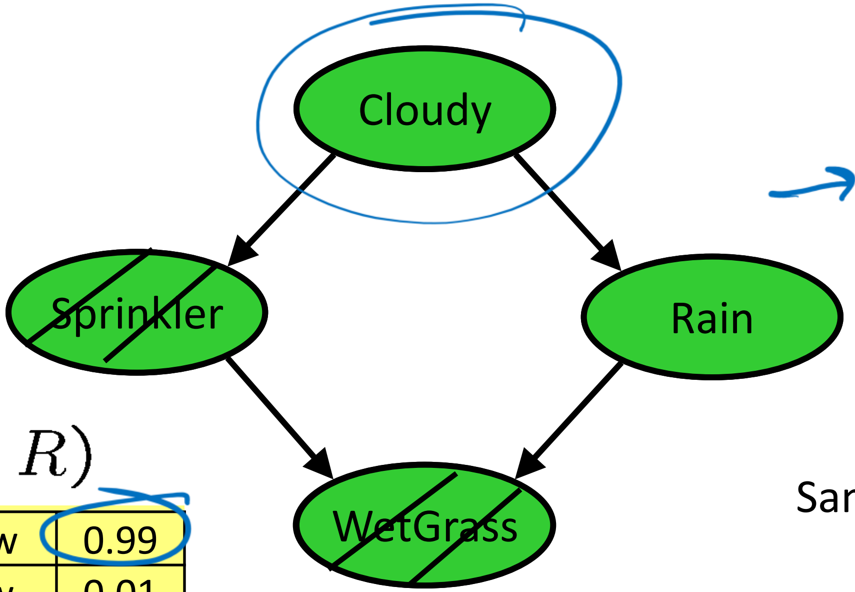
?c ?r +s +w

$P(C)$

+c	0.5
-c	0.5

$P(S|C)$

+c	+s	0.1
	-s	0.9
-c	+s	0.5
	-s	0.5



$P(R|C)$

+c	+r	0.8
	-r	0.2
-c	+r	0.2
	-r	0.8

$P(W|S,R)$

+s	+r	+w	0.99
		-w	0.01
	-r	+w	0.90
		-w	0.10
-s	+r	+w	0.90
		-w	0.10
	-r	+w	0.01
		-w	0.99

Samples:

+c, +s, +r, +w

...

$\boxed{+c\ +s\ +r\ +w}$

$$w = 1.0 \times 0.1 \times 0.99$$

Likelihood Weighting

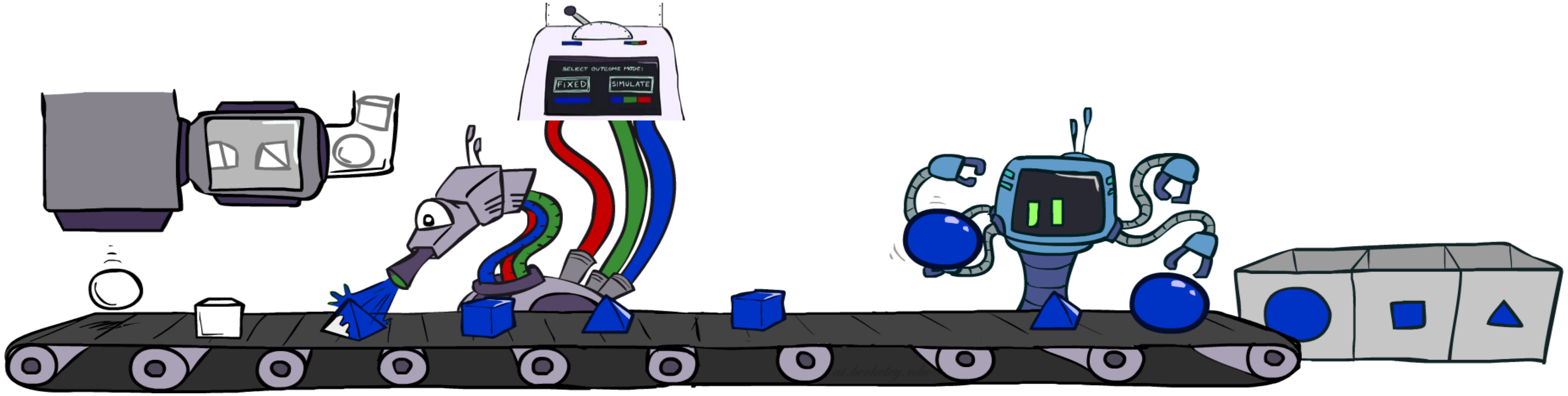
IN: evidence instantiation

w = 1.0

for i=1, 2, …, m

- if X_i is an evidence variable
  - X_i = observation x_i for X_i
  - Set w = w * $P(x_i \mid Parents(X_i))$
- else
  - Sample x_i from $P(X_i \mid Parents(X_i))$

return $(x_1, x_2, \ldots, x_m)$, w



Likelihood Weighting

No evidence:
Prior Sampling

Input: no evidence

for $i=1, 2, \ldots, m$

- Sample x_i from $P(X_i \mid Parents(X_i))$

return $(x_1, x_2, \ldots, x_m)$

Some evidence:
Likelihood Weighted Sampling

Input: evidence instantiation

$w = 1.0$

for $i=1, 2, \ldots, m$

if X_i is an evidence variable

- X_i = observation x_i for X_i
- Set $w = w * P(x_i \mid Parents(X_i))$

else

- Sample x_i from $P(X_i \mid Parents(X_i))$

return $(x_1, x_2, \ldots, x_m)$, w

All evidence:
Likelihood Weighted

Input: evidence instantiation

$w = 1.0$

for $i=1, 2, \ldots, m$

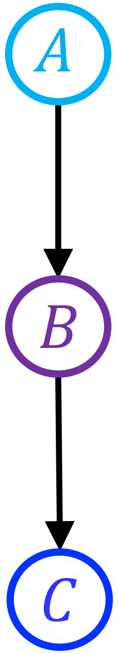
- Set $w = w * P(x_i \mid Parents(X_i))$

return w

Remember Poll 2

How many $\{-a, +b, -c\}$ samples out of N=1000 should we expect?

A. 1
B. 50
C. 125
D. 200
E. I have no idea



$P(A)$

+a	1/2
-a	1/2

$P(B|A)$

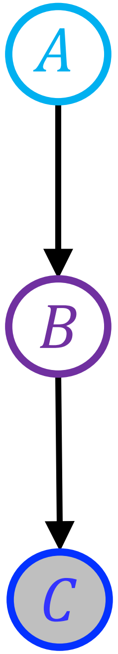
+a	+b	1/10
	-b	9/10
-a	+b	1/2
	-b	1/2

$P(C|B)$

+b	+c	4/5
	-c	1/5
-b	+c	1
	-c	0

Likelihood Weighting

$-a$	$+a$
$+b$	$-b$
$w = 4/5$	$w = 1$



$P(A)$

+a	1/2
-a	1/2

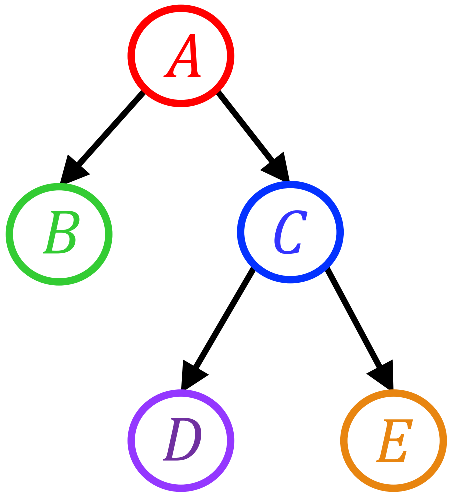
$P(B|A)$

+a	+b	1/10
	-b	9/10
-a	+b	1/2
	-b	1/2

$P(C|B)$

+b	$\boxed{+c}$	4/5
	-c	1/5
-b	$\boxed{+c}$	1
	-c	0

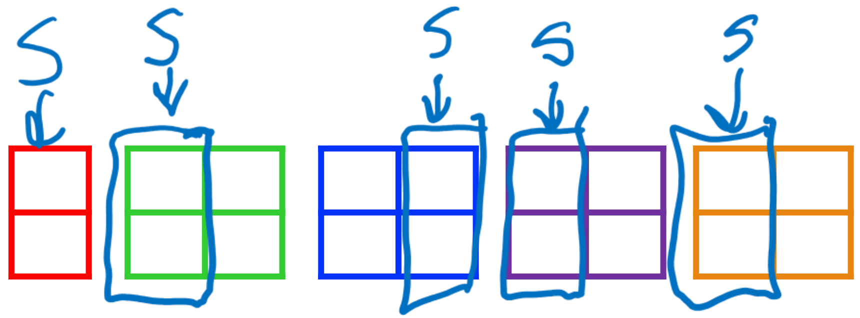
Likelihood Weighting Distribution



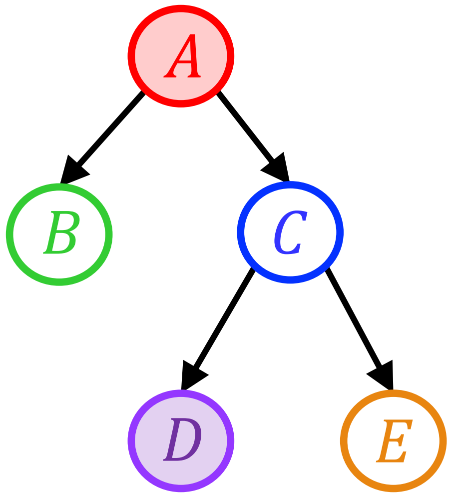
Consistency of likelihood weighted sampling distribution

Joint from Bayes nets

$$P(A,B,C,D,E) = P(A)\, P(B|A)\, P(C|A)\, P(D|C)\, P(E|C)$$



Likelihood Weighting Distribution

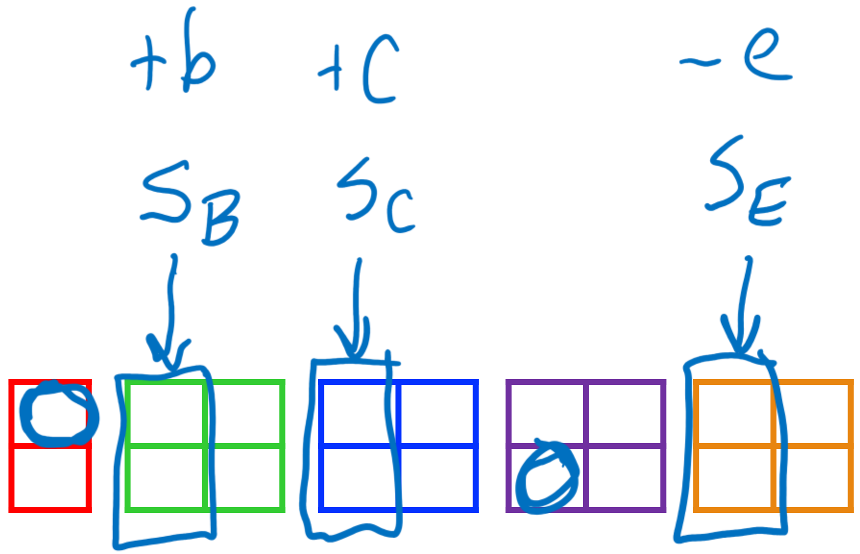


Consistency of likelihood weighted sampling distribution

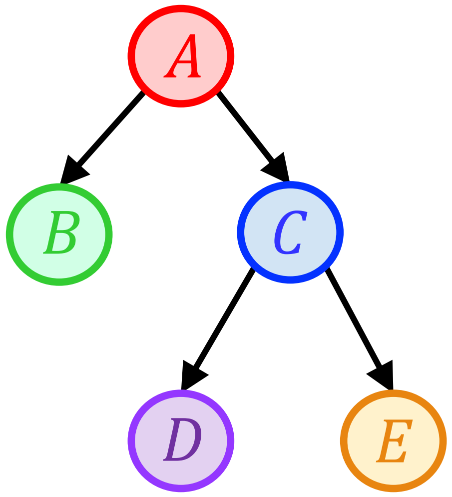
Evidence: $+a, -d$

Joint from Bayes nets

$$P(A, B, C, D, E) = P(+a)\, P(B|+a)\, P(C|+a)\, P(-d|C)\, P(E|C)$$



Likelihood Weighting Distribution

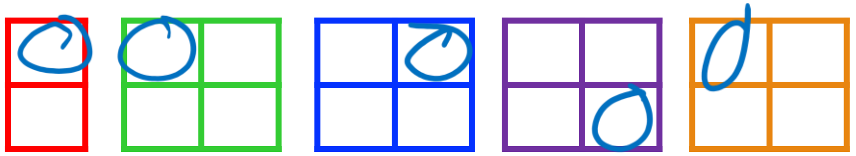


Consistency of likelihood weighted sampling distribution

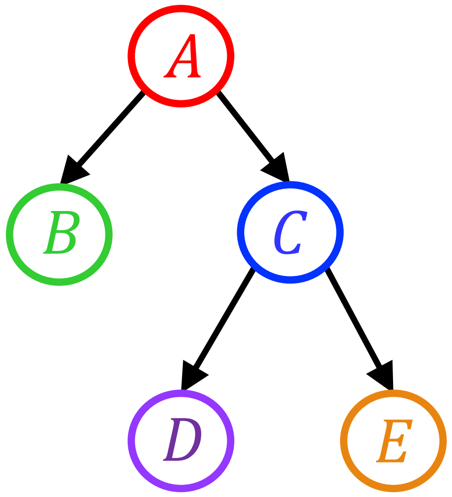
Evidence: $+a, +b, -c, -d, +e$

Joint from Bayes nets

$$P(A,B,C,D,E) = P(+a)\, P(+b|+a)\, P(-c|+a)\, P(-d|-c)\, P(+e|-c)$$



Likelihood Weighting Distribution

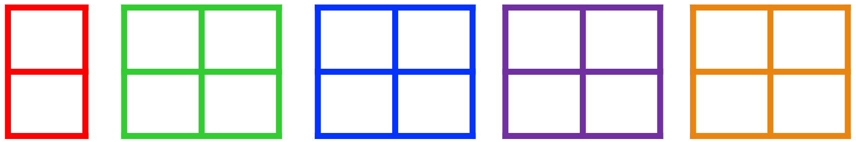


Consistency of likelihood weighted sampling distribution

Evidence: None

Joint from Bayes nets

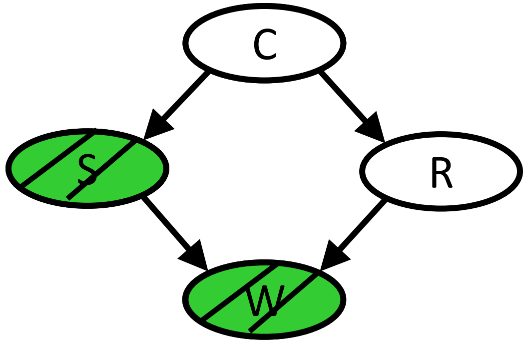
$$P(A,B,C,D,E) = P(A)\,P(B|A)\,P(C|A)\,P(D|C)\,P(E|C)$$



Likelihood Weighting

Sampling distribution if z sampled and e fixed evidence

$$S_{WS}(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{l} P(z_i | \text{Parents}(Z_i))$$



Now, samples have weights

$$w(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{m} P(e_i | \text{Parents}(E_i))$$

Together, weighted sampling distribution is consistent

$$S_{\mathrm{WS}}(z, e) \cdot w(z, e) = \prod_{i=1}^{l} P(z_i | \text{Parents}(z_i)) \prod_{i=1}^{m} P(e_i | \text{Parents}(e_i))$$

$$= P(\mathbf{z}, \mathbf{e})$$

Poll 4

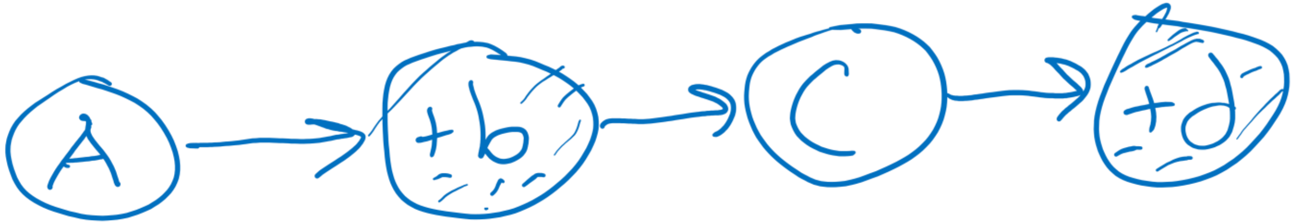
$P(A\ C \mid +b, +d)$

Given a fixed query, two identical samples from likelihood weighted sampling will have the same exact weights.

A. True
B. False
C. It depends
D. I don't know

$+a$	$+b$	$+c$	$+d$	W
$+a$	$+b$	$+c$	$+d$	

$W = P(+b \mid +a)\, P(+d \mid +c)$

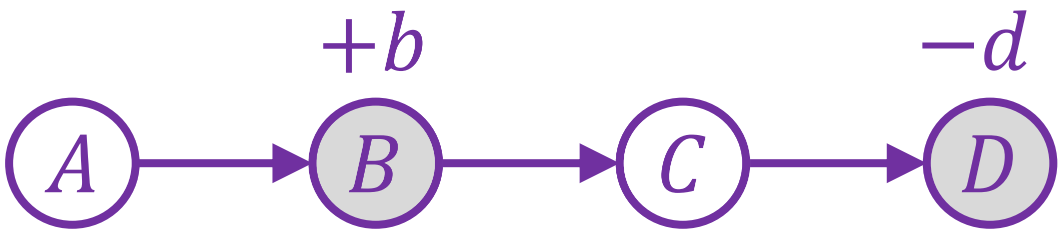


Poll 4

Given a fixed query, two identical samples from likelihood weighted sampling will have the same exact weights.

A. True
B. False
C. It depends
D. I don't know

Example:
$P(A, C \mid +b, -d)$



Poll 5

evidence: +c

What does the following likelihood weighted value approximate?

$$\underline{\text{weight}_{(+a,-b,+c)}} \cdot \boxed{\frac{N(+a,-b,+c)}{N}}$$

A. $P(+a, -b, +c)$

B. $P(+a, -b \mid + c)$

C. I'm not sure

$A \to B \to C$

evid

$w = P(+c \mid -b)$

$P(A)\,P(B \mid A)$

$P(+a)\,P(-b \mid +a)$

Poll 5

What does the following likelihood weighted value approximate?

$$\text{weight}_{(+a,-b,+c)} \cdot \frac{N(+a,-b,+c)}{N}$$

A. $P(+a,-b,+c)$

B. $P(+a,-b \mid +c)$

C. I'm not sure

$$weight_{(x_1,x_2,x_3,e_1,e_2,e_3)} = \prod_i P(e_i \mid Parents(e_i))$$

$$\frac{N(x_1,x_2,x_3,e_1,e_2,e_3)}{N} \approx \prod_j P(x_j \mid Parents(x_j))$$

$$weight_{(x_1,x_2,x_3,e_1,e_2,e_3)} \cdot \frac{N(x_1,x_2,x_3,e_1,e_2,e_3)}{N} \approx P(x_1,x_2,x_3,e_1,e_2,e_3)$$

Likelihood Weighting

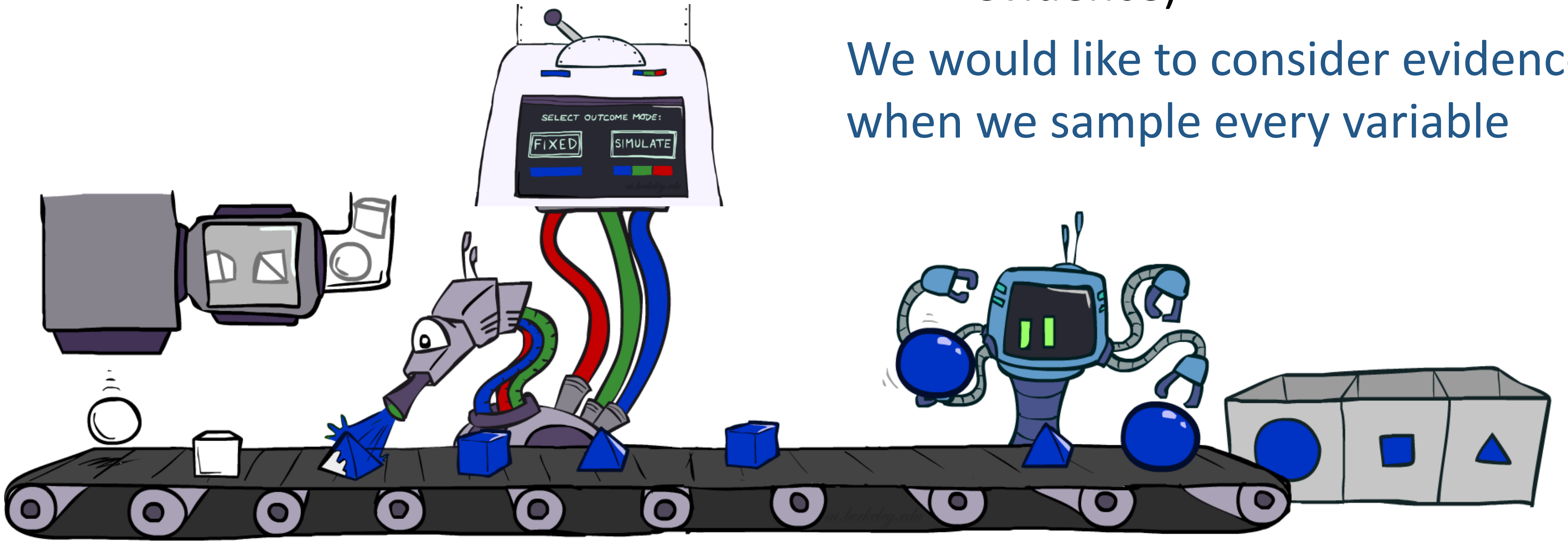
Likelihood weighting is good

- We have taken evidence into account as we generate the sample
- E.g. here, W's value will get picked based on the evidence values of S, R
- More of our samples will reflect the state of the world suggested by the evidence

Likelihood weighting doesn’t solve all our problems

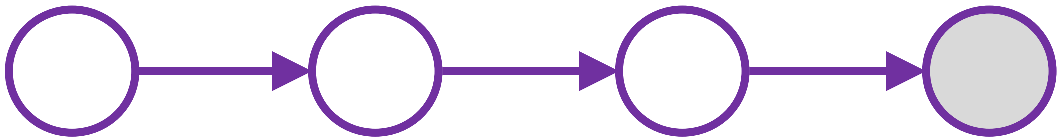
- Evidence influences the choice of downstream variables, but not upstream ones (C isn’t more likely to get a value matching the evidence)

We would like to consider evidence when we sample every variable



Likelihood Weighting

$P(Q_1, Q_2\ H_1, H_2, e_1, e_2)$



Mellon

I will eat $w = 0,00001$

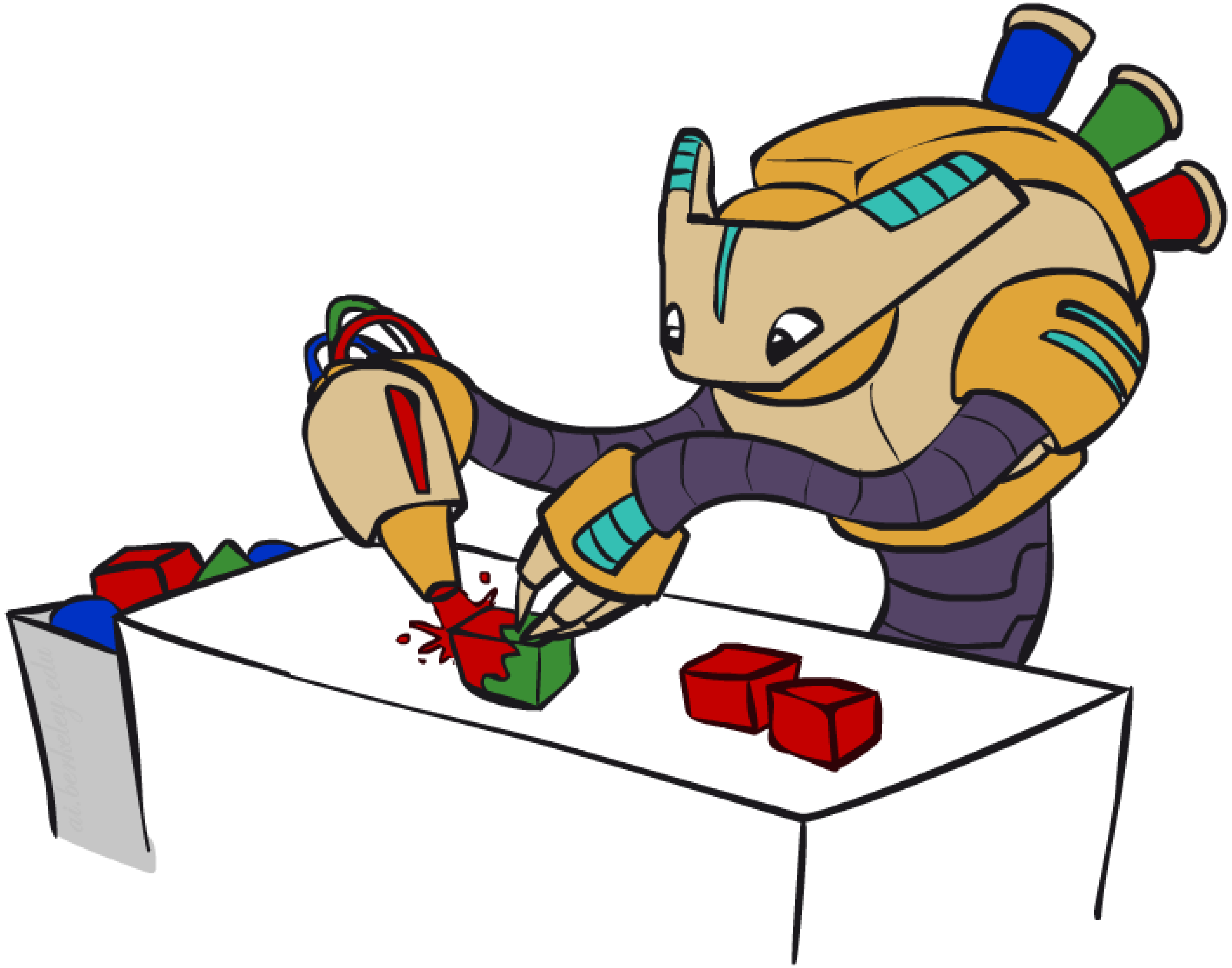
Likelihood weighting doesn't solve all our problems

- Evidence influences the choice of downstream variables, but not upstream ones (C isn't more likely to get a value matching the evidence)

We would like to consider evidence when we sample every variable

→ Gibbs sampling

Gibbs Sampling



Gibbs Sampling

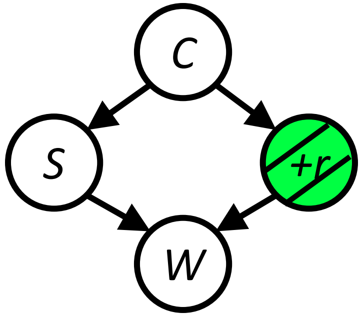
Procedure: keep track of a full instantiation $x_1, x_2, \ldots, x_m$.

1. Start with an arbitrary instantiation consistent with the evidence.
2. Sample one variable at a time, conditioned on all the rest, but keep evidence fixed.
3. Keep repeating this for a long time.

Gibbs Sampling Example: P(S | +r)

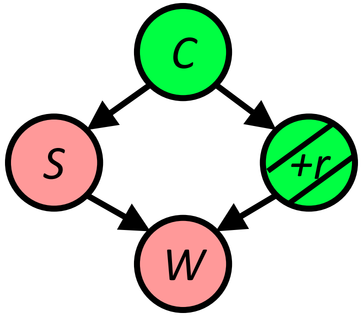
Step 1: Fix evidence

- R = +r



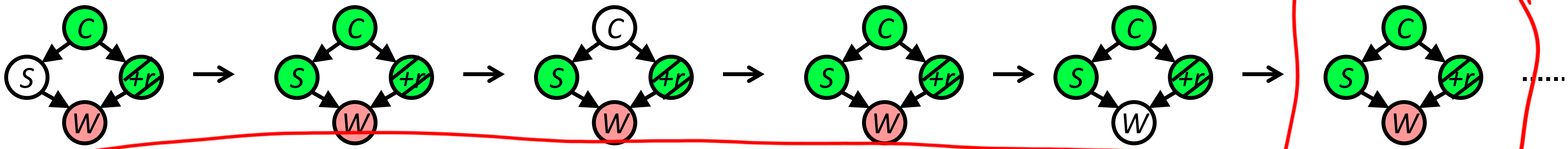
Step 2: Initialize other variables

- Randomly



Steps 3: Repeat

- Choose a non-evidence variable X
- Resample X from P(X | all other variables)

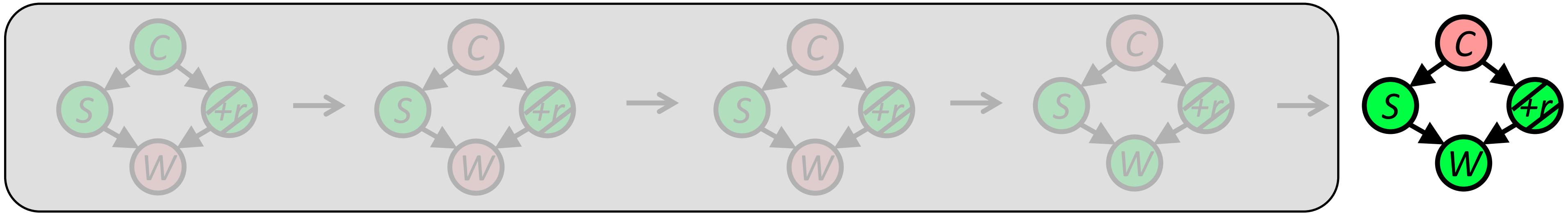


Sample from $P(S|+c,-w,+r)$ Sample from $P(C|+s,-w,+r)$ Sample from $P(W|+s,+c,+r)$

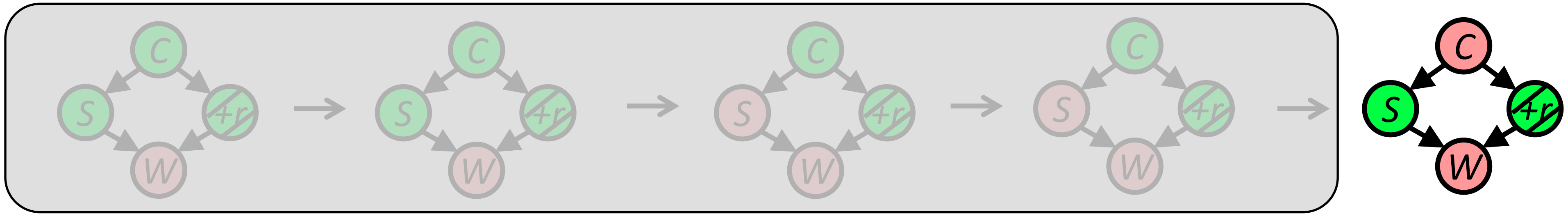
Gibbs Sampling Example: P(S | +r)

Keep only the last sample from each iteration:

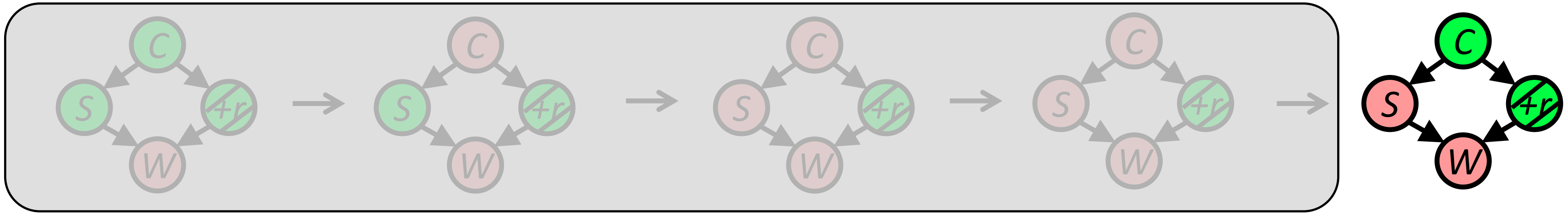
1.



2.

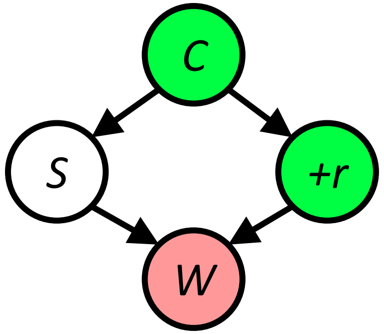


3.



Efficient Resampling of One Variable

Sample from P(S | +c, +r, -w)



$$
\begin{aligned}
P(S|+c,+r,-w) &= \frac{P(S,+c,+r,-w)}{P(+c,+r,-w)} \\
&= \frac{P(S,+c,+r,-w)}{\sum_s P(s,+c,+r,-w)} \\
&= \frac{P(+c)P(S|+c)P(+r|+c)P(-w|S,+r)}{\sum_s P(+c)P(s|+c)P(+r|+c)P(-w|s,+r)} \\
&= \frac{P(+c)P(S|+c)P(+r|+c)P(-w|S,+r)}{P(+c)P(+r|+c)\sum_s P(s|+c)P(-w|s,+r)} \\
&= \frac{P(S|+c)P(-w|S,+r)}{\sum_s P(s|+c)P(-w|s,+r)}
\end{aligned}
$$

Many things cancel out – only CPTs with S remain!

More generally: only CPTs that have resampled variable need to be considered, and joined together

Gibbs Sampling

Procedure: keep track of a full instantiation $x_1, x_2, \ldots, x_m$.

1. Start with an arbitrary instantiation consistent with the evidence.
2. Sample one variable at a time, conditioned on all the rest, but keep evidence fixed.
3. Keep repeating this for a long time.

Property: in the limit of repeating this infinitely many times the resulting sample is coming from the correct distribution

Rationale: both upstream and downstream variables condition on evidence.

In contrast: likelihood weighting only conditions on upstream evidence, and hence weights obtained in likelihood weighting can sometimes be very small. Sum of weights over all samples is indicative of how many "effective" samples were obtained, so want high weight.

Further Reading on Gibbs Sampling

Gibbs sampling produces sample from the query distribution P(Q | e) in limit of re-sampling infinitely often

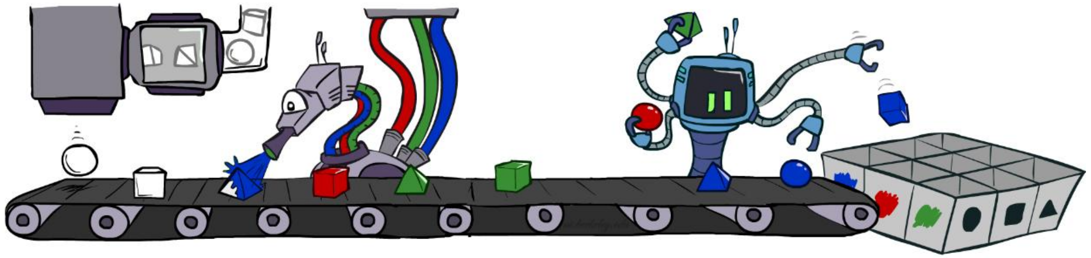
Gibbs sampling is a special case of more general methods called Markov chain Monte Carlo (MCMC) methods

- Metropolis-Hastings is one of the more famous MCMC methods (in fact, Gibbs sampling is a special case of Metropolis-Hastings)

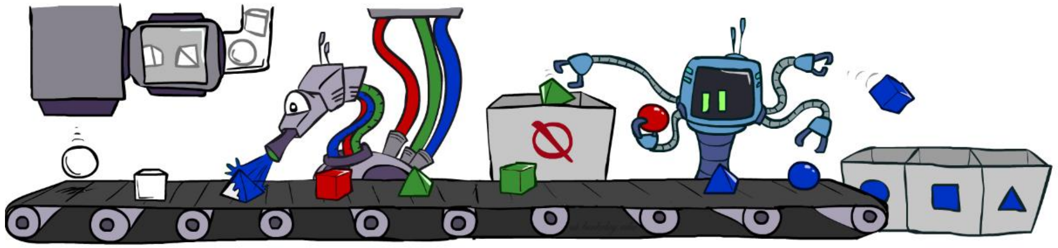
You may read about Monte Carlo methods – they're just sampling

Bayes Net Sampling Summary

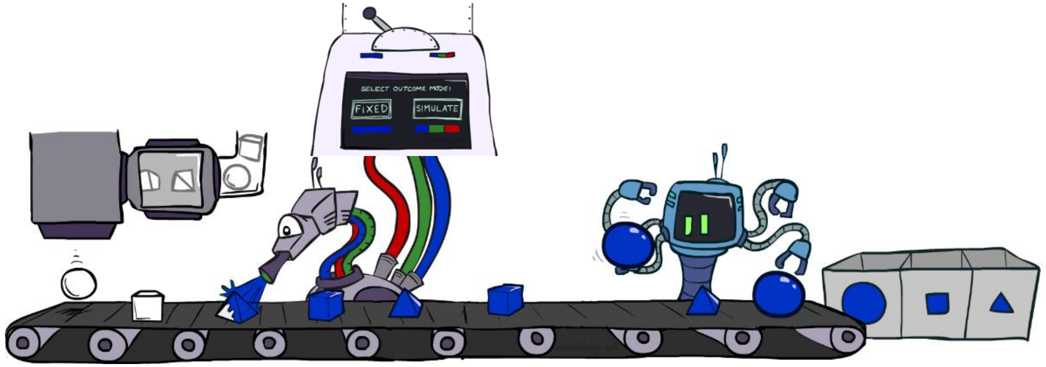
Prior Sampling P(Q, E)



Rejection Sampling P(Q | e)



Likelihood Weighting P(Q , e)



Gibbs Sampling P(Q | e)