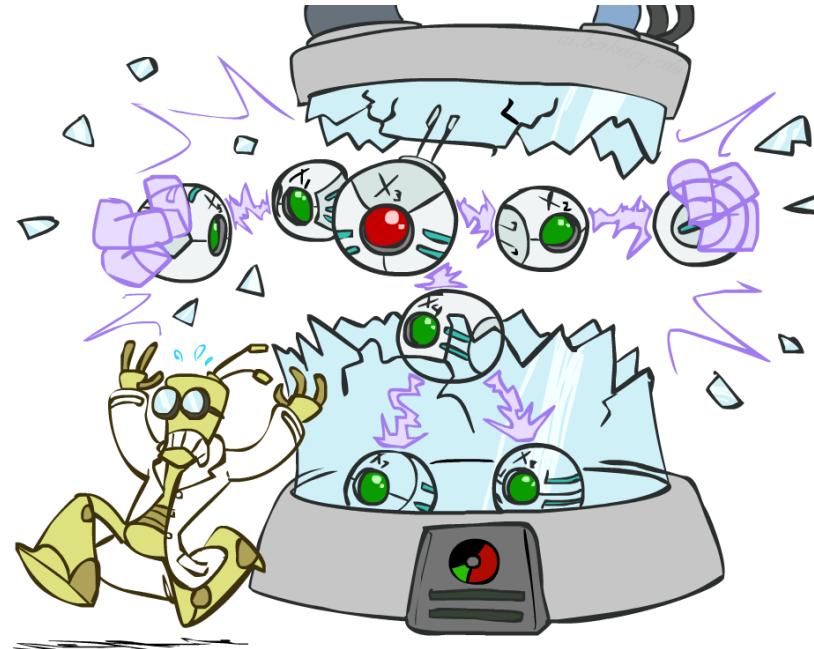


AI: Representation and Problem Solving

Bayes Nets: Independence



Instructors: Tuomas Sandholm and Vincent Conitzer

Slide credits: CMU AI and <http://ai.berkeley.edu>

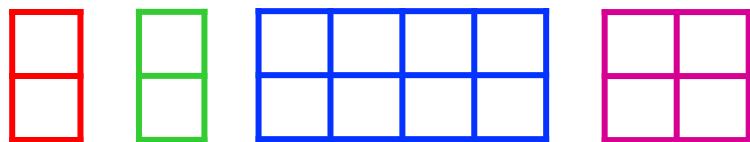
Announcements

- P4 due tonight!

Bayesian Networks

- One node per random variable
- Directed Acyclic Graph
- One CPT per node: $P(\text{node} \mid \text{Parents}(\text{node}))$

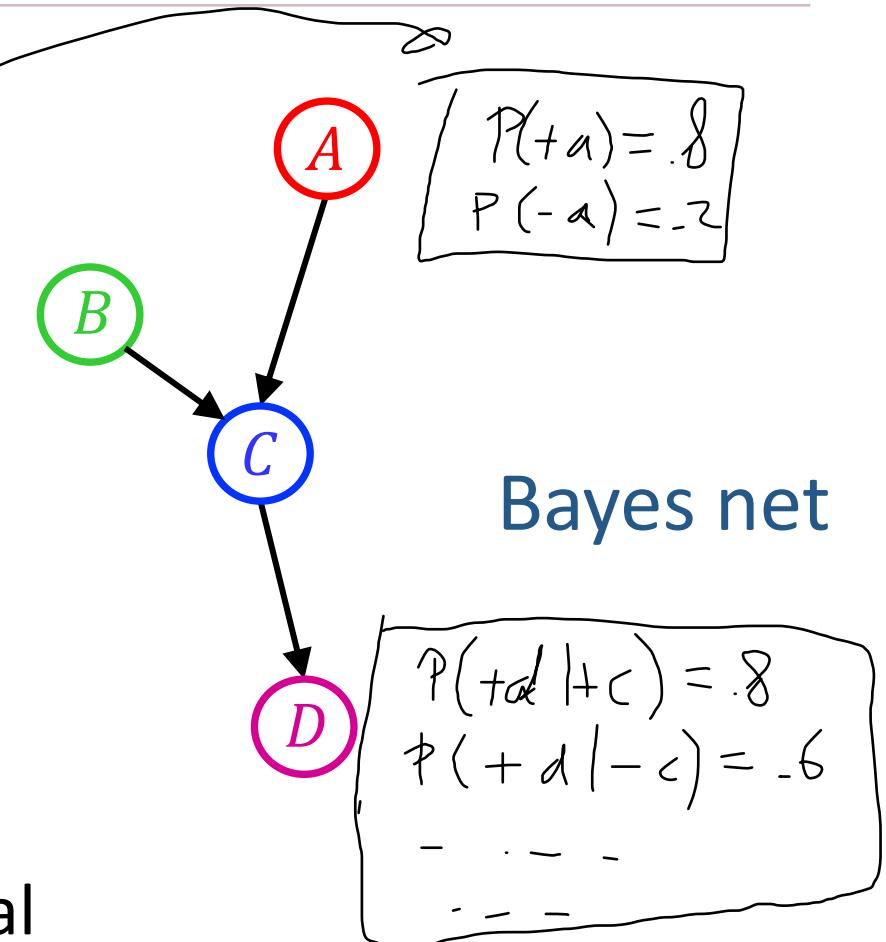
conditional probability tables



$$P(A, B, C, D) = P(A) P(B) P(C|A, B) P(D|C)$$

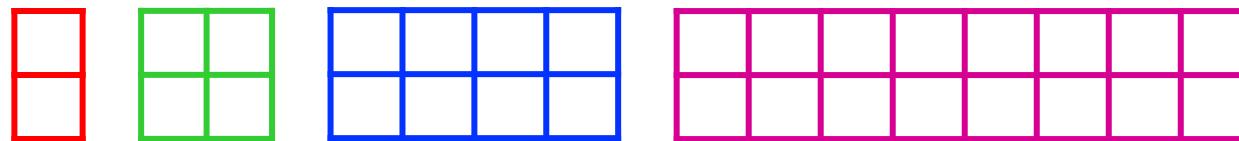
Encode joint distributions as product of conditional distributions on each variable

$$P(X_1, \dots, X_N) = \prod_i P(X_i \mid \text{Parents}(X_i))$$



Any distribution can be a Bayes net

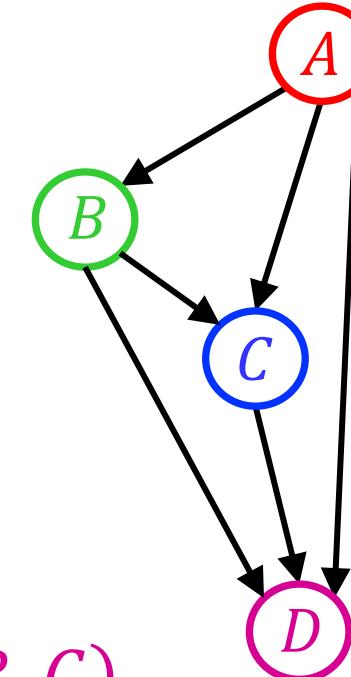
- One node per random variable
- Directed-Acyclic-Graph
- One CPT per node: $P(\text{node} \mid \text{Parents}(\text{node}))$



$$P(A, B, C, D) = P(A) P(B|A) P(C|A, B) P(D|A, B, C)$$

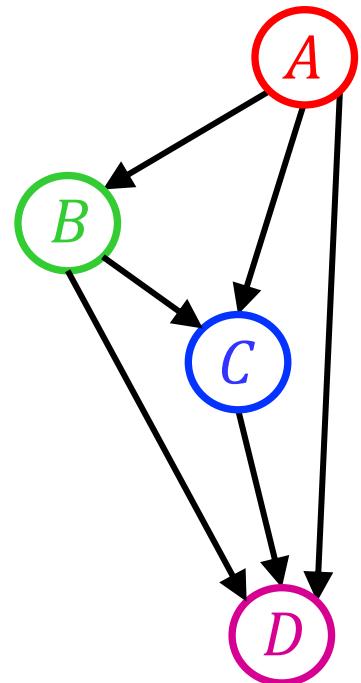
Encode joint distributions as product of conditional distributions on each variable

$$P(X_1, \dots, X_N) = \prod_i P(X_i \mid \text{Parents}(X_i))$$

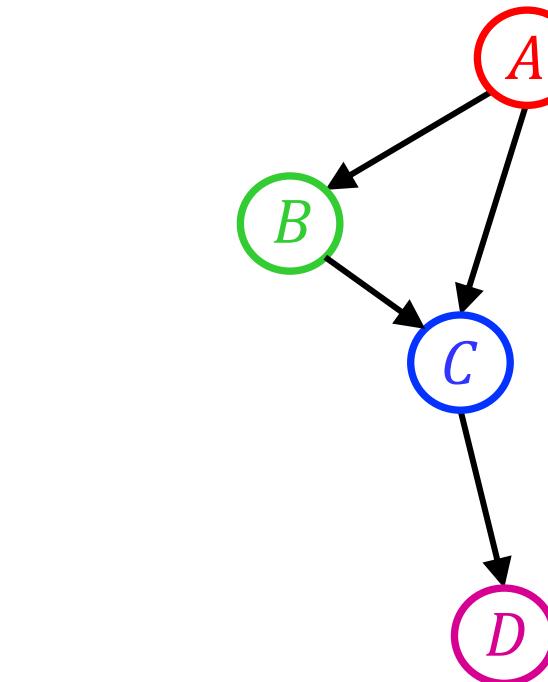


Bayes net

Joint distribution from Bayes nets



$$P(A) P(B|A) P(C|A, B) P(D|A, B, C)$$



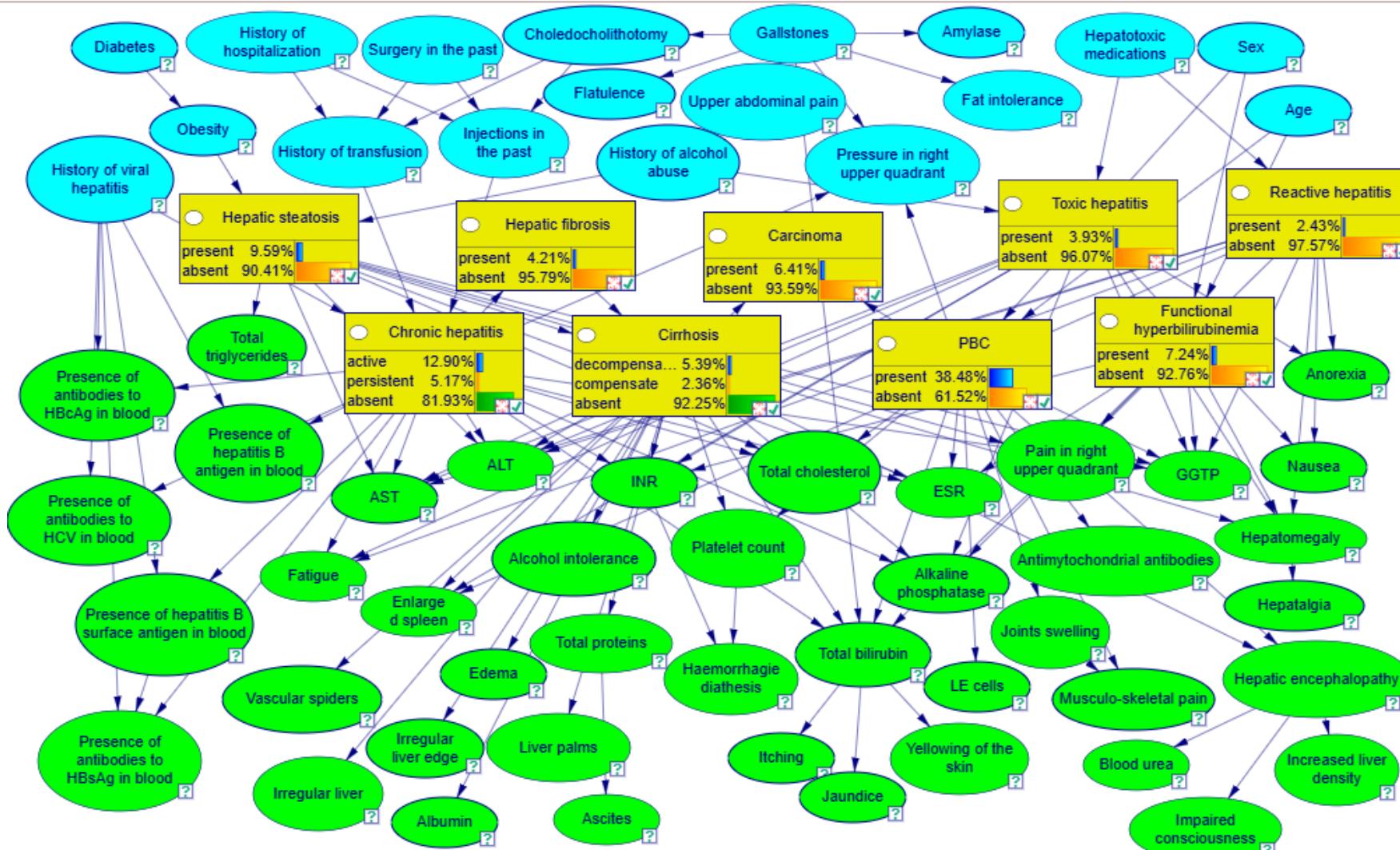
$$P(A) P(B|A) P(C|A, B) P(D|C)$$

If these encode the same distribution then
 $P(D | A, B, C) = P(D | C)$

The power of conditional independencies

- For a Bayes net $P(X_1, \dots, X_N) = \prod_i P(X_i | \text{Parents}(X_i))$
- When $|\text{Parents}(X_i)|$ is small, the conditional probability tables are much smaller
 - Makes them easier to estimate from data

Example: Liver Disorders



<https://demo.bayesfusion.com/bayesbox.html>

Independence

- Two variables X and Y are *independent* if

$$\forall x, y \quad P(x, y) = P(x) P(y)$$

- This says that their joint distribution **factors** into a product of two simpler distributions

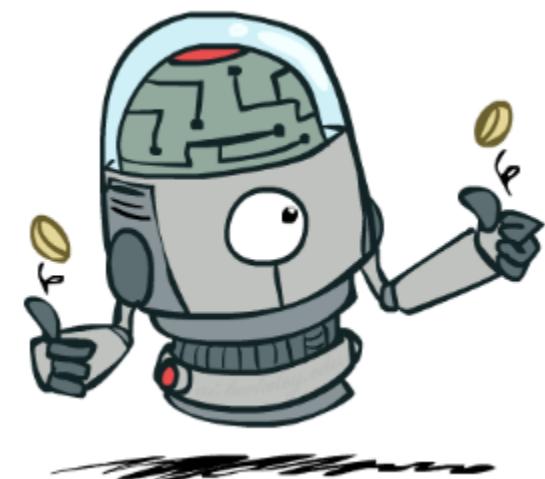
- Combine with product rule $P(x, y) = P(x | y)P(y)$ we obtain another form:

$$\forall x, y \quad P(x | y) = P(x) \quad \text{or} \quad \forall x, y \quad P(y | x) = P(y)$$

- Example: two dice rolls R_1 and R_2

$$P(R_1=5, R_2=5) = P(R_1=5) P(R_2=5) = 1/6 \times 1/6 = 1/36$$

$$P(R_2=5 | R_1=5) = P(R_2=5)$$



Example: Independence

- n fair, independent coin flips:

$$P(X_1)$$

H	0.5
T	0.5

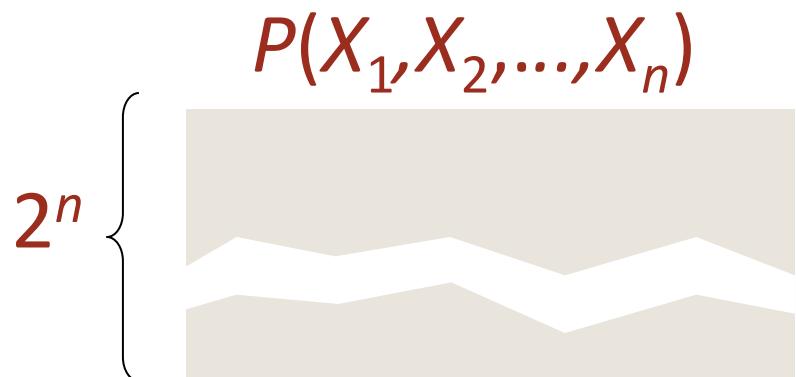
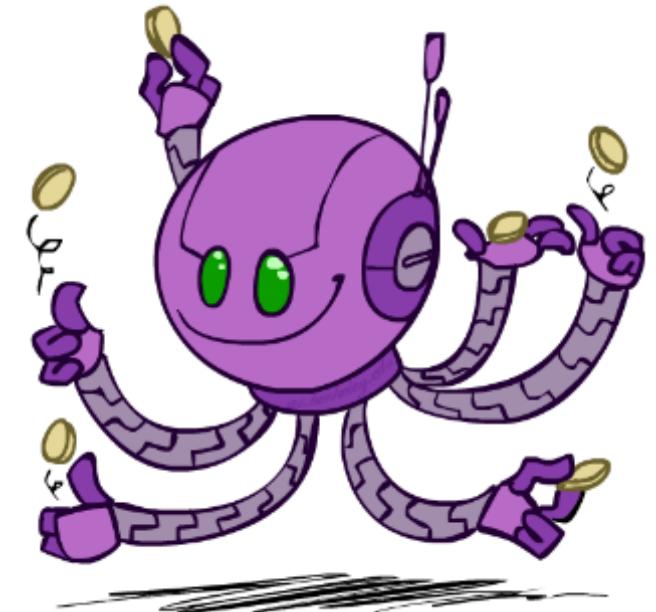
$$P(X_2)$$

H	0.5
T	0.5

...

$$P(X_n)$$

H	0.5
T	0.5



Poll 1

- Are T and W independent?

$P(T)$

T	P
hot	0.5
cold	0.5

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P(W)$

W	P
sun	0.6
rain	0.4

$$P(\text{hot, sun}) = 0.4$$

$$P(\text{hot}) * P(\text{sun}) = 0.5 * 0.6 = 0.3$$

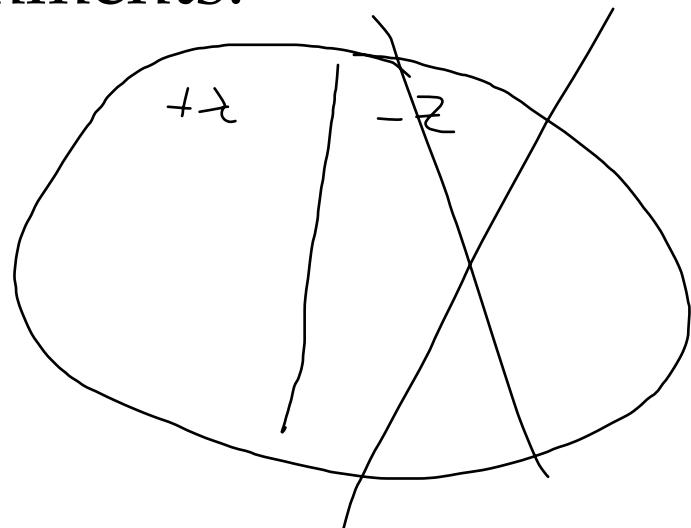
Conditional Independence

- Absolute (unconditional) independence very rare
- *Conditional independence* is our most basic and robust form of structural knowledge about uncertain environments.
- X is conditionally independent of Y given Z if and only if:

$$\forall x, y, z \quad P(x \mid y, z) = P(x \mid z)$$

or, equivalently, if and only if

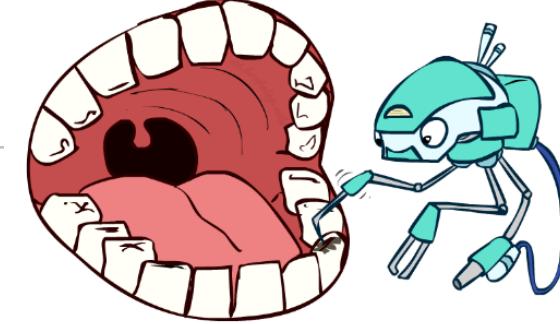
$$\forall x, y, z \quad P(x, y \mid z) = P(x \mid z) P(y \mid z)$$



Conditional independence

- X and Y are independent if
 - $P(X, Y) = P(X) P(Y)$, or
 - $P(X \mid Y) = P(X)$
- X and Y are **conditionally independent given Z** if
 - $P(X, Y \mid Z) = P(X \mid Z) P(Y \mid Z)$, or
 - $P(X \mid Y, Z) = P(X \mid Z)$

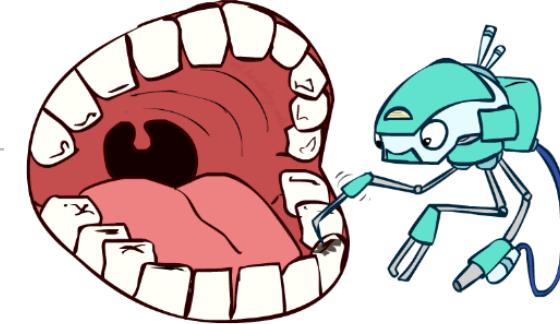
Conditional independence



- $P(\text{Toothache, Cavity, (p)Robe})$
- If I have a cavity, the probability that the probe catches it **doesn't** depend on whether I have a toothache:
 - $P(+r \mid +\text{toothache}, +\text{cavity}) = P(+r \mid +\text{cavity})$
- The same independence holds if I don't have a cavity:
 - $P(+r \mid +\text{toothache}, -\text{cavity}) = P(+r \mid -\text{cavity})$
- Probe is *conditionally independent* of Toothache given Cavity:
 - $P(R \mid T, C) = P(R \mid C)$

$$R \perp\!\!\!\perp T \mid C \quad \checkmark$$
$$R \perp\!\!\!\perp T \quad \times$$

Conditional independence



Equivalent statements:

- $P(\text{Toothache} \mid \text{Probe, Cavity}) = P(\text{Toothache} \mid \text{Cavity}) \uparrow$
- $P(\text{Toothache, Probe} \mid \text{Cavity}) = \boxed{P(\text{Toothache} \mid \text{Cavity})} P(\text{Probe} \mid \text{Cavity})$
 $P(T, R \mid C) = P(R \mid C) \cancel{P(T \mid \cancel{R}))} P(T \mid R, C)$
- One can be derived from the other easily

Independence Rules

- Independence

If A and B are independent, then:

$$P(A, B) = P(A)P(B)$$

$$P(A | B) = P(A)$$

$$P(B | A) = P(B)$$

- Conditional independence

If A and B are conditionally independent given C, then:

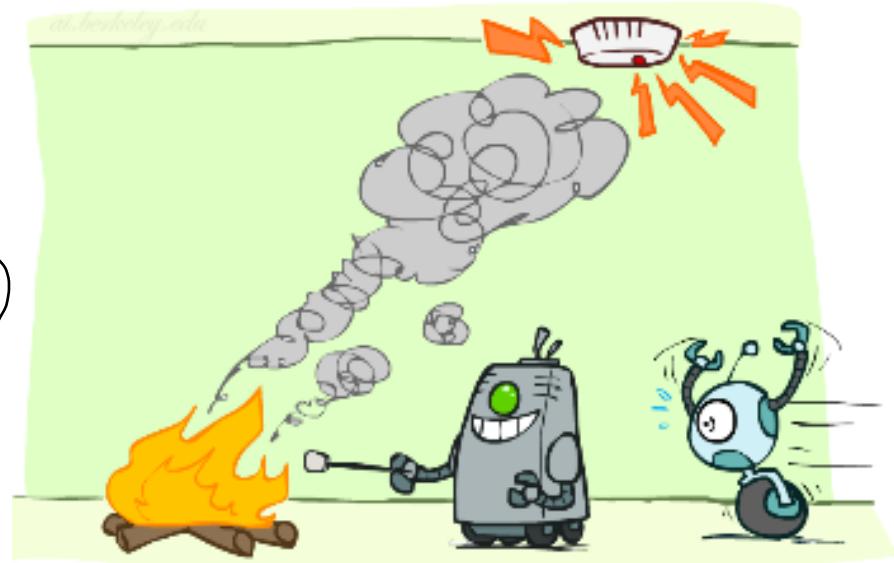
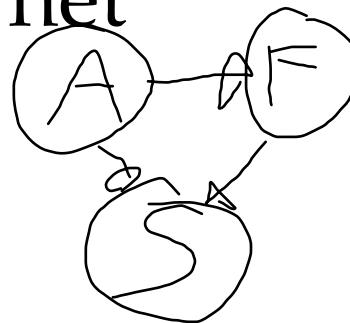
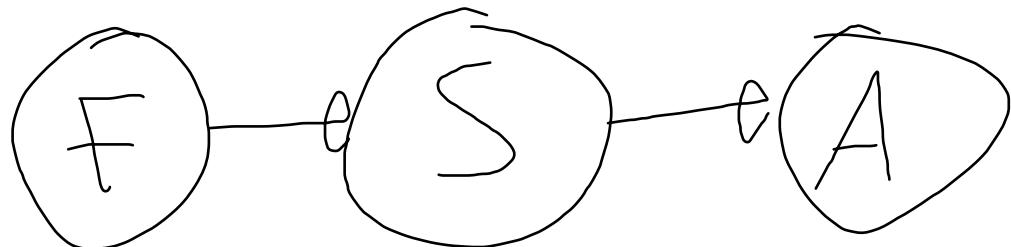
$$P(A, B | C) = P(A | C)P(B | C)$$

$$P(A | B, C) = P(A | C)$$

$$P(B | A, C) = P(B | C)$$

Conditional Independence and Bayes Nets

- Fire, Smoke, Alarm
- Causal story to create Bayes net

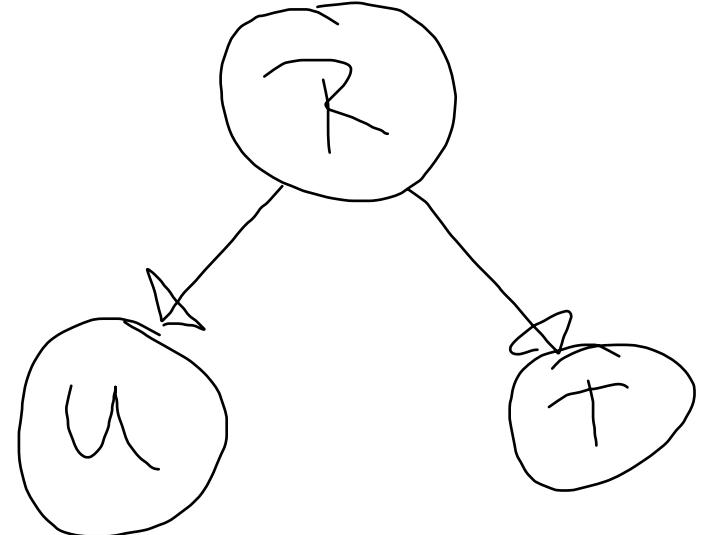


- From Bayes Net: $P(S, F, A) = P(F) P(S | F) \mathbf{P(A | S)}$
- Joint distribution: $P(S, F, A) = P(F) P(S | F) \mathbf{P(A | S, F)}$

Conditional Independence and Bayes Nets

- What about this domain:

- Traffic
- Umbrella
- Raining



Conditional Independence and the Chain Rule

- Chain rule:

$$P(x_1, x_2, \dots, x_n) = \prod_i P(x_i \mid x_1, \dots, x_{i-1})$$

- Trivial decomposition:

- $P(\text{Rain}, \text{Traffic}, \text{Umbrella}) =$

- With assumption of conditional independence:

- $P(\text{Rain}, \text{Traffic}, \text{Umbrella}) = P(R) P(U|R) P(T|R)$

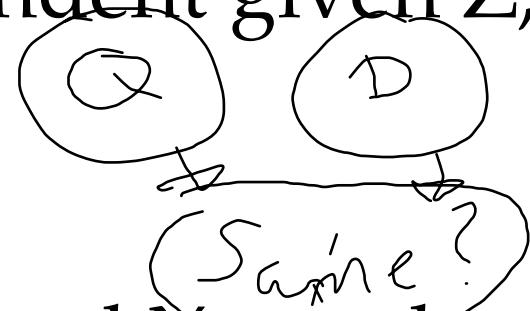
- Bayes nets / graphical models help us express conditional independence assumptions



Poll 2

Choose the true statement(s):

- (A) If X and Y are conditionally independent given Z , then X and Y are independent
- (B) If X and Y are independent, then X and Y are also conditionally independent given Z
- (C) Neither is true



Example: Traffic II

- Let's build a causal Bayes net!

- Variables

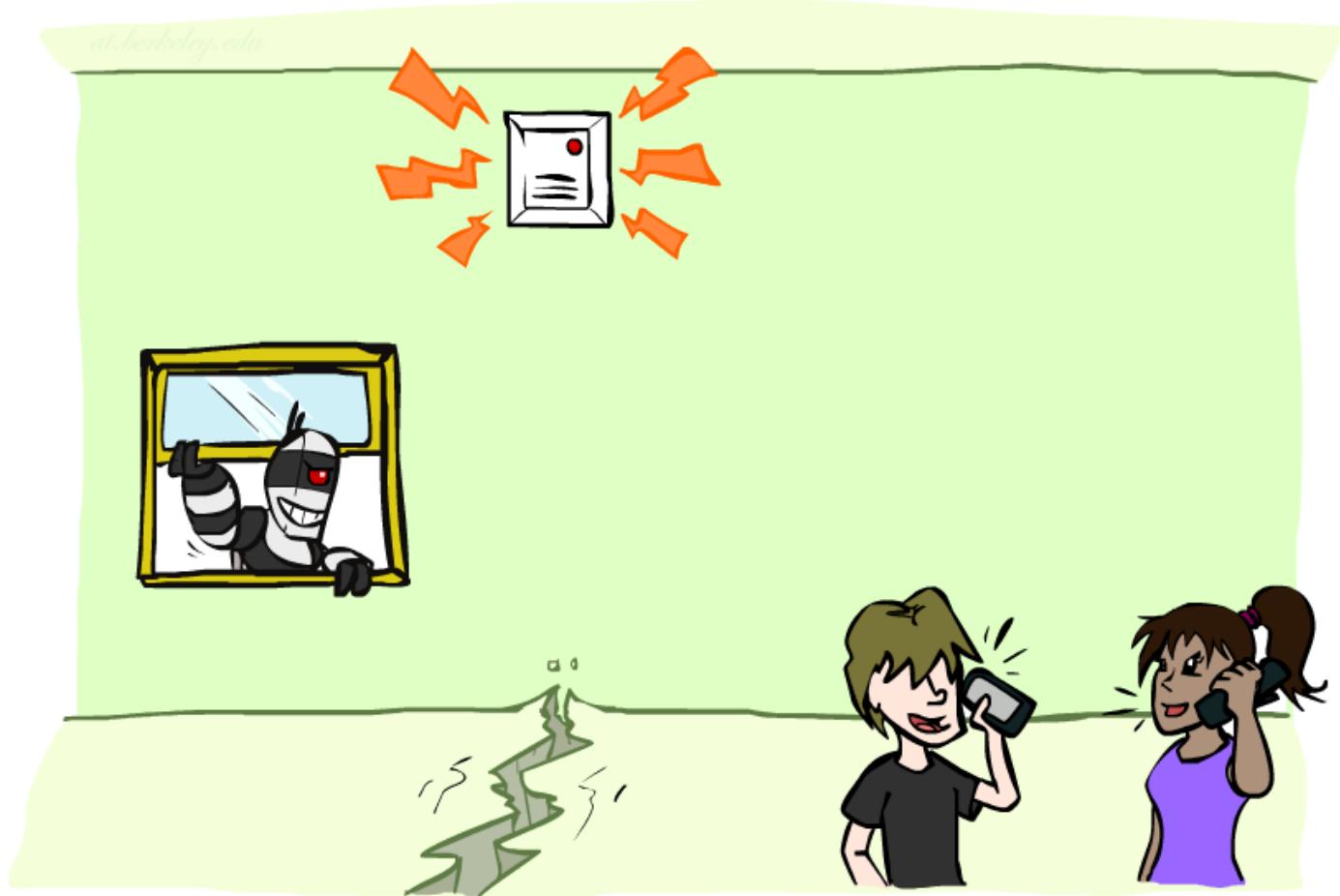
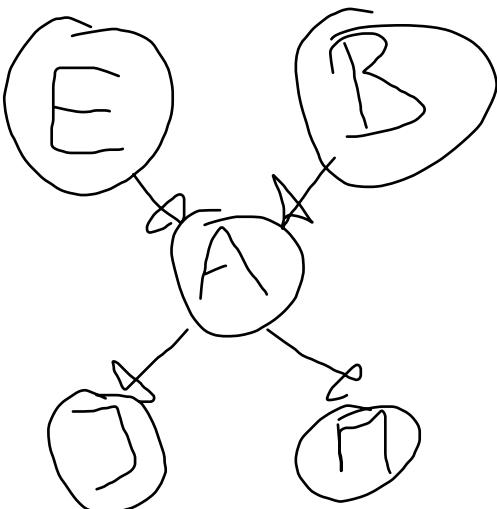
- T: Traffic
- R: It rains
- L: Low pressure
- D: Roof drips
- H: Hockey game
- C: Cavity



Example: alarm network

- Variables

- B: Burglary
- A: Alarm goes off
- M: Mary calls
- J: John calls
- E: Earthquake!



Analyzing the structure of the alarm network

- Joint distribution factorization example

- Generic chain rule

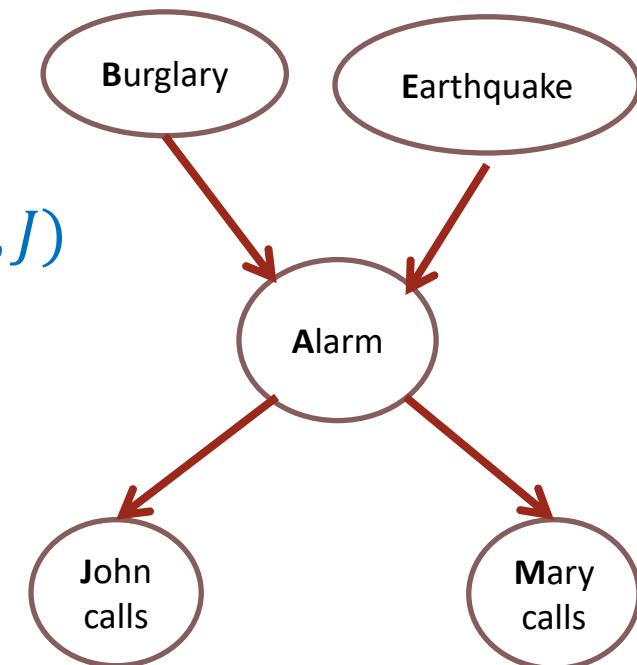
- $$P(X_1 \dots X_n) = \prod_i P(X_i | X_1 \dots X_{i-1})$$

$$P(B, E, A, J, M) = P(B) P(E|B) P(A|B, E) P(J|B, E, A) P(M|B, E, A, J)$$

$$P(B, E, A, J, M) = P(B) P(E) P(A|B, E) P(J|A) P(M|A)$$

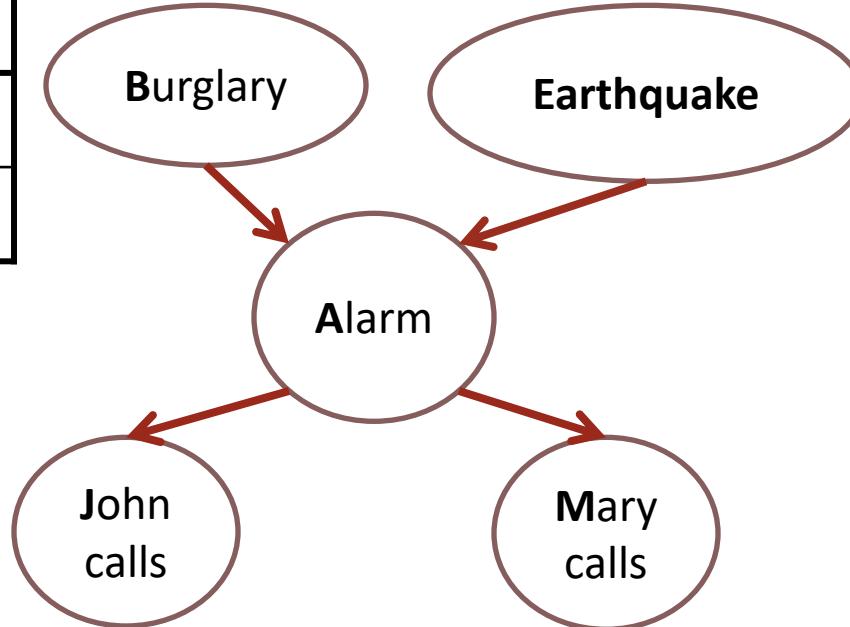
- Bayes nets

- $$P(X_1 \dots X_n) = \prod_i P(X_i | \text{Parents}(X_i))$$

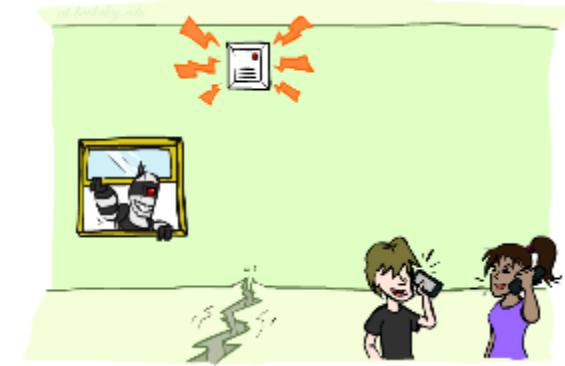


Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999



E	P(E)
+e	0.002
-e	0.998



A	J	P(J A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

B	E	A	P(A B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

Common Effect

- Chain rule:

$$P(x_1, x_2, \dots, x_n) = \prod_i P(x_i | x_1, \dots, x_{i-1})$$

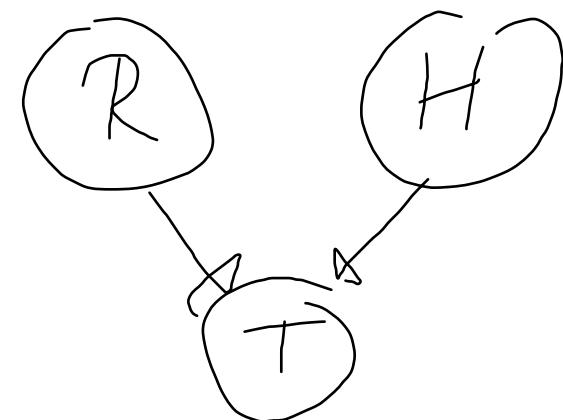
- Trivial decomposition:

$$P(\text{Rain}, \text{Hockey}, \text{Traffic}) = P(\text{Rain}) P(\text{Hockey} | \cancel{\text{Rain}}) P(\text{Traffic} | \text{Rain}, \text{Hockey})$$

- With assumption of conditional independence:

- $P(\text{Rain}, \text{Hockey}, \text{Traffic}) =$

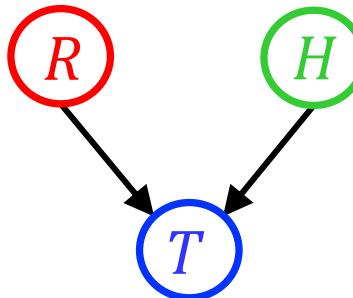
- Bayes nets / graphical models help us express conditional independence assumptions



Common Effect

- Chain rule:

$$P(x_1, x_2, \dots, x_n) = \prod_i P(x_i | x_1, \dots, x_{i-1})$$



- Trivial decomposition:

$$P(\text{Rain}, \text{Hockey}, \text{Traffic}) = P(\text{Rain}) P(\text{Hockey} | \text{Rain}) P(\text{Traffic} | \text{Rain}, \text{Hockey})$$



- With assumption of conditional independence:

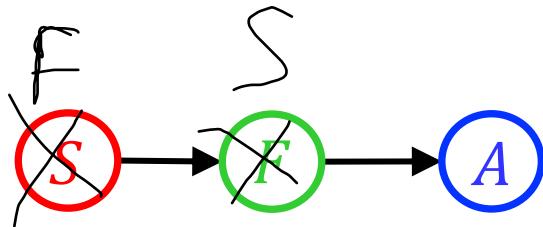
$$P(\text{Rain}, \text{Hockey}, \text{Traffic}) = P(\text{Rain}) P(\text{Hockey}) P(\text{Traffic} | \text{Rain}, \text{Hockey})$$

- Bayes nets / graphical models help us express conditional independence assumptions

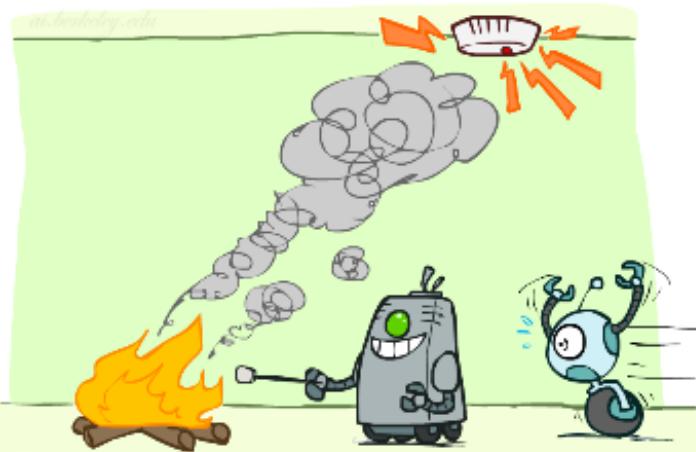
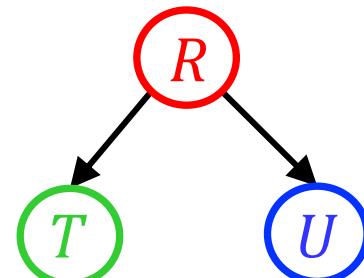
Conditional Independence Semantics

- Important local relationships within a Bayes net

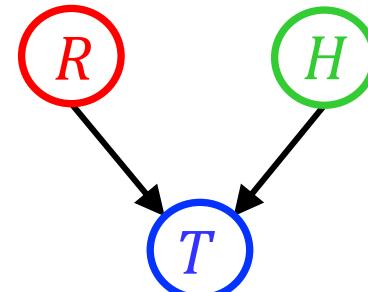
Causal Chain



Common Cause

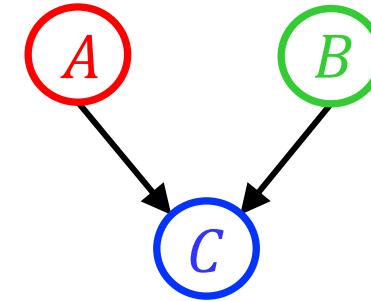
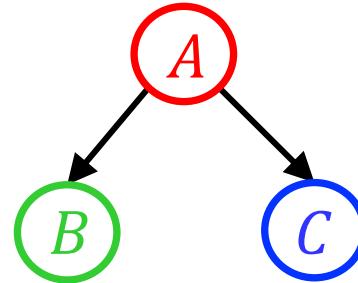


Common Effect



Poll 3: Conditional independence from Bayes Nets

- Match the product of CPTs to the Bayes net.



- I. $P(A) P(B|A) P(C|B)$

$P(A) P(B|A) P(C|A)$

$P(A) P(B) P(C|A, B)$

- II. $P(A) P(B) P(C|A, B)$

$P(A) P(B|A) P(C|B)$

$P(A) P(B|A) P(C|A)$

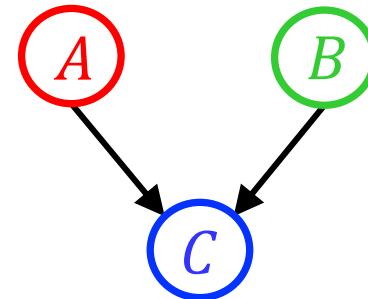
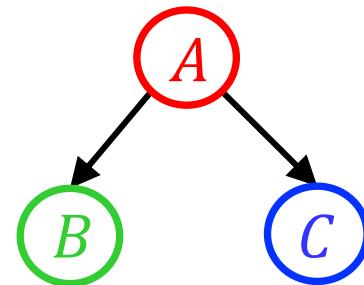
- III. $P(A) P(B|A) P(C|B)$

$P(A|B, C) P(B) P(C)$

$P(A) P(B|A) P(C|B)$

Conditional Independence Semantics

- For the following Bayes nets, write the joint $P(A, B, C)$
 1. Using the chain rule (with top-down order A,B,C)
 2. Using Bayes net semantics (product of CPTs)



Conditional Independence Semantics

- For the following Bayes nets, we write the joint $P(A, B, C)$
 - Using the chain rule (with top-down order A,B,C)
 - Using Bayes net semantics (product of CPTs)



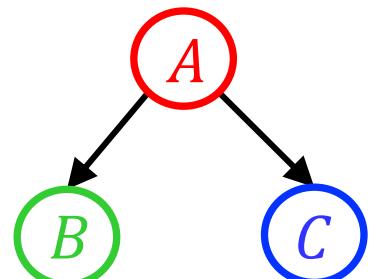
$P(A) P(B|A) P(C|A, B)$

$P(A) P(B|A) P(C|B)$

Assumption:

$$P(C|A, B) = P(C|B)$$

C is independent from A given B



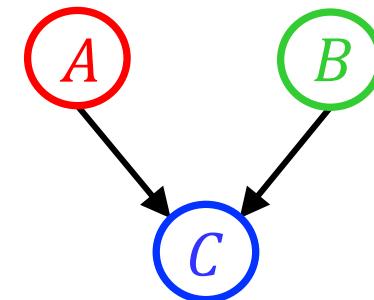
$P(A) P(B|A) P(C|A, B)$

$P(A) P(B|A) P(C|A)$

Assumption:

$$P(C|A, B) = P(C|A)$$

C is independent from B given A



$P(A) P(B|A) P(C|A, B)$

$P(A) P(B) P(C|A, B)$

Assumption:

$$P(B|A) = P(B)$$

A is independent from B given {}

Causal Chains

- This configuration is a “causal chain”



X: Low pressure Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

Guaranteed X independent of Z ?
No!

- One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed
 - Low pressure *always* causes rain
 - Rain *always* causes traffic
 - High pressure *always* causes no rain
 - No rain *always* causes no traffic
- Then:
$$P(+z | +x) = 1$$
$$P(+z | -x) = 0$$

Causal Chains

- This configuration is a “causal chain”
- Guaranteed X independent of Z given Y?



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

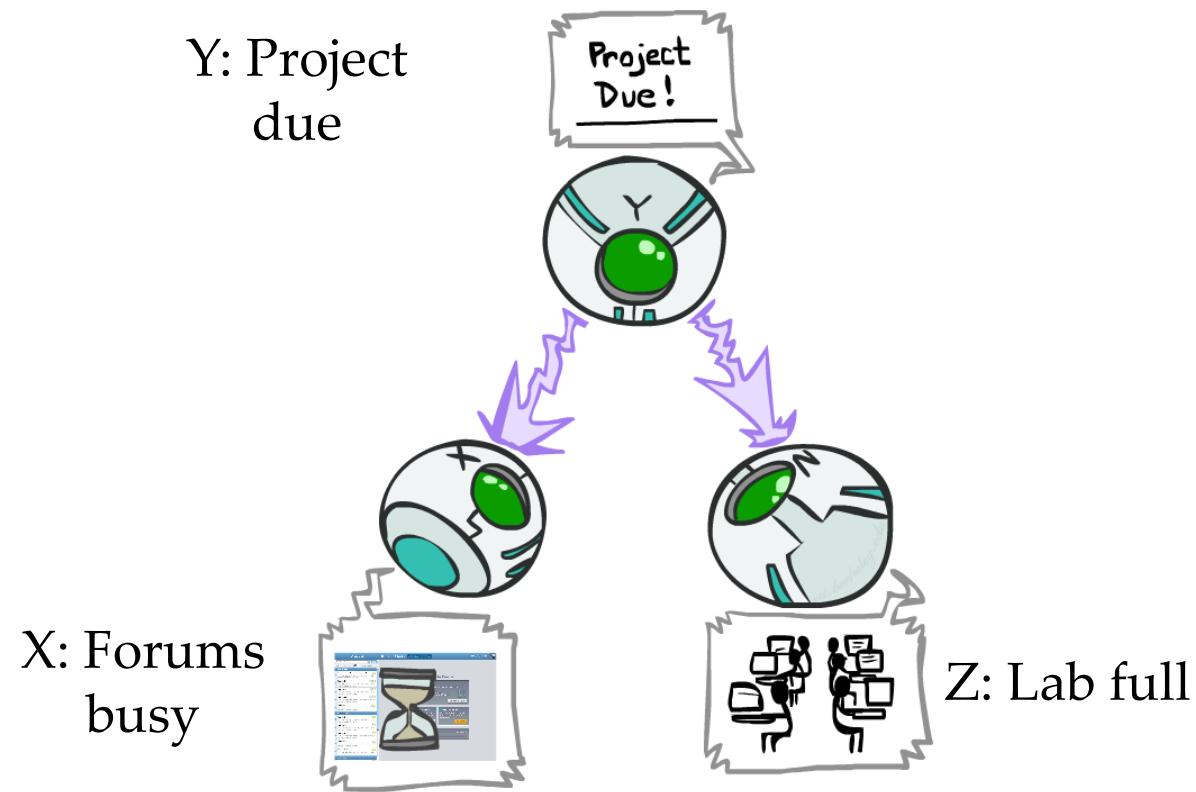
$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)} \\ &= P(z|y) \end{aligned}$$

Yes!

- Evidence along the chain “blocks” the influence from the beginning to the end

Common Cause

- This configuration is a “common cause”



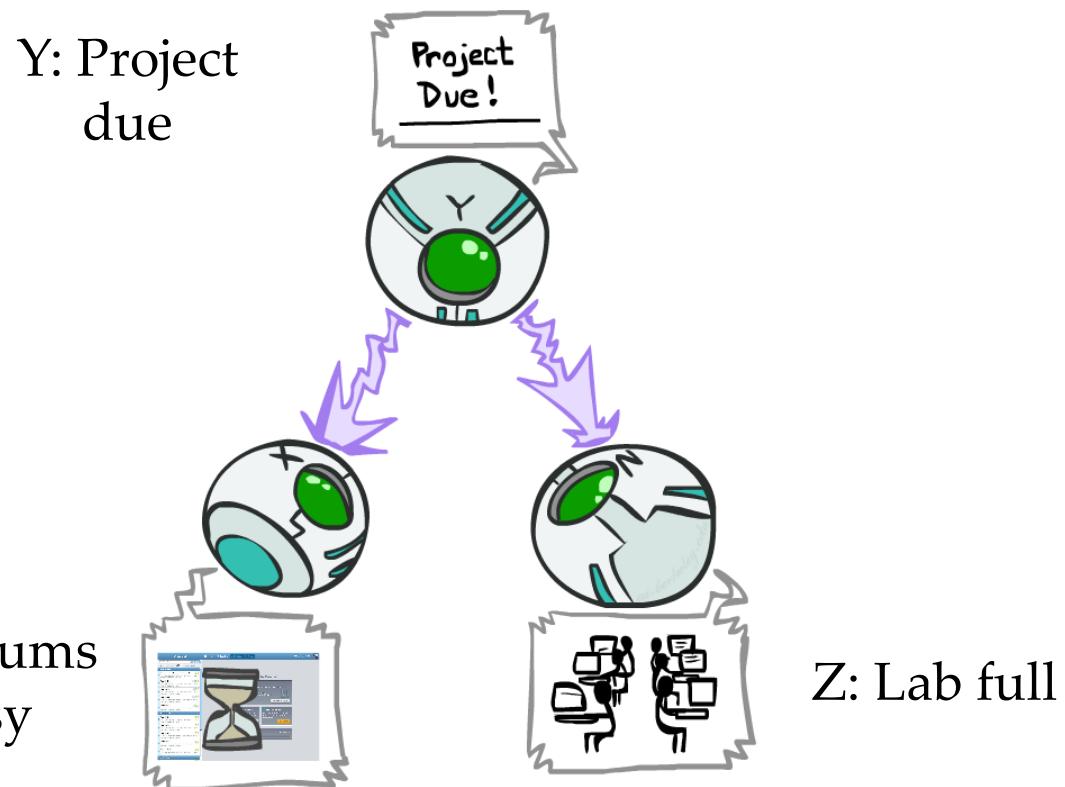
$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Guaranteed X independent of Z ? *No!*

- One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.
- Example:
 - Project due *always* causes both forums busy and lab full; not project due *always* causes both forums not busy and lab not full
- Then:
$$P(+x | +z) = 1$$
$$P(+x | -z) = 0$$

Common Cause

- This configuration is a “common cause”
- Guaranteed X and Z independent given Y?



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

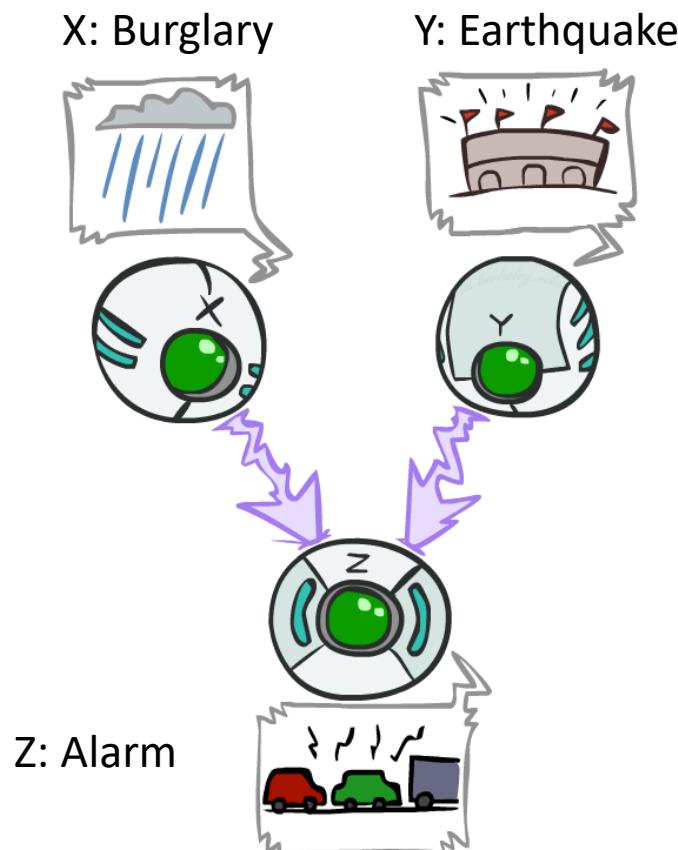
$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)} \\ &= P(z|y) \end{aligned}$$

Yes!

- Observing the cause blocks influence between effects.

Common Effect

- Last configuration: two causes of one effect (v-structures)



- Are X and Y independent?
 - *Yes*: burglary and earthquake cause alarm, but they are not correlated
 - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
 - *No*: observing alarm puts the burglary and the earthquake in competition as explanation.
- This is backwards from the other cases
 - Observing an effect **activates** influence between possible causes. **Explaining away** effect.

Bayes Net Independence



Answering Independence Questions

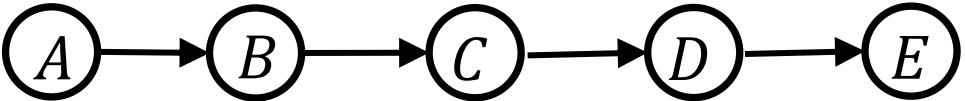
- Is A independent from E?



- Is A independent from E given C?

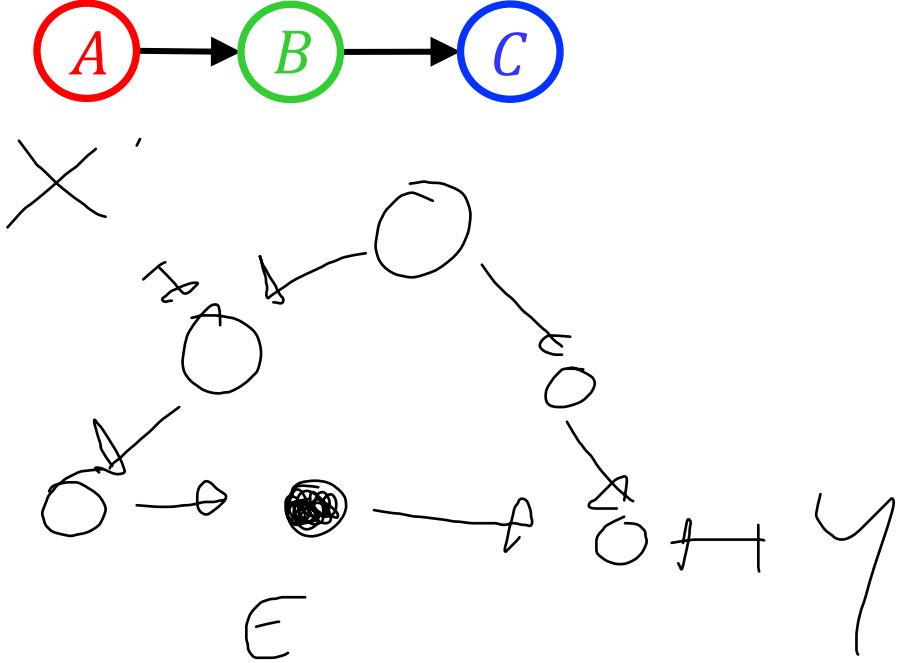


- Is A independent from C given E?

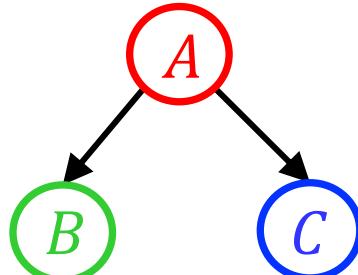


Summary from before

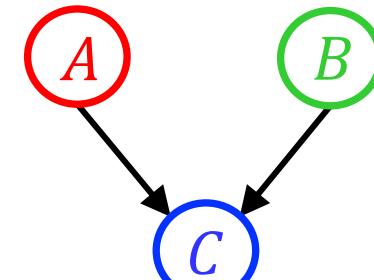
Causal chain



Common cause



Common effect
(v-structure)



Given a Bayes net, which conditional independences hold?

- Question: Given a Bayes net, are X and Y conditionally independent given some “evidence” variables (E, F, G...)?
- Influence is generally exerted via edges in Bayes net
- Idea: trace **undirected paths** from X to Y and see if X and Y exert influence on each other via any path
 - No active paths => conditional independence

Active and inactive paths

- A path is active if **each consecutive triple of nodes on the path** is active
 - If we get inactivity (independence) at any point, the influence of X on Y is blocked
- Shaded nodes are evidence (or “given” nodes)
- Causal chain



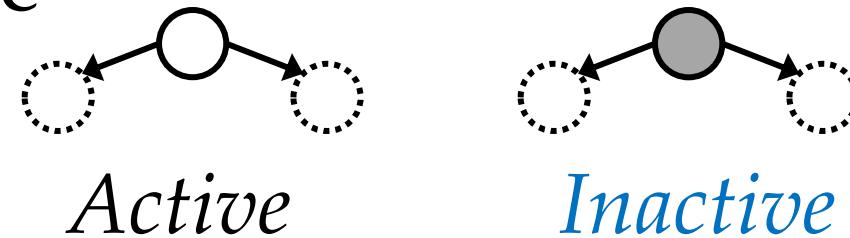
Active

Inactive

- *Knowing about intermediate node gives independence*

Active and inactive paths

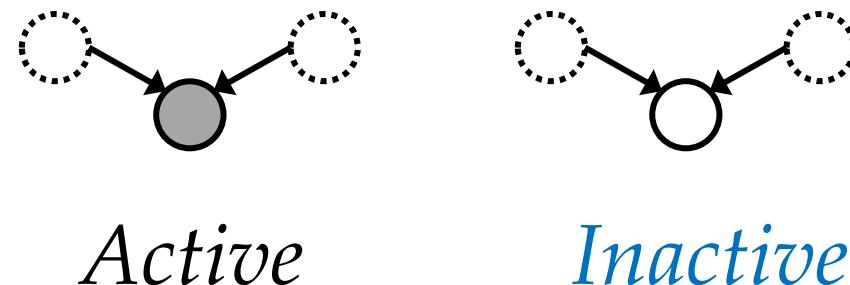
- A path is active if **each consecutive triple of nodes on the path** is active
 - If we get inactivity (independence) at any point, the influence of X on Y is blocked
- Shaded nodes are evidence (or “given” nodes)
- Common cause



- *Children are dependent, but become independent when conditioned on parent*

Active and inactive paths

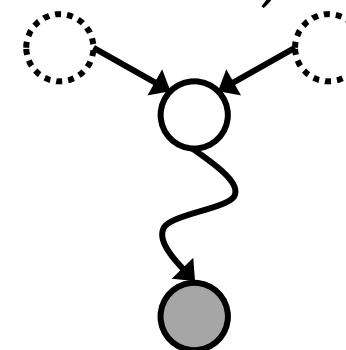
- A path is active if **each consecutive triple of nodes on the path** is active
 - If we get inactivity (independence) at any point, the influence of X on Y is blocked
- Shaded nodes are evidence (or “given” nodes)
- Common effect



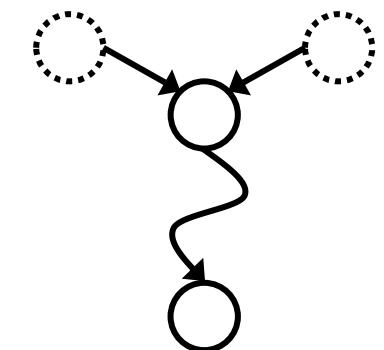
- *Parents were independent, but become dependent if the child is known (explaining away effect)*

Active and inactive paths

- A path is active if **each consecutive triple of nodes on the path** is active
 - If we get inactivity (independence) at any point, the influence of X on Y is blocked
- Shaded nodes are evidence (or “given” nodes)
- Common effect
- Explaining away effect also holds if: some **descendant** of child is observed!



Active



Inactive

Summary

- Question: Are X and Y conditionally independent given evidence variables $\{Z\}$?
- Consider all (undirected) paths from X to Y
- A path is active if **each consecutive triple** is active:
 - Causal chain $A \rightarrow B \rightarrow C$ where B is unobserved (either direction)
 - Common cause $A \leftarrow B \rightarrow C$ where B is unobserved
 - Common effect (aka v-structure) $A \rightarrow B \leftarrow C$ where B or *one of its descendants* is observed
- No active paths \Rightarrow independence

Important note

- We look at all paths along **undirected** edges
- But when going down a path and looking at triplets, we need to look at the **direction** of the edges
- Common cause and common effect induce opposite phenomena: observing parent causes independence, observing child causes dependence

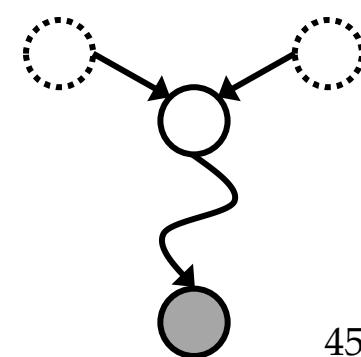
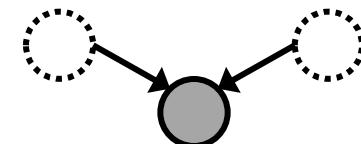
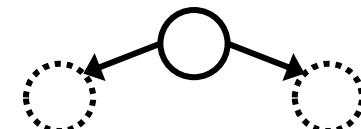
Bayes Ball

- Question: Are X and Y conditionally independent given evidence variables $\{Z\}$?

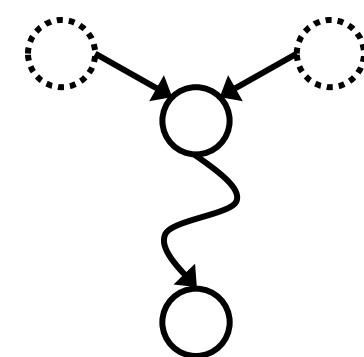
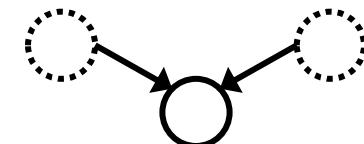
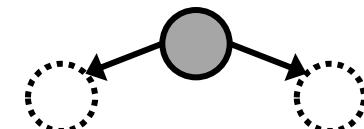


Shachter, Ross D. "Bayes-Ball: Rational Pastime (for Determining Irrelevance and Requisite Information in Belief Networks and Influence Diagrams)." *Proceedings of the Fourteenth Conference on Uncertainty in Artificial Intelligence*. 1998.

Active Paths



Inactive Paths

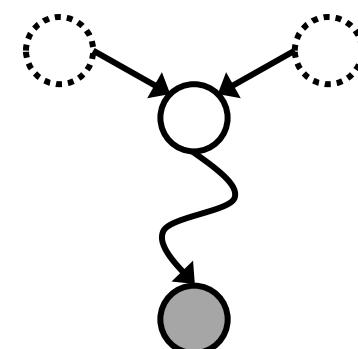
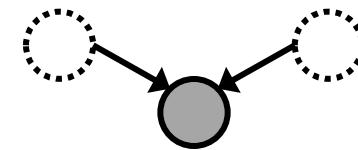
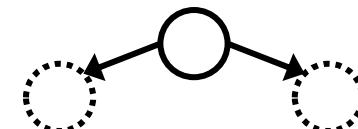


Bayes Ball

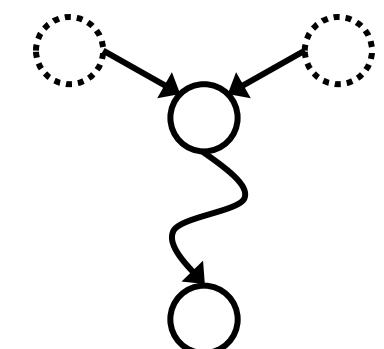
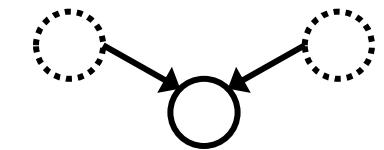
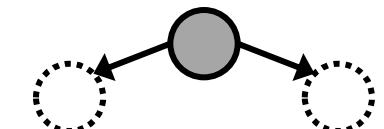
- Question: Are X and Y conditionally independent given evidence variables $\{Z\}$?

1. Shade in Z
2. Drop a ball at X
3. The ball can pass through any *active* path and is blocked by any *inactive* path (ball can move either direction on an edge)
4. If the ball can reach Y, then X and Y are NOT conditionally independent given Z

Active Paths

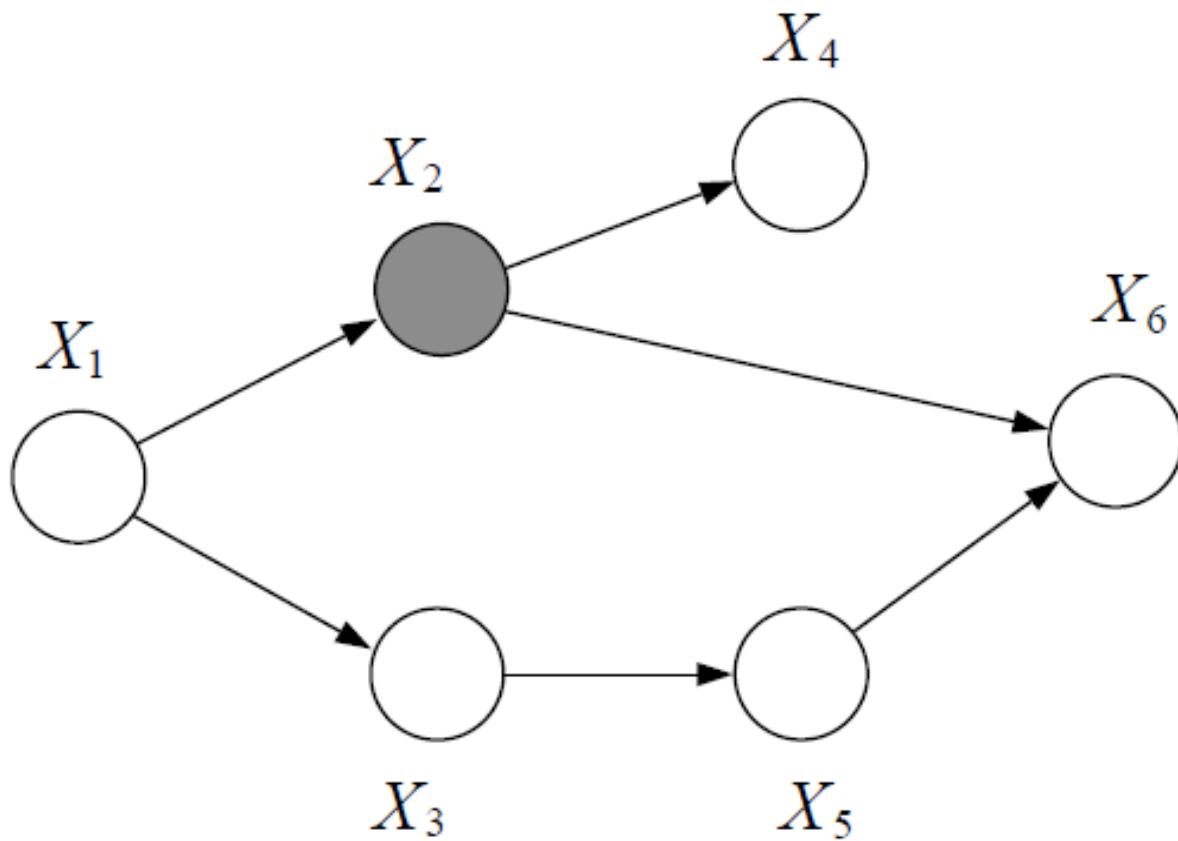


Inactive Paths



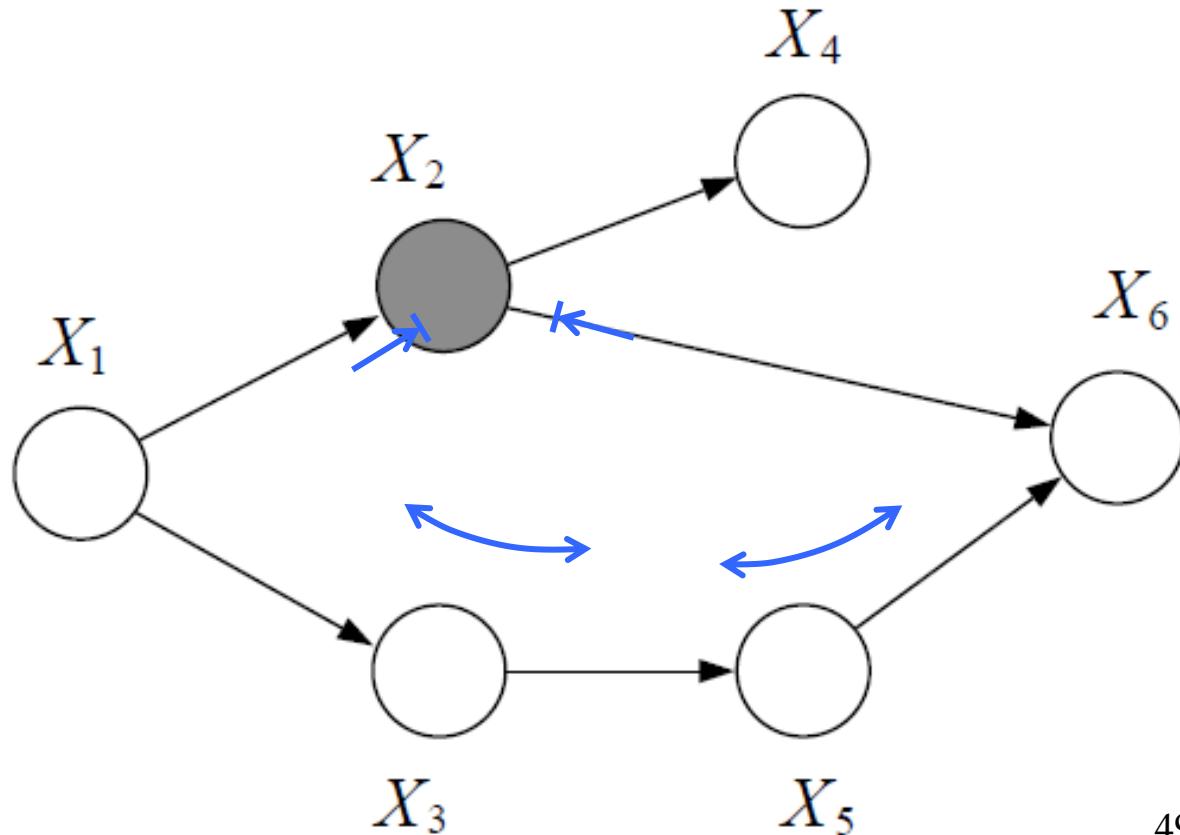
Question

- Is X_1 independent from X_6 given X_2 ?



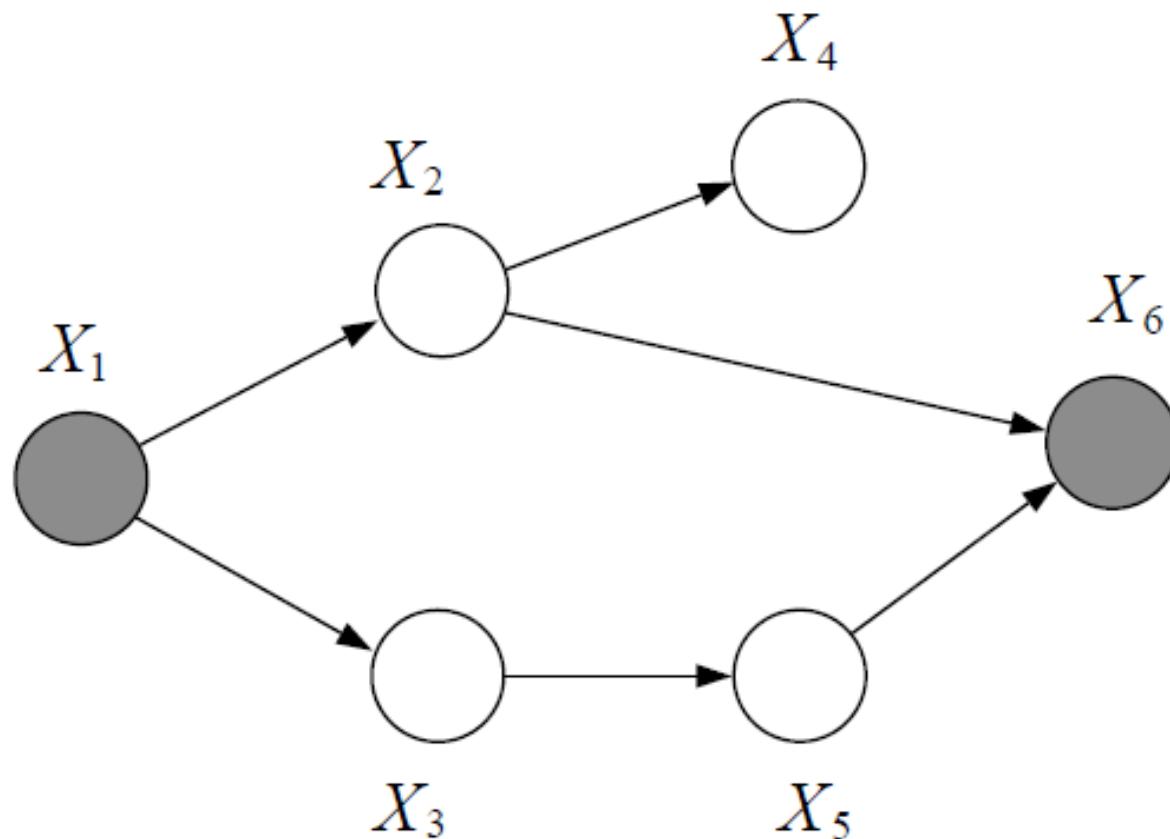
Question

- Is X_1 independent from X_6 given X_2 ?
- No, the Bayes ball can travel through X_3 and X_5 .



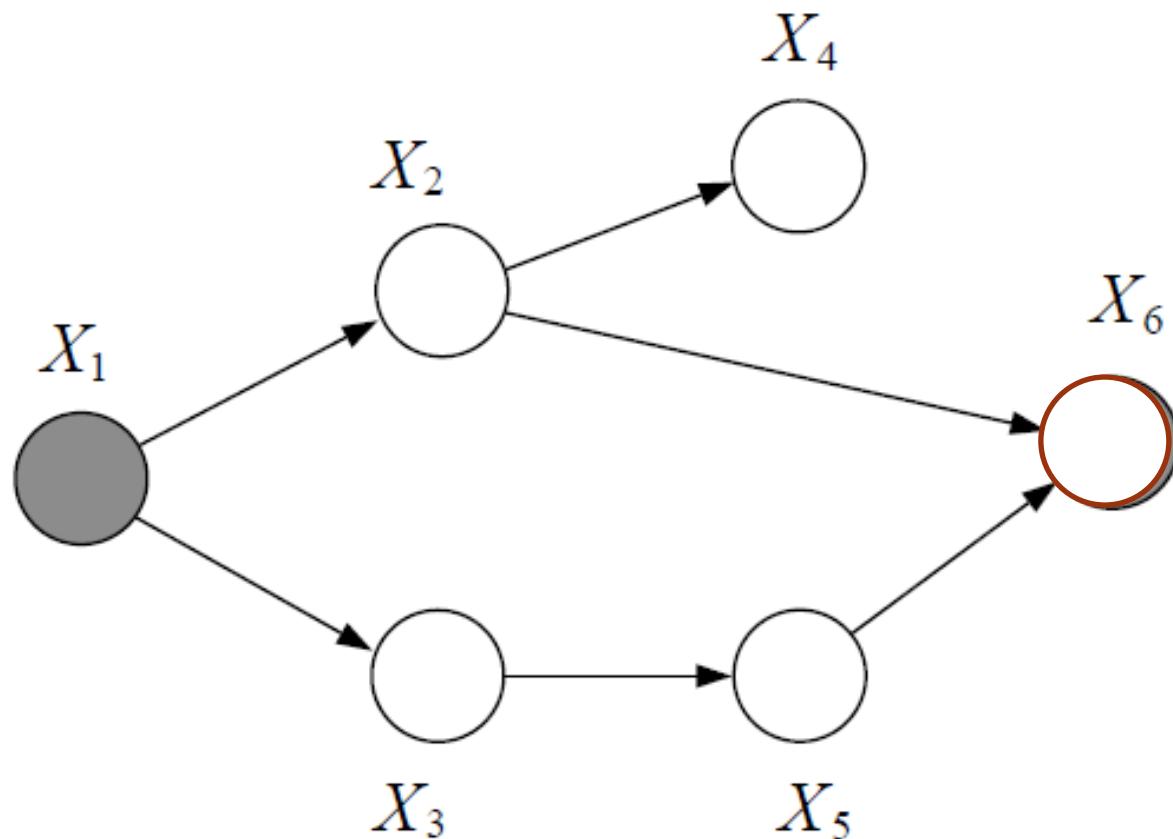
Poll 4

- Is X_2 independent from X_3 given X_1 and X_6 ?



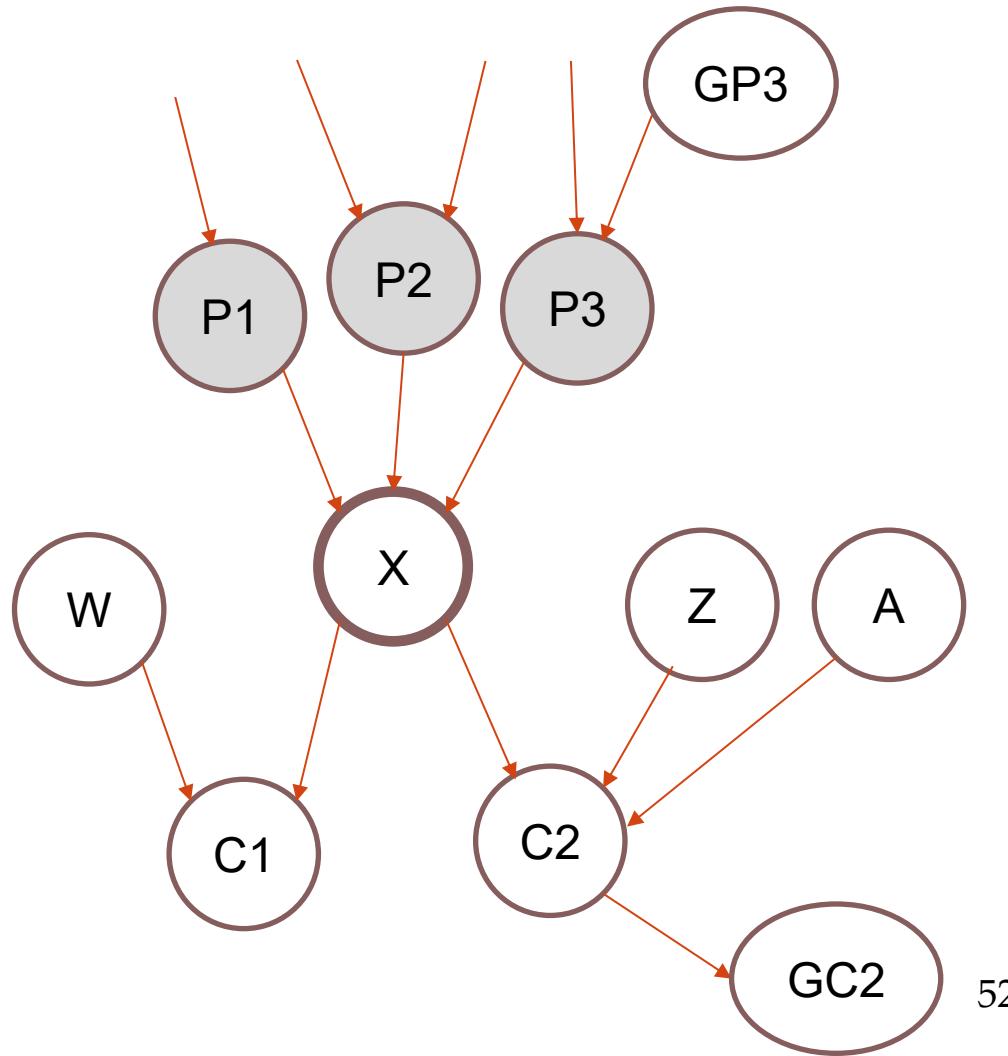
What if?

- Is X_2 independent from X_3 given X_1 and X_6 ?



Conditional independence semantics

- *Every variable is conditionally independent of its non-descendants given its parents*

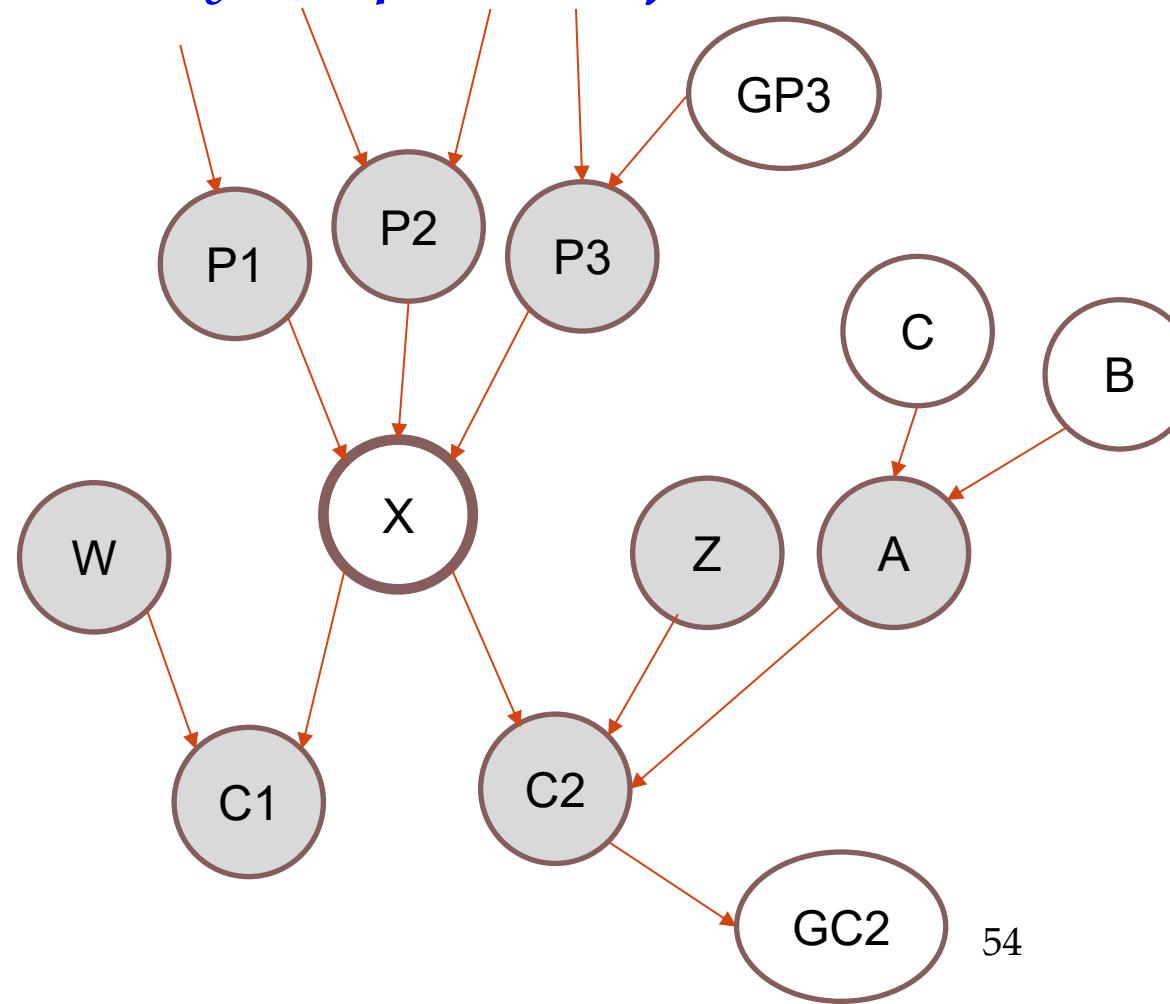


Markov blanket

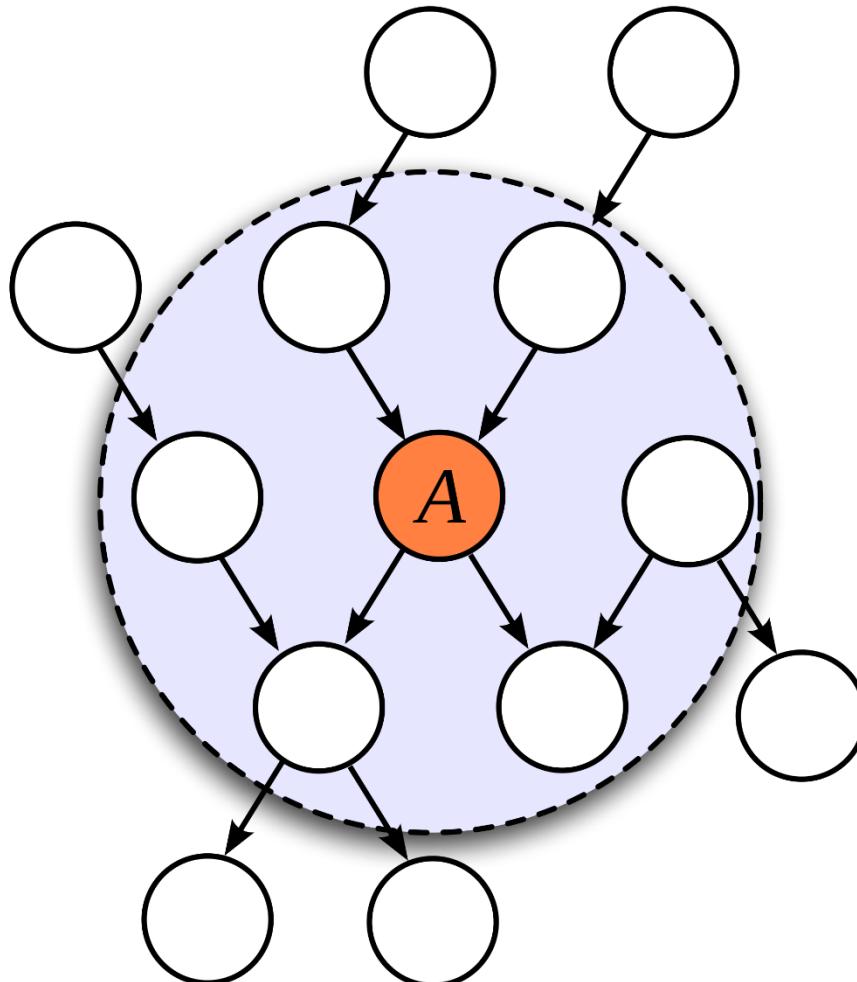
- Markov blanket of X - subset of variables such that all other variables are independent of X conditioned on the blanket

Markov blanket

- A variable's Markov blanket consists of parents, children, children's other parents
- *Every variable is conditionally independent of all other variables given its Markov blanket*



Markov blanket



Summary – things to know

- Answer queries from joint distribution (marginalization, conditioning)
- Independence, conditional independence implied by a Bayes net
 - “Bayes ball” method
 - Reasoning about causal chain, common cause, common effect