Warm-up: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Healthy Squad Goals

- $2000 \leq \text{Calories} \leq 2500$
- $\text{Sugar} \leq 100$ g
- $\text{Calcium} \geq 700$ mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

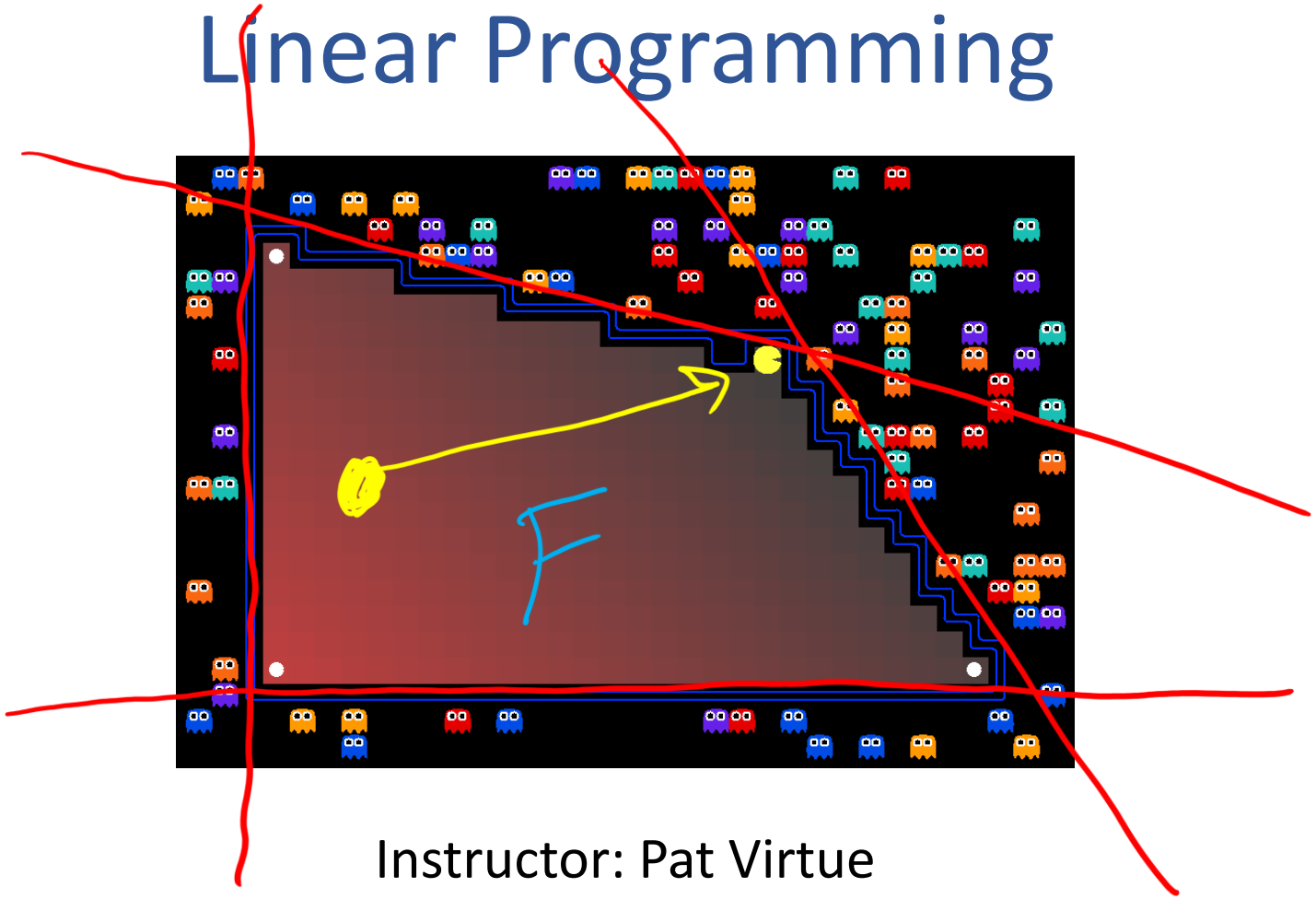
How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Questions about schedule update?

9/5 Thu	Informed Search	AIMA Ch. 3.5-6	pptx (inked) pdf (inked)
9/10 Tue	Adversarial Search	AIMA Ch. 5.1-2, 5.5	pptx (inked) pdf (inked)
9/12 Thu	Contraint Satisfaction Problems	AIMA Ch. 6.1-3, 6.5 CSP Demo	pptx (inked) pdf (inked) Video: Forward Checking Video: AC-3 Video: Ordering: MRV and LCV
9/17 Tue	Optimization & Linear Programming	Boyd and Vandenberghe Ch. 2.2.1, 2.2.4, 4.3-4.3.1 Desmos Demos: • Prereq: Dot Product • LP: Cost at points • LP: Zero cost • LP: Cost contours • LP: Constraint • LP: Cost with one constraint • LP	
9/19 Thu	Solving LPs & Integer Programming		
9/24 Tue	*Tentative:* Local Search & Ethics	AIMA Ch. 4.1, 6.4	
9/26 Thu	Logical Agents	AIMA Ch. 7.1-7	
10/1 Tue	MIDTERM 1 EXAM	In class	

AI: Representation and Problem Solving

Linear Programming



Instructor: Pat Virtue

Slide credits: CMU AI with drawings from http://ai.berkeley.edu

Warm-up: What to eat?

$A\vec{x} = b$ $a_{11}x_1 + a_{12}x_2$ $a_{21}x_1 + a_{22}x_2$

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

x_1

x_2

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

x_1 ?

x_2 ?

Optimization

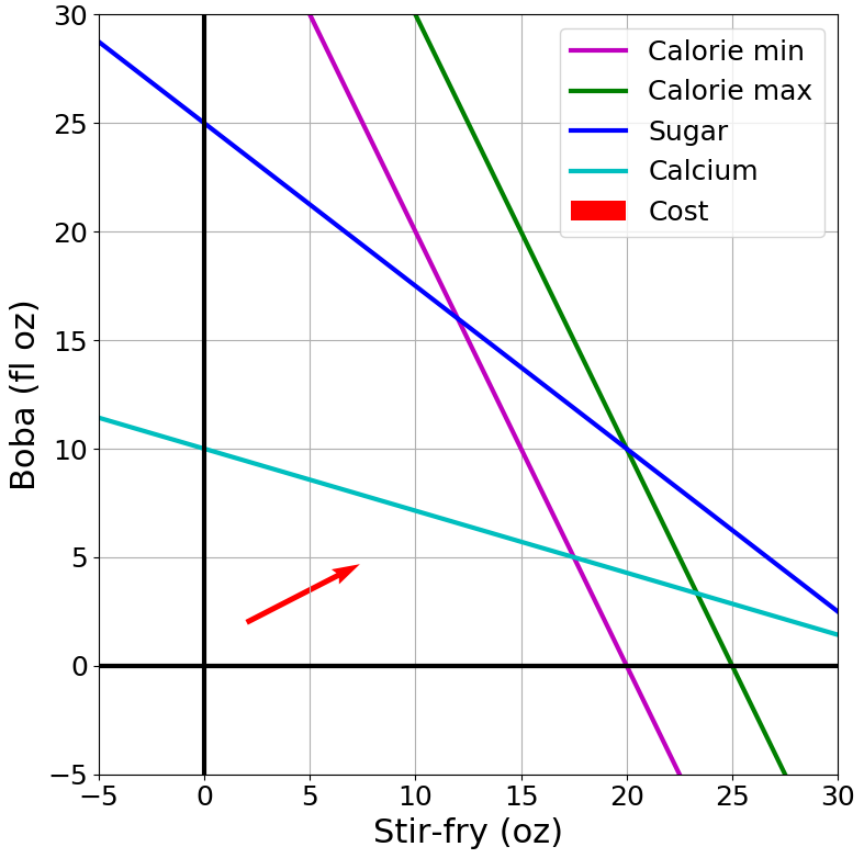
Problem Description

Optimization Representation

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

Graphical Representation



Constraint Satisfaction Problems

x^T

Map coloring

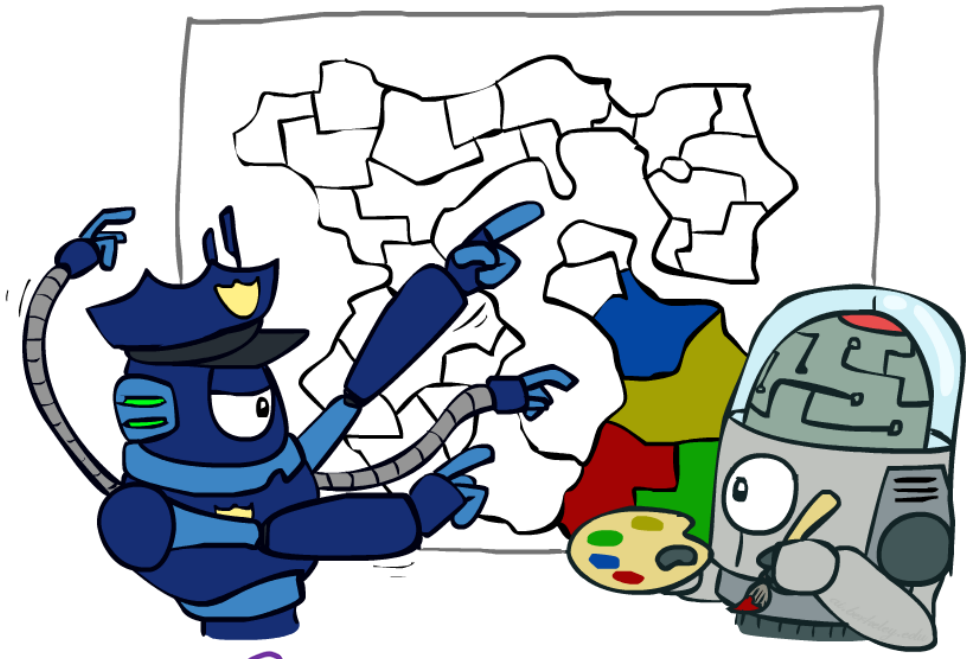
vectors

Any **x**

s.t. **x** satisfies constraints

"such that"

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1, x_2]^T$$



$\vec{x} \in \mathbb{R}^2$

$A \in \mathbb{R}^{m \times n}$

N

$m \begin{bmatrix} a_{11} & \\ & \end{bmatrix}$

Assume vectors are column vectors

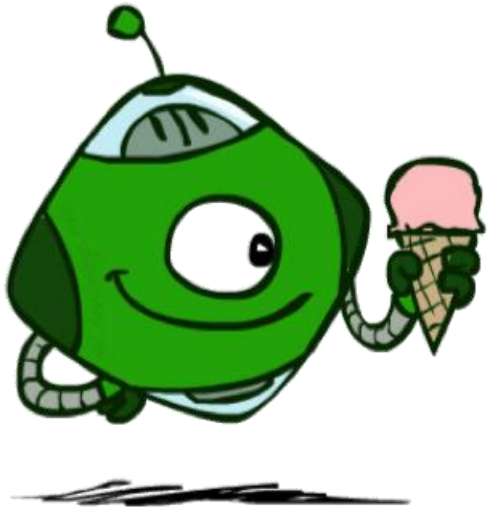
Notation Alert!

Optimization Formulation

Diet Problem

Any $\mathbf{x}$

s.t. $\mathbf{x}$ satisfies constraints



Healthy Squad Goals

- $2000 \leq \text{Calories} \leq 2500$
- $\text{Sugar} \leq 100$ g
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Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

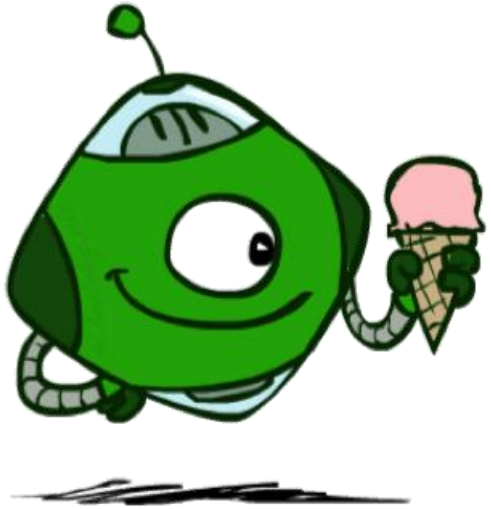
Optimization Formulation

Diet Problem

$$\min_{\mathbf{x}} \quad cost(\mathbf{x})$$

Objective function

$$\text{s.t.} \quad \mathbf{x} \text{ satisfies constraints}$$



Healthy Squad Goals

- $2000 \leq \text{Calories} \leq 2500$
- $\text{Sugar} \leq 100$ g
- $\text{Calcium} \geq 700$ mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

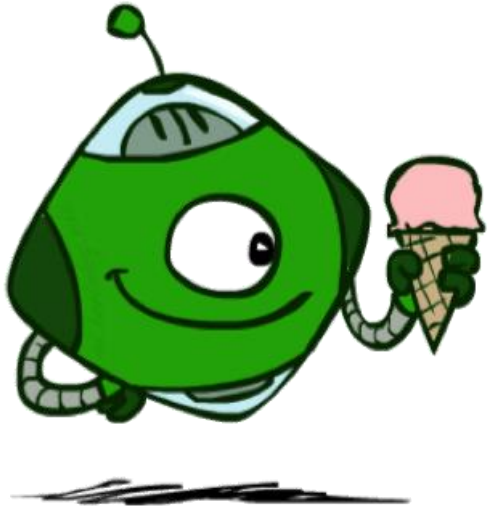
Notation Alert!

Optimization Formulation

Diet Problem

$$\min_{\mathbf{x}} \quad cost(\mathbf{x})$$

$$\text{s.t.} \quad calories(\mathbf{x}) \; contained$$
$$sugar(\mathbf{x}) \leq limit$$
$$\text{calcium}(\mathbf{x}) \geq limit$$



[Healthy Squad Goals]

- $2000 \leq \text{Calories} \leq 2500$
- $\text{Sugar} \leq 100$ g
- $\text{Calcium} \geq 700$ mg

$\vec{x}$

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

x_1

x_2

Optimization Formulation

Diet Problem

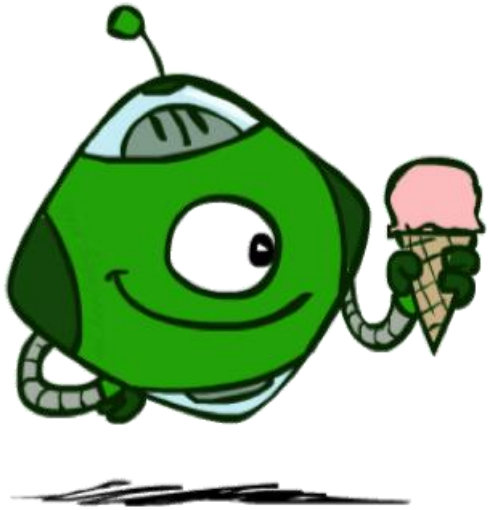
$c_1 x_1 + c_2 x_2$

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\begin{aligned}
\text{s.t.} \quad & 100\,x_1 + 50\,x_2 \geq 2000 \\
& 100\,x_1 + 50\,x_2 \leq 2500 \\
& 3\,x_1 + 4\,x_2 \leq 100 \\
& 20\,x_1 + 70\,x_2 \geq 700
\end{aligned}$$

$a_1 x_1 + a_2 x_2 \geq 2$

$a_1 x_1 + a_2 x_2 \leq 2$



[Healthy Squad Goals](#)

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Fat = 2

Food	Cost	Calories	Sugar	Calcium	F
Stir-fry (per oz)	c_1 1	100	3	20	a_1
Boba (per fl oz)	c_2 0.5	50	4	70	a_2

x_1

x_2

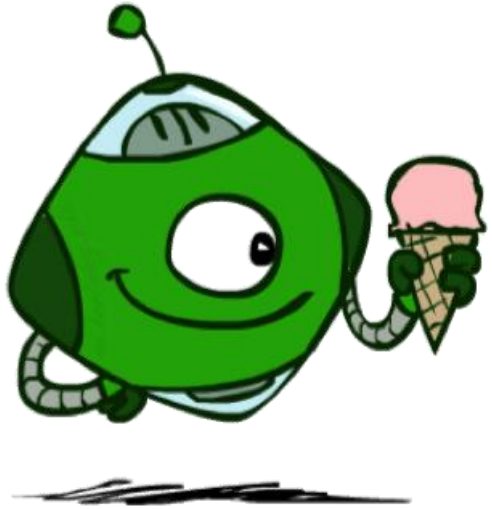
Notation Alert!

Optimization Formulation

Diet Problem

$$\min_{x_1, x_2} \quad c_1 x_1 + c_2 x_2 \longrightarrow c^T x$$

$$\text{s.t.} \quad \begin{aligned} a_{1,1} x_1 + a_{1,2} x_2 &\geq b_1 \\ a_{2,1} x_1 + a_{2,2} x_2 &\leq b_2 \\ a_{3,1} x_1 + a_{3,2} x_2 &\leq b_3 \\ a_{4,1} x_1 + a_{4,2} x_2 &\geq b_4 \end{aligned}$$



Cost

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad \begin{matrix} c_1 \\ c_2 \end{matrix}$$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & \\ \vdots & \\ a_{4,1} & a_{4,2} \end{bmatrix}$$

Notation Alert!

Stir-fry Boba

$$A = \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix}$$

Limit

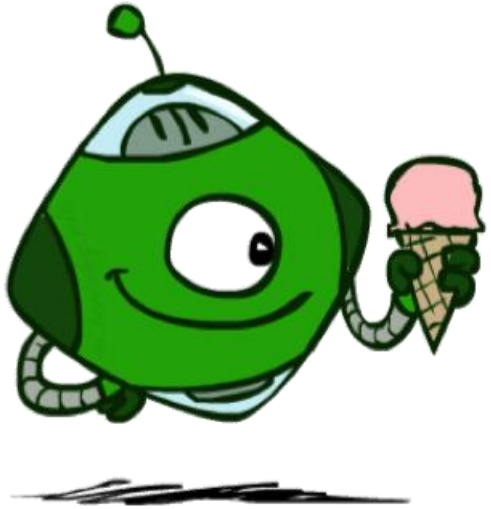
Calorie min
Calorie max
Sugar
Calcium

Optimization Formulation

Diet Problem

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad \begin{aligned} a_{1,1}\, x_1 + a_{1,2}\, x_2 &\geq b_1 \\ a_{2,1}\, x_1 + a_{2,2}\, x_2 &\leq b_2 \\ a_{3,1}\, x_1 + a_{3,2}\, x_2 &\leq b_3 \\ a_{4,1}\, x_1 + a_{4,2}\, x_2 &\geq b_4 \end{aligned}$$



Cost

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Notation Alert!

$$A = \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix}$$

Stir-fry Boba

Limit

$$\mathbf{b} = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix}$$

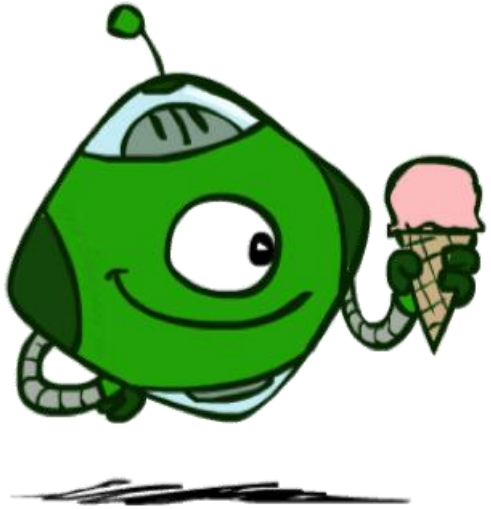
Calorie min
Calorie max
Sugar
Calcium

Optimization Formulation

Diet Problem

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad -a_{1,1}\, x_1 - a_{1,2}\, x_2 \le -b_1$$
$$a_{2,1}\, x_1 + a_{2,2}\, x_2 \le b_2$$
$$a_{3,1}\, x_1 + a_{3,2}\, x_2 \le b_3$$
$$-a_{4,1}\, x_1 - a_{4,2}\, x_2 \le -b_4$$



Cost

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Stir-fry Boba

$$A = \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix}$$

Limit

$$\mathbf{b} = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix}$$

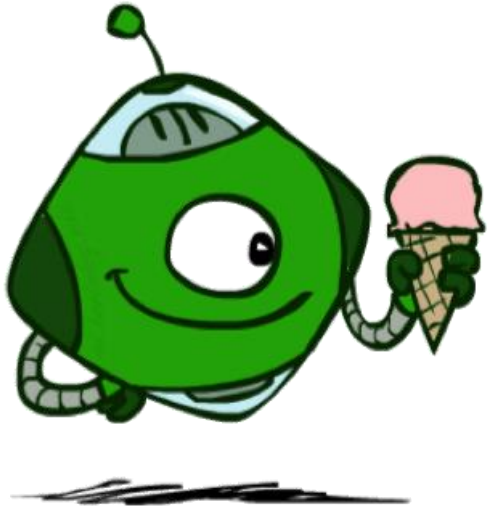
Calorie min
Calorie max
Sugar
Calcium

Optimization Formulation

Diet Problem

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad \begin{aligned} a_{1,1}\, x_1 + a_{1,2}\, x_2 &\leq b_1 \\ a_{2,1}\, x_1 + a_{2,2}\, x_2 &\leq b_2 \\ a_{3,1}\, x_1 + a_{3,2}\, x_2 &\leq b_3 \\ a_{4,1}\, x_1 + a_{4,2}\, x_2 &\leq b_4 \end{aligned}$$



Cost

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Stir-fry Boba

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}$$

Limit

$$\mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

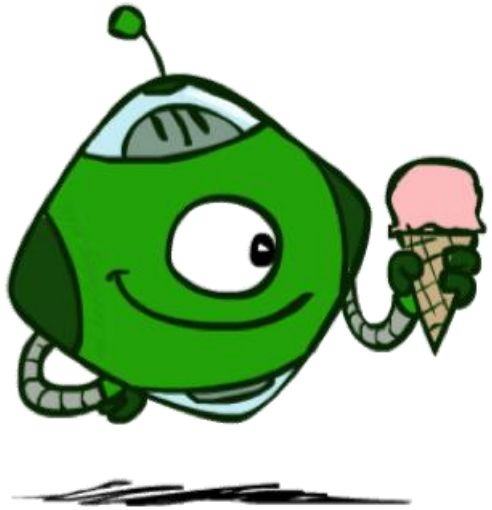
Calorie min
Calorie max
Sugar
Calcium

Optimization Formulation

Diet Problem

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$



Cost

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$[4\times 2][2\times 1]$

$[4\times 1] \leq [4\times 1]^4$

Notation Alert!

Stir-fry, Boba

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}$$

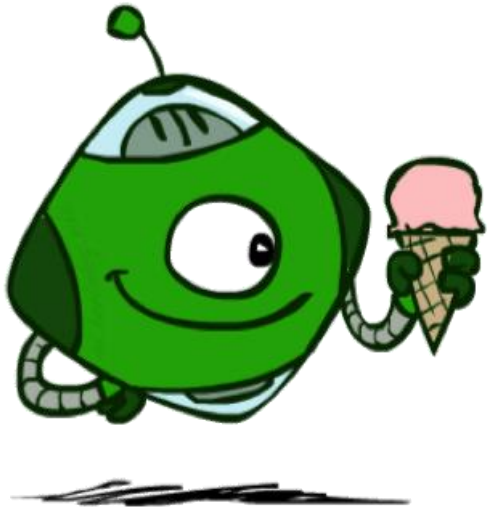
Limit 4×1

$$\mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium

Poll 1

What has to increase to add more nutrition constraints?



$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

Select all that apply

A) length $\mathbf{x}$

B) length $\mathbf{c}$

C) height A 75 - 86%

D) width A

E) length $\mathbf{b}$ 75 - 86%

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
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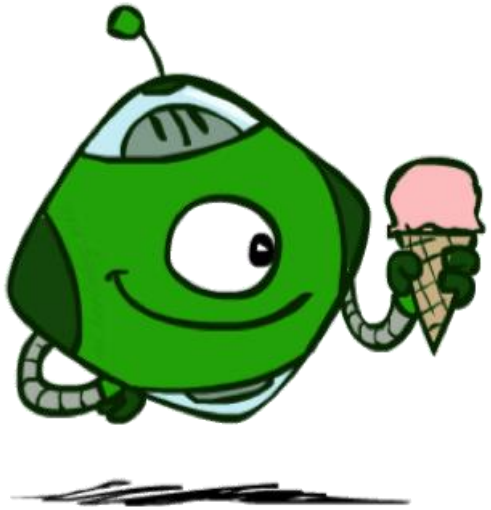
Poll 1

What has to increase to add more nutrition constraints?



$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

$Ax = b$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}$$

a_{51} a_{52}

$$\mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

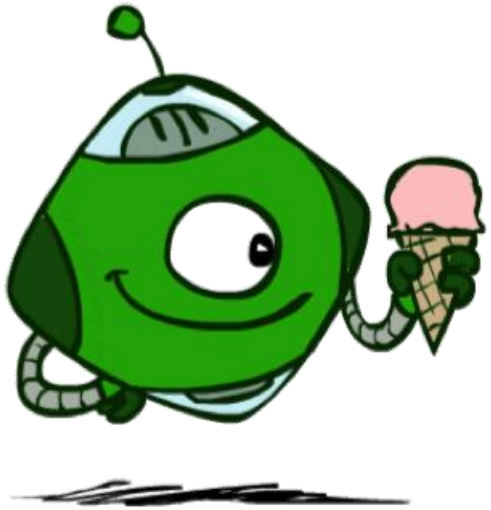
b_5

Poll 2

What has to increase to add more menu items?

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$



Select all that apply

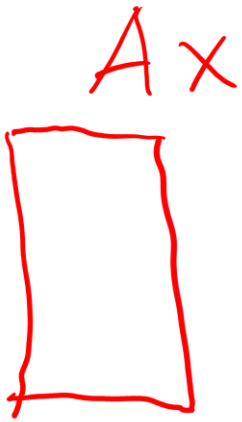
A) length $\mathbf{x}$

B) length $\mathbf{c}$

C) height A

D) width A

E) length $\mathbf{b}$

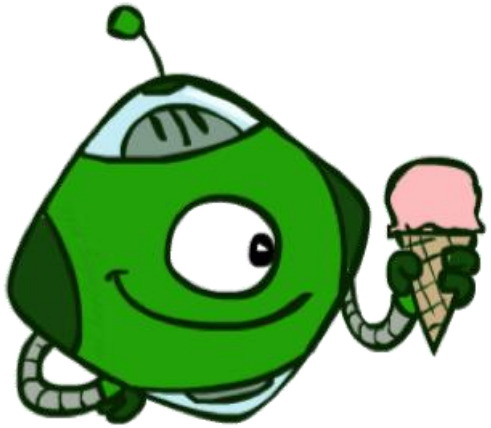


Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

x_3

Poll 2

What has to increase to add more nutrition constraints?



$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Question

If $A \in \mathbb{R}^{M \times N}$, which of the following also equals N?

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

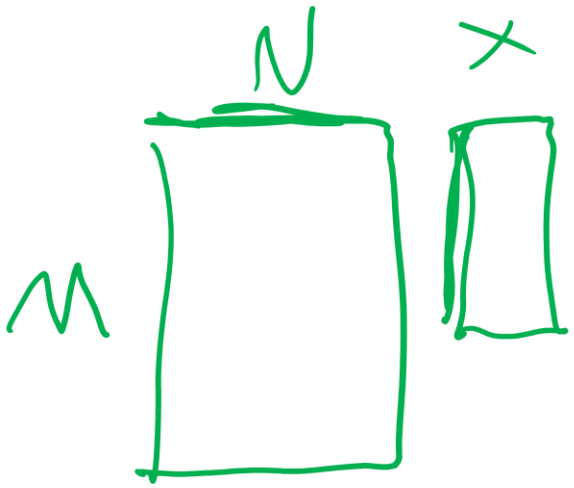
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

Select all that apply

A) length $\mathbf{x}$

B) length $\mathbf{c}$

C) length $\mathbf{b}$



Notation Alert!

Linear Programming

Linear objective with linear constraints

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

As opposed to general optimization

$$\min_{\mathbf{x}} \quad f_0(\mathbf{x})$$
$$\text{s.t.} \quad f_i(\mathbf{x}) \leq 0, \quad i = 1 \ldots M$$
$$\mathbf{a}_i^\top \mathbf{x} = \mathbf{b}_i, \quad i = 1 \ldots P$$

Linear Programming

Different formulations

Inequality form

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

General form

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x} + \mathbf{d}$$
$$\text{s.t.} \quad G\mathbf{x} \preceq \mathbf{h}$$
$$A\mathbf{x} = \mathbf{b}$$

Standard form

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad \mathrm{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \succeq 0$$

Important to pay attention to form!

Can switch between formulations!

Optimization

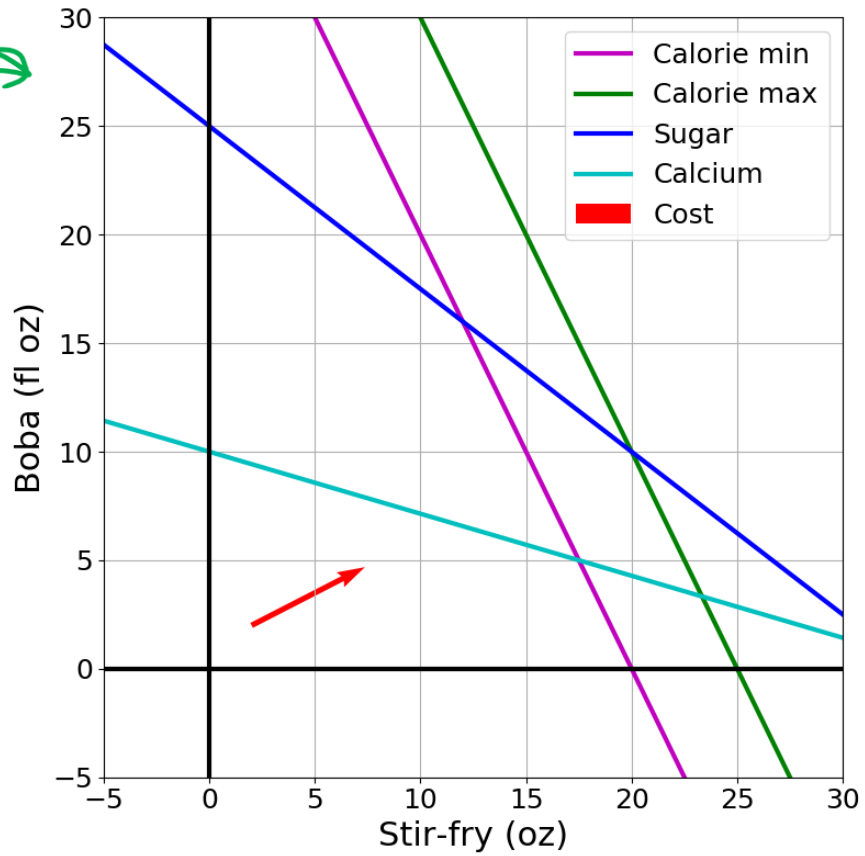
Problem Description

Optimization Representation

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

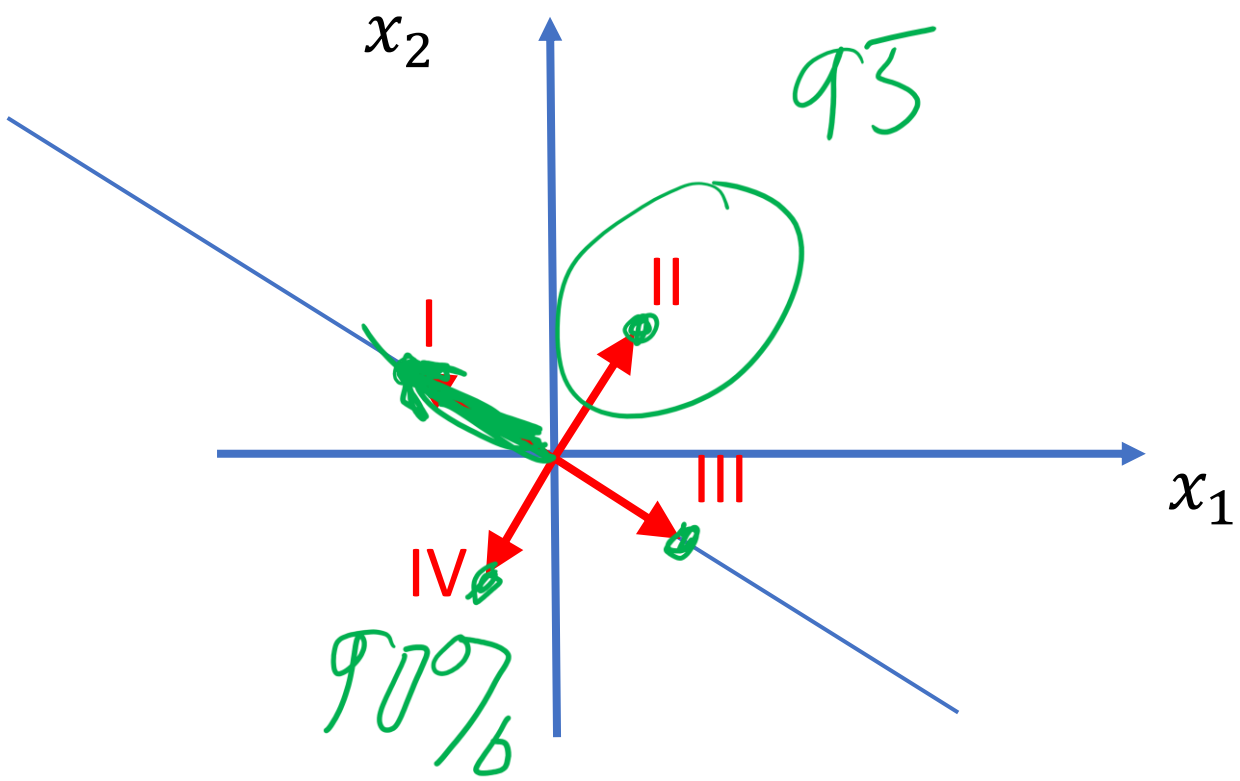
Graphical Representation



Poll 3

Which of these points have cost $\mathbf{c}^\top \mathbf{x} = 0$?

for cost vector: $\mathbf{c} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$



Cost Contours

Given the cost vector $[c_1, c_2]^\top$ where will

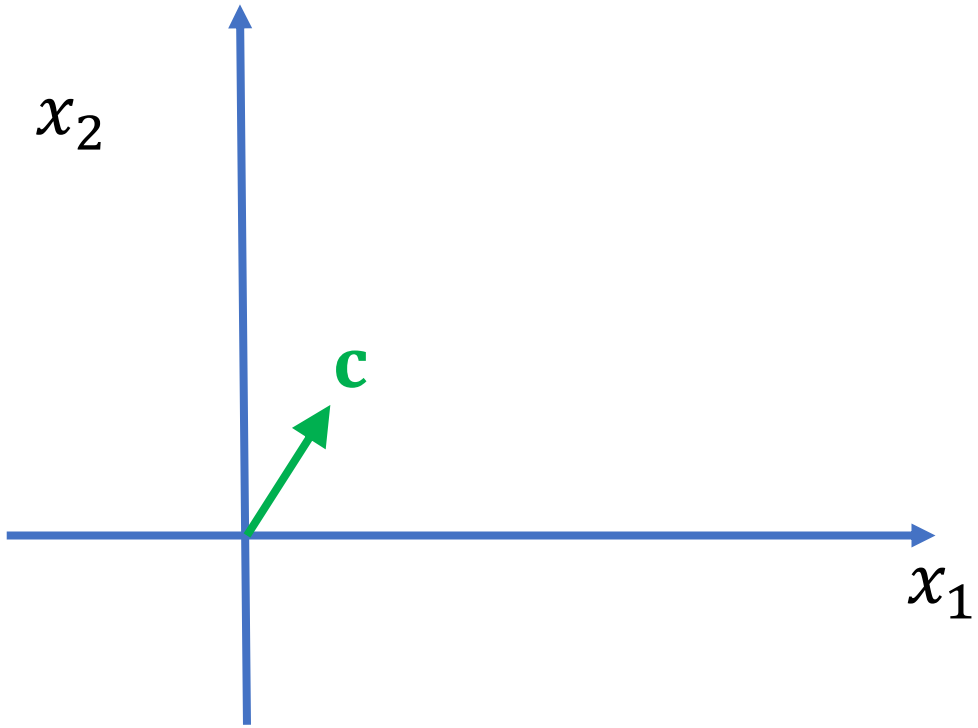
$\mathbf{c}^\top\mathbf{x} = 0$?

$\mathbf{c}^\top\mathbf{x} = 1$?

$\mathbf{c}^\top\mathbf{x} = 2$?

$\mathbf{c}^\top\mathbf{x} = -1$?

$\mathbf{c}^\top\mathbf{x} = -2$?

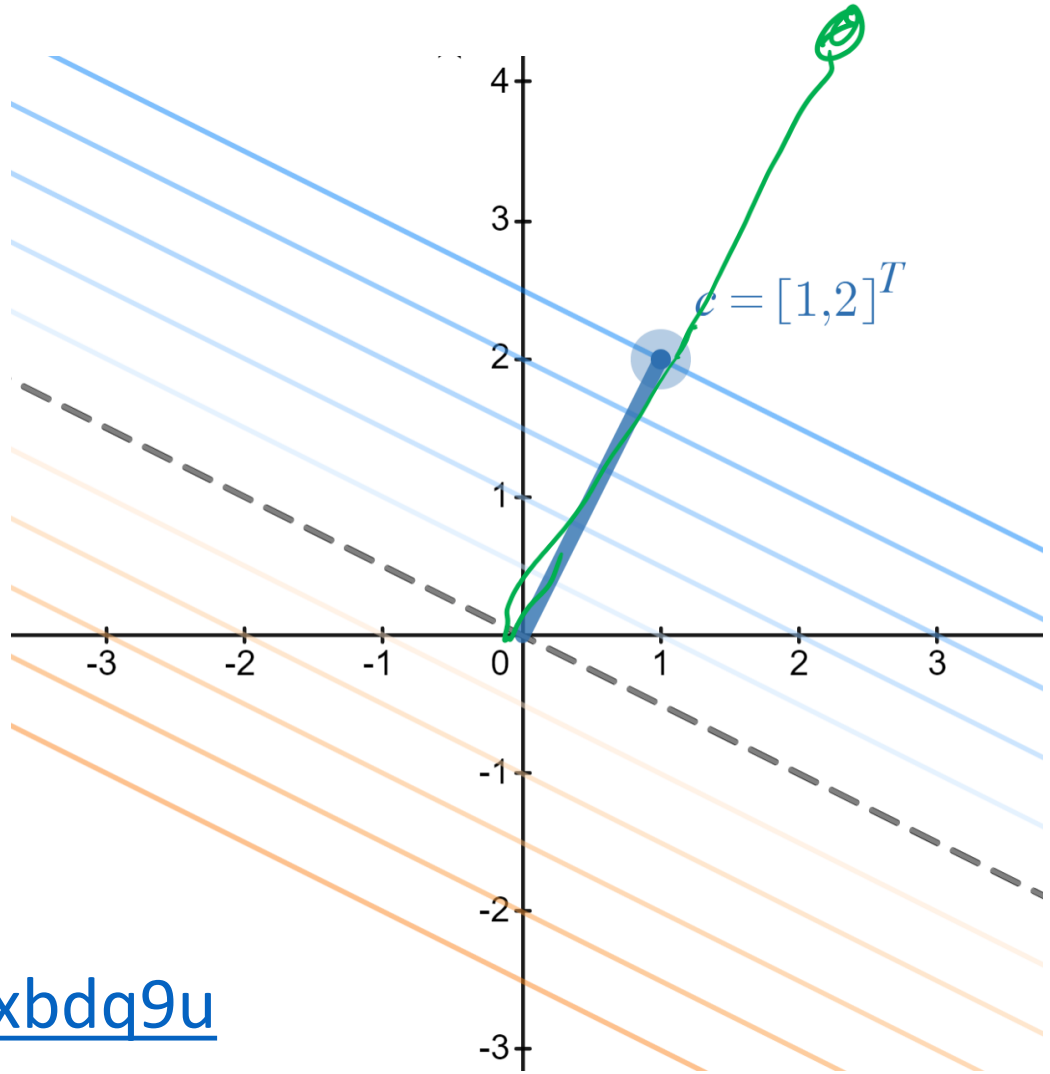


Question

As the magnitude of $\mathbf{c}$ increases, the distance between the contours lines of the objective $\mathbf{c}^T\mathbf{x}$:

A) Increases

B) Decreases



https://www.desmos.com/calculator/8d9kxbdq9u

Graphics Representation

Geometry / Algebra I Quiz

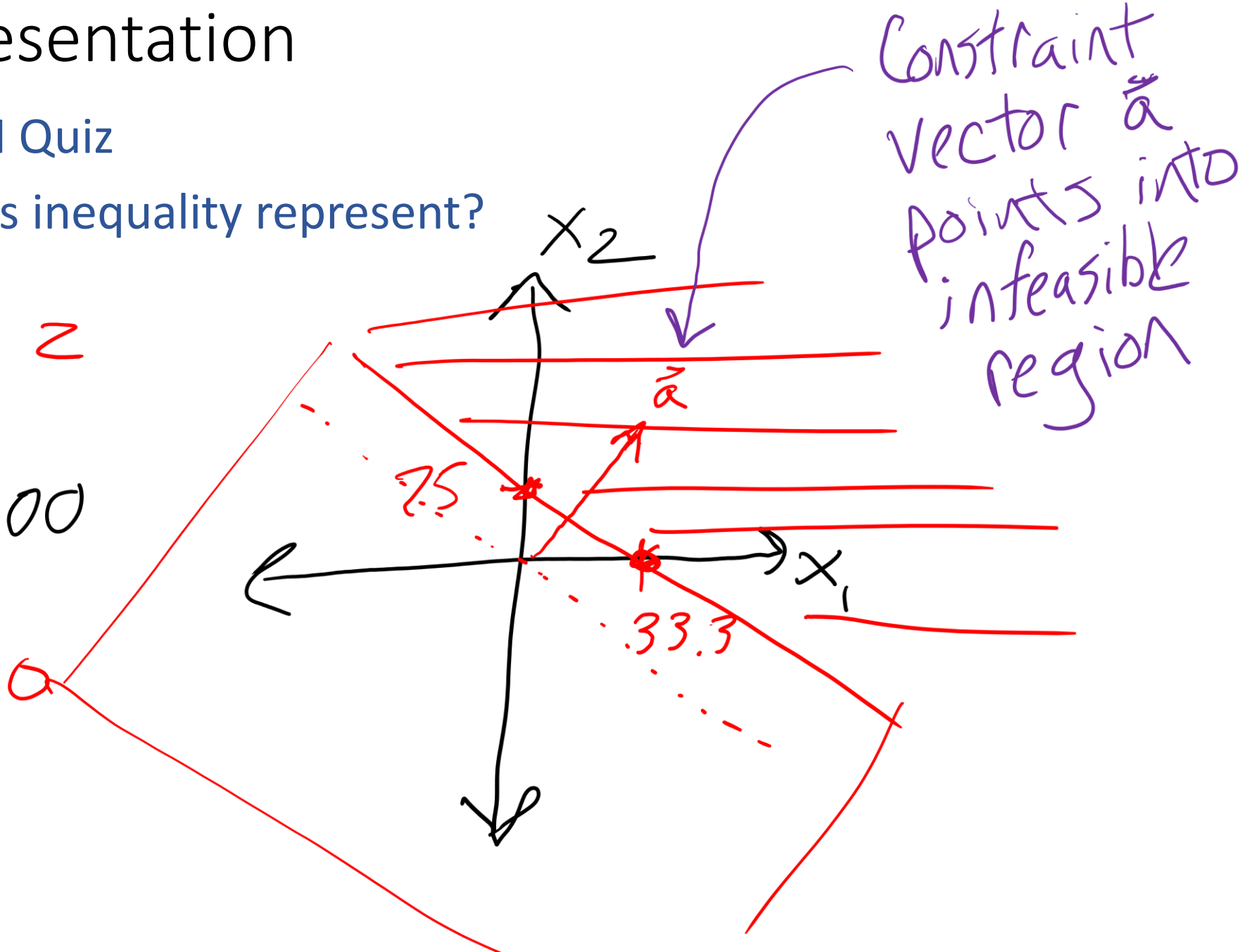
What shape does this inequality represent?

$$a_1 x_1 + a_2 x_2 \leq b_1$$

$$3x_1 + 4x_2 = 100$$

$$\vec{a}^T \vec{x} =$$

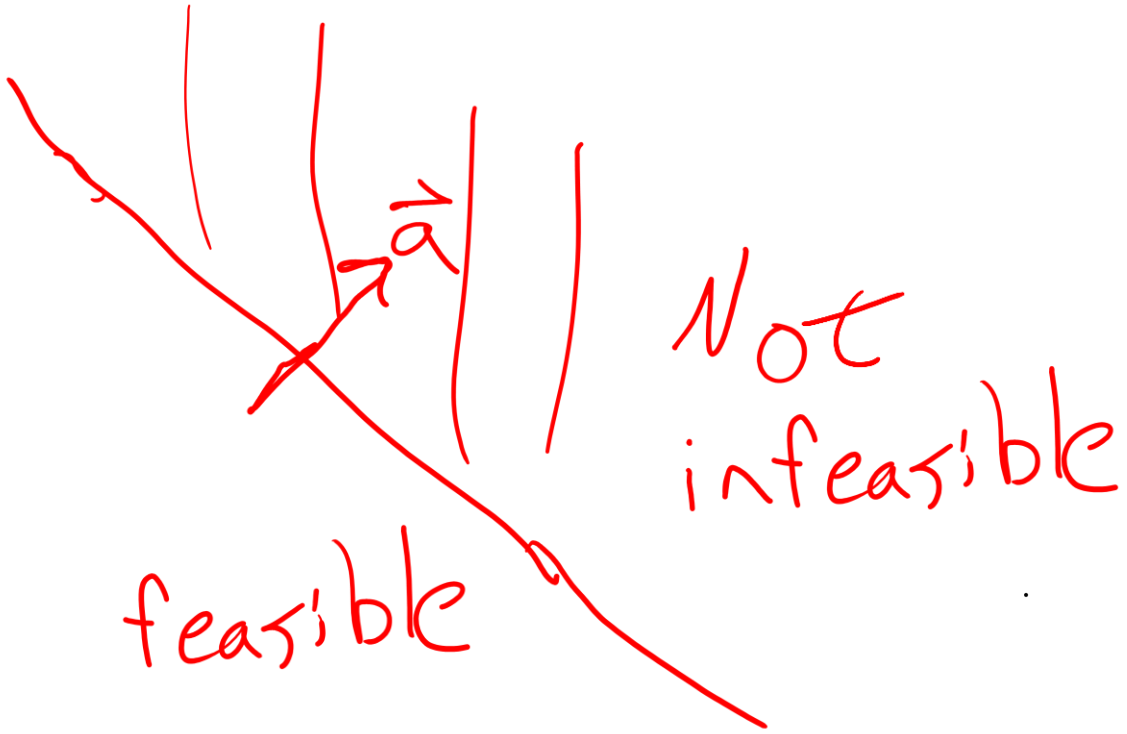
$$\vec{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



Graphics Representation

Geometry / Algebra I Quiz

What shape do these represent?



1. $a_1 x_1 + a_2 x_2 = b_1$

2. $a_1 x_1 + a_2 x_2 \leq b_1$

3. $a_{1,1} x_1 + a_{1,2} x_2 \leq b_1$
   $a_{2,1} x_1 + a_{2,2} x_2 \leq b_2$
   $a_{3,1} x_1 + a_{3,2} x_2 \leq b_3$
   $a_{4,1} x_1 + a_{4,2} x_2 \leq b_4$

Feasible region:
All points x that satisfy the constraints

Graphics Representation

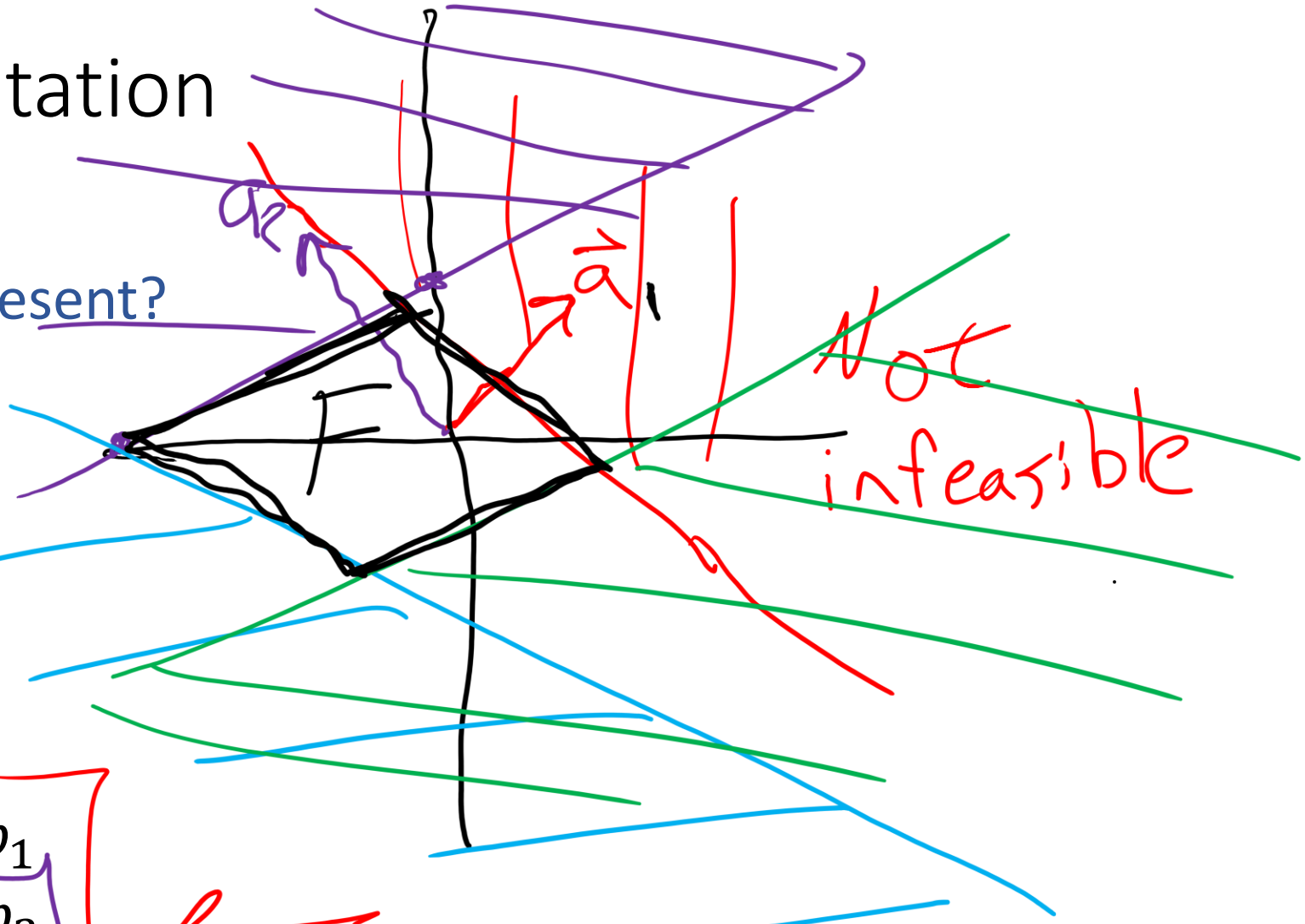
Geometry / Algebra I Quiz

What shape do these represent?

1. $a_1 x_1 + a_2 x_2 = b_1$

2. $a_1 x_1 + a_2 x_2 \leq b_1$

3. $a_{1,1} x_1 + a_{1,2} x_2 \leq b_1$
   $a_{2,1} x_1 + a_{2,2} x_2 \leq b_2$
   $a_{3,1} x_1 + a_{3,2} x_2 \leq b_3$
   $a_{4,1} x_1 + a_{4,2} x_2 \leq b_4$



Feasible region:
All points x that satisfy the constraints

Graphics Representation

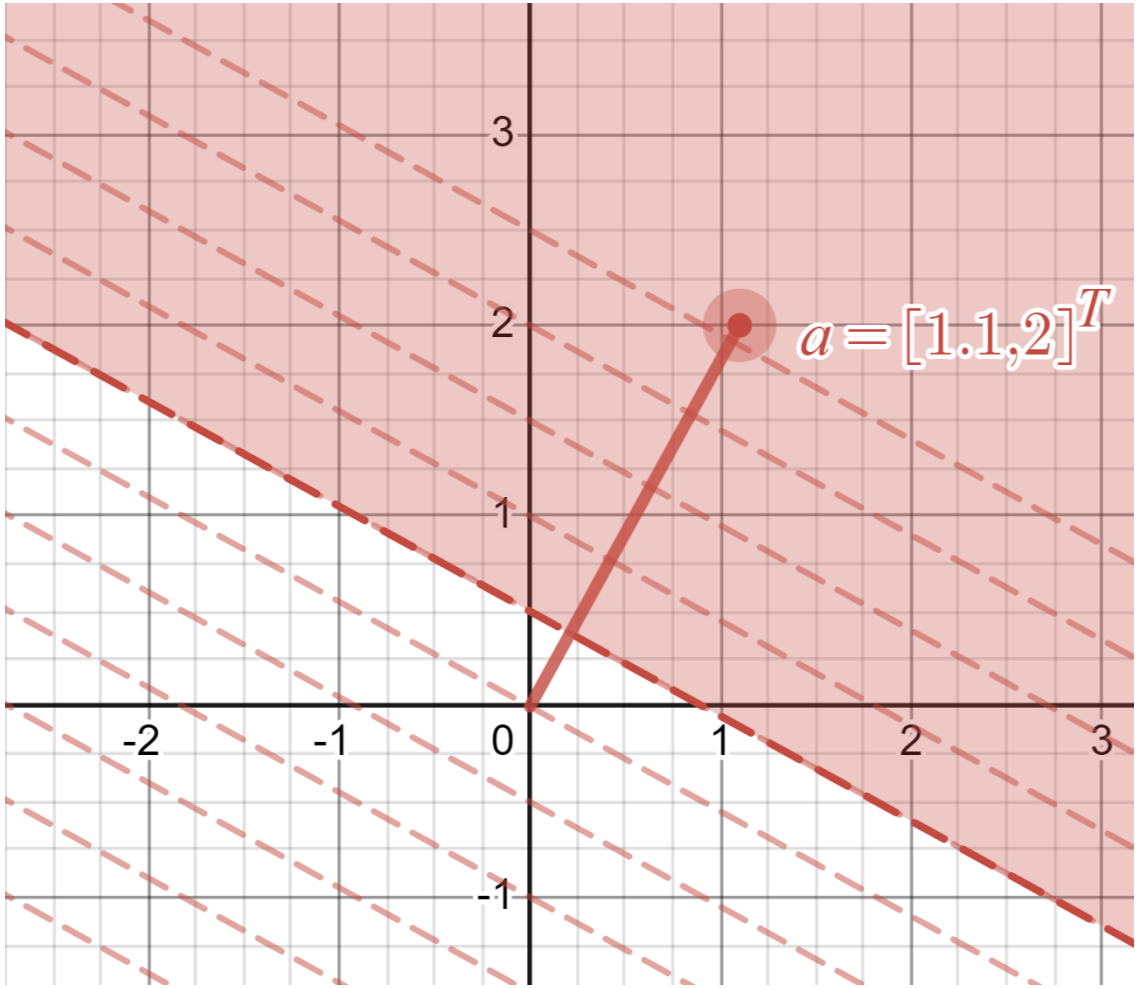
Geometry / Algebra I Quiz

What shape do these represent?

1. $a_1 x_1 + a_2 x_2 = b_1$

2. $a_1 x_1 + a_2 x_2 \leq b_1$

3.



https://www.desmos.com/calculator/lp0rqsb1w6

Graphics Representation

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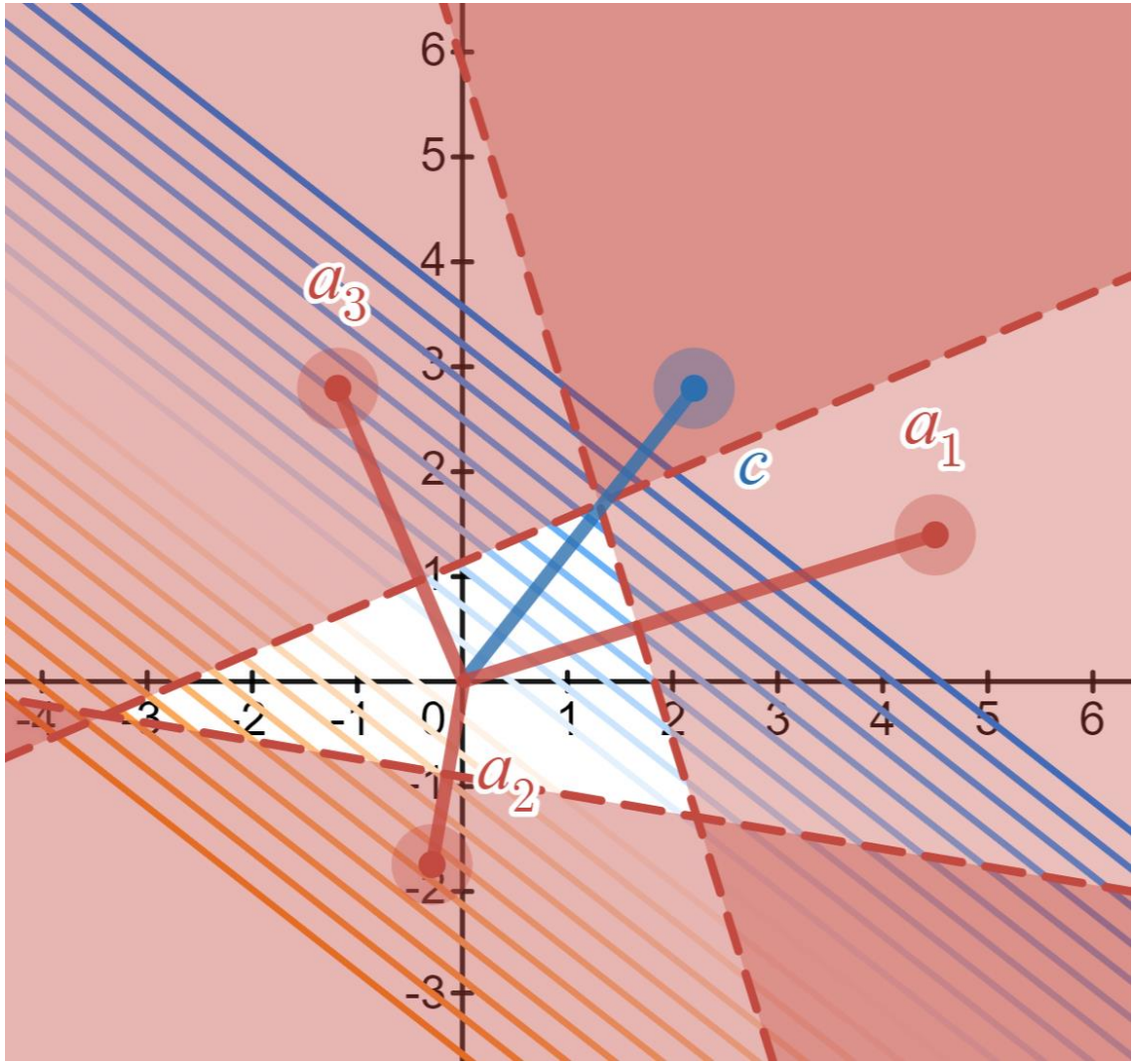
$$a_{3,1}\, x_1 + a_{3,2}\, x_2 \leq b_3$$
$$a_{4,1}\, x_1 + a_{4,2}\, x_2 \leq b_4$$



https://www.desmos.com/calculator/plp1thgsbh