Play Rock Paper Scissors with the people around you

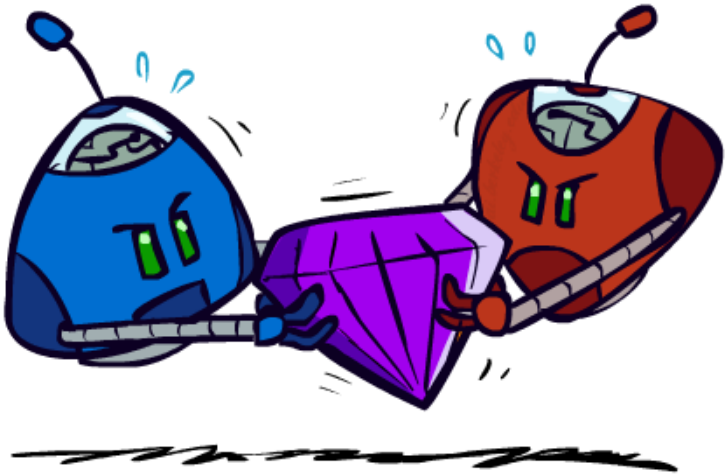
What are the rules?

What is the best action to take if the other player plays Rock?

What is the best overall strategy?

AI: Representation and Problem Solving

Game Theory: Equilibrium



Instructors: Tuomas Sandholm and Nihar Shah

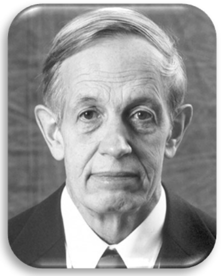
Slide credits: CMU AI

Image credit: ai.berkeley.edu

From Games to Game Theory

The study of mathematical models of conflict and cooperation between intelligent decision makers

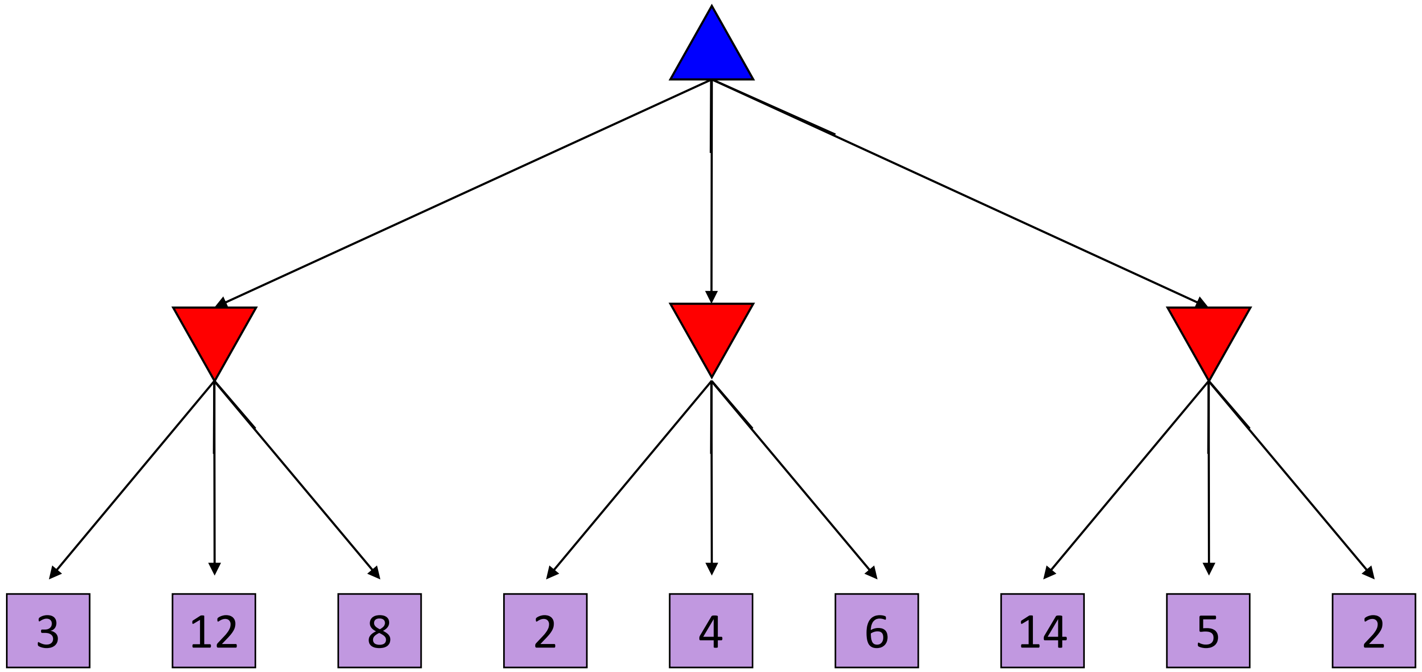
Used in economics, political science, biology, computer science, …



John Nash
Winner of Nobel Memorial Prize in Economic Sciences

Recall: Adversarial Search

Zero-sum, perfect information, two player games with turn-taking moves



Simultaneous-Move Games and Payoff Matrices

Rock-Paper-Scissors (RPS)

- Rock beats Scissors
- Scissors beats Paper
- Paper beats Rock

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

2-player normal-form game with finite set of actions taken simultaneously represented in a (bi)matrix

Player 1 is row player (typically first number)

Player 2 is column player (typically second number)

Rock, Paper, Scissors, Lizard, Spock

CBS, Big Bang Theory

https://www.youtube.com/watch?v=iSHPVCBsnLw

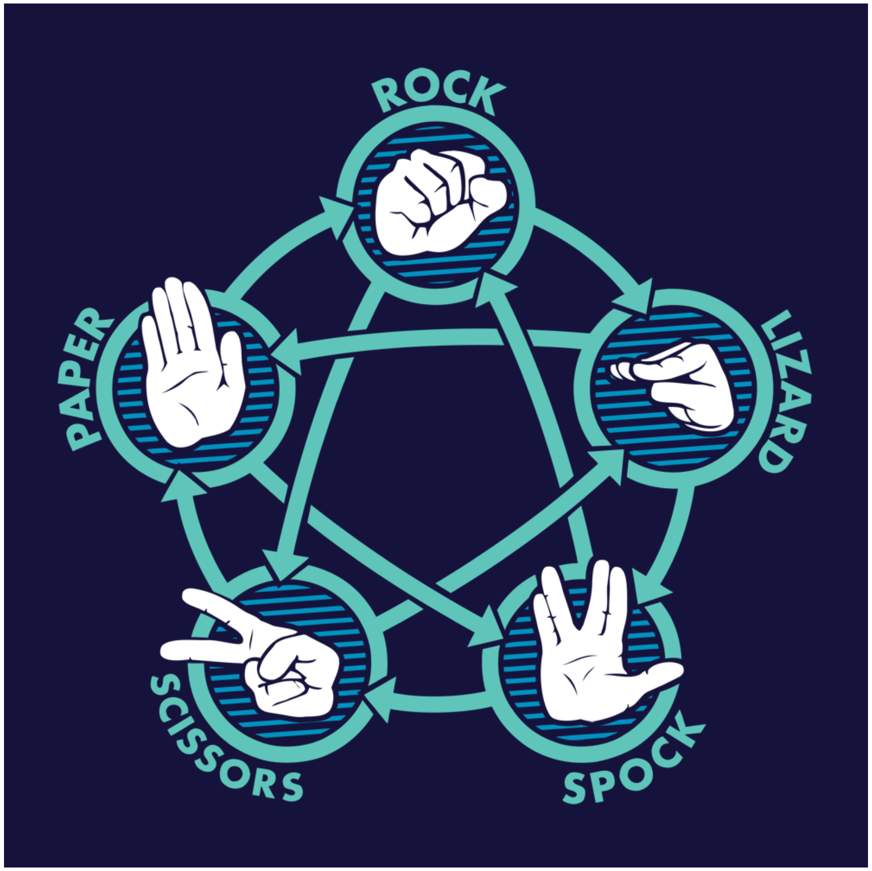


Image credit: https://www.snorgtees.com/rock-paper-scissors-lizard-spock

Simultaneous-Move Games and Payoff Matrices

Football vs Concert (FvsC)

- Historically known as Battle of Sexes
- If football together: Alex ☺☺, Berry ☺
- If concert together: Alex ☺, Berry ☺☺
- If not together: Alex ☹, Berry ☹

Fill in the payoff matrix, row payoff first then column!

Berry

Alex

	Football	Concert
Football	2, 1	-1, -1
Concert	-1, -1	1, 2

Normal-Form Games

A game in normal form consists of the following elements

- Set of players
- Set of actions for each player
- Payoffs / Utility functions
  - Determines the utility for each player given the actions chosen by all players (referred to as action profile)
- Bimatrix game is special case: two players, finite action sets

Players move simultaneously and the game ends immediately afterwards

What are the players, set of actions and utility functions of Football vs Concert (FvsC) game?

Classical Games and Payoff Matrices

Prisoner's Dilemma (PD)

- If both Cooperate with each other: 1 year in jail each
- If one Defects to police, one Cooperates: 0 year for (D), 3 years for (C)
- If both Defects to police: 2 years in jail each
- Let's play!

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Variation: Split or Steal

https://youtu.be/p3Uos2fzIJ0

https://www.youtube.com/watch?v=S0qjK3TWZE8

Zero-sum vs General-sum

Zero-sum Game

- No matter what actions are chosen by the players, the utilities for all the players sum up to zero or a constant

General-sum Game

- The sum of utilities of all the players need not be a constant

Which ones are general-sum games?

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissor	-1,1	1,-1	0,0

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Strategy

Pure strategy: choose an action deterministically

Mixed strategy: choose actions according to a probability distribution

- Notation: $s = (0.3, 0.7, 0)$
- Support: set of actions chosen with non-zero probability

Notation Alert! We use s to represent strategy here (not states)

Does your AI play a deterministic strategy or a mixed strategy?

What is the support size of your AI's strategy?

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Expected Utility

Notation Alert!

Use a, s, u to represent action, strategy, utility of a player

Use $\mathbf{a}, \mathbf{s}, \mathbf{u}$ to represent action, strategy, utility profile

Given the strategies of all players,

Expected Utility for player i u_i=

$$\sum_{\mathbf{a}} \text{Prob(action profile } \mathbf{a}) \times \text{Utility for player } i \text{ in } \mathbf{a}$$

Can skip action profiles with probability 0 or utility 0

If Alex's strategy $s_A = \left(\frac{1}{2}, \frac{1}{2}\right)$, Berry's strategy $s_B = (1,0)$

What is the probability of action profile $\mathbf{a}$ =(Concert,Football)?

What is Alex's expected utility overall?

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Best Response

Best Response (BR): Given the strategies or actions of all players but player i (denoted as $\mathbf{s}_{-i}$ or $\mathbf{a}_{-i}$), Player i's best response to $\mathbf{s}_{-i}$ or $\mathbf{a}_{-i}$ is the set of actions or strategies of player i that can lead to the highest expected utility for player i

In RPS, what is Player 1's best response to Rock (i.e., assuming Player 2 plays Rock)? What about to (.5 Rock, .5 Paper, 0 Scissors)?

In Prisoner's Dilemma, what is Player 1's best response to Cooperate? What is Player 1's best response to Defect? What about to (.5 Cooperate, .5 Defect)?

Player 2

Player 1	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Best Response

Best Response (BR): Given the strategies or actions of all players but player i (denoted as $\mathbf{s}_{-i}$ or $\mathbf{a}_{-i}$), Player i's best response to $\mathbf{s}_{-i}$ or $\mathbf{a}_{-i}$ is the set of actions or strategies of player i that can lead to the highest expected utility for player i

What is Alex's best response to Berry's mixed strategy $s_B = \left(\frac{1}{2}, \frac{1}{2}\right)$?

		Berry	
		1/2	1/2
		Football	Concert
Alex	Football	$\underline{2}$,1	$\underline{0}$,0
	Concert	$\underline{0}$,0	$\underline{1}$,2

Poll

In Rock-Paper-Scissors, if $s_1 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, which actions or strategies are player 2’s best responses to s_1?

- [✓ A.] Rock $\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{3}(-1) = 0$
- [✓ B.] Paper 0
- [✓ C.] Scissors 0
- ~~D. Lizard~~
- [✓ E.] $s_2 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ $\cdot$ 0
- [✓ F.] $s_2 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ 0

$$\sum_{(r,c)} \underbrace{P(r,c)}_{P(r)\cdot P(c)} \cdot u_1(r,c)$$

$$= \sum_c P(c) \underbrace{\sum_r P(r)\, u_1(r,c)}$$

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Best Response

Proposition: A mixed strategy is BR iff all actions in the support are BR

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Dominance

s_i and $s_i{}'$ are two strategies for player i

s_i strictly dominates $s_i{}'$if s_i is always better than $s_i{}'$, no matter what strategies are chosen by other players

s_i strictly dominates s_i' if

$$u_i\,(s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i}), \forall \mathbf{s}_{-i}$$

always better

s_i very weakly dominates s_i' if

$$u_i\,(s_i, \mathbf{s}_{-i}) \geq u_i(s_i', \mathbf{s}_{-i}), \forall \mathbf{s}_{-i}$$

never worse

s_i weakly dominates s_i' if

$$u_i\,(s_i, \mathbf{s}_{-i}) \geq u_i(s_i', \mathbf{s}_{-i}), \forall \mathbf{s}_{-i}$$

and $\exists \mathbf{s}_{-i}, u_i\,(s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i})$

never worse and
sometimes better

Dominance

Can you find any dominance relationships between the pure strategies in these games?

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissor	-1,1	1,-1	0,0

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Dominance

If s_i strictly dominates $s_i', \forall s_i' \in S_i \backslash \{s_i\}$,

is s_i a best response to $\mathbf{s}_{-i}, \forall \mathbf{s}_{-i}$?

Yes. Remember:

- s_i strictly dominates s_i' if

$$u_i\ (s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i}), \forall \mathbf{s}_{-i}$$

Rewriting the statement at the top:

$$u_i\ (s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i}), \forall \mathbf{s}_{-i} \quad \forall s_i' \in S_i \backslash \{s_i\}$$

So… for any $\mathbf{s}_{-i}$

$$u_i\ (s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i}), \forall s_i' \in S_i \backslash \{s_i\}$$

This is the definition of best response ☺

That is, s_i leads to the highest utility compared to all other responses, ${s_i}'$

Solution Concepts in Games

How should one player play and what should we expect all the players to play?

- Dominant strategy and dominant strategy equilibrium
- Nash Equilibrium
- Stackelberg Equilibrium

Dominant Strategy

A strategy could be (always better / never worse / never worse and sometimes better) than any other strategy

s_i is a (strictly/very weakly/weakly) dominant strategy if it (strictly/very weakly/weakly) dominates $s_i', \forall s_i' \in S_i \backslash \{s_i\}$

Focus on single player's strategy

Doesn't always exist

Is there a strictly dominant strategy for player 1 in PD?

Player 2

		Cooperate	Defect
Player 1	Cooperate	-1,-1	-3,0
	Defect	0,-3	-2,-2

Dominant Strategy Equilibrium

Sometimes called dominant strategy solution

Every player plays a dominant strategy

Focus on strategy profile for all players

Note: Doesn't always exist

What is the dominant strategy equilibrium for PD?

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Solution Concepts in Games

How should one player play and what should we expect all the players to play?

- Dominant strategy and dominant strategy equilibrium
- Nash Equilibrium
- Stackelberg Equilibrium

Nash Equilibrium

Nash Equilibrium (NE)

Every player’s strategy is a best response to others’ strategy profile

In other words, one cannot gain by unilateral deviation

Pure Strategy Nash Equilibrium (PSNE)

- $a_i \in BR(\mathbf{a}_{-i}), \forall i$

Mixed Strategy Nash Equilibrium

- At least one player use a randomized strategy
- $s_i \in BR(\mathbf{s}_{-i}), \forall i$

Nash Equilibrium

What are the PSNEs in these games?

What is the mixed strategy NE in RPS?

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissor	-1,1	1,-1	0,0

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	~~0,0~~	1,2

Nash Equilibrium

Theorem (Nash 1951): NE always exists in finite games

- Finite number of players, finite number of actions
- NE: can be pure or mixed
- Proof: Through Brouwer's fixed point theorem

Find PSNE

Find pure strategy Nash Equilibrium (PSNE)

- Enumerate all action profiles
- For each action profile, check if it is NE
  - For each player, check other available actions to see if he should deviate
- Other approaches?

Player 2

Player 1

	L	C	R
U	10, 3	1, 5	5, 4
M	3, 1	2, 4	5, 2
D	0, 10	1, 8	7, 0

Find PSNE

A strictly dominated strategy is one that is always worse than some other strategy

Strictly dominated strategies cannot be part of an NE **Why?**

Which are the strictly dominated strategies for player 1?

How about player 2?

Player 2

Player 1		L	C	R
	U	10, 3	1, 5	5, 4
	M	3, 1	2, 4	5, 2
	D	0, 10	1, 8	7, 0

Find PSNE through Iterative Removal

Remove strictly dominated actions (pure strategies) and then find PSNE in the remaining game

Can have new strictly dominated actions in the remaining game!

Repeat the process until no actions can be removed

This is the Iterative Removal algorithm (also known as Iterative Elimination of Strictly Dominated Strategies)

Find PSNE in this game using iterative removal

Player 2

Player 1

	L	C	R
U	10, 3	1, 5	5, 4
M	3, 1	2, 4	5, 2
D	0, 10	1, 8	7, 0

Find Mixed Strategy Nash Equilibrium

Big idea:

A NE occurs when there’s no incentive to change actions

Ensure that the expected utility of other player’s actions is equal

Can we still apply iterative removal?

- Yes! The removed strategies cannot be part of any NE
- You can always apply iterative removal first

Berry

		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

Find Mixed Strategy Nash Equilibrium

How to find mixed strategy NE (after iterative removal)?

q Berry $1-q$

Alex		Football	Concert
	p Football	2,1	0,0
	$(1-p)$ Concert	0,0	1,2

$$1\cdot p + 0(1-p) = 0p + 2(1-p)$$

$$2q + 0(1-q) = 0q + 1(1-q)$$

$$\left(p = \frac{2}{3} \quad q = \frac{1}{3} \right)$$

If $s_A = (p, 1-p)$ and $s_B = (q, 1-q)$ with $0 < p, q < 1$ is a NE, what are the necessary conditions for p and q?

Find Mixed Strategy Nash Equilibrium

How to find mixed strategy NE (after iterative removal)?

Berry

Alex	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

If $s_A = (p, 1-p)$ and $s_B = (q, 1-q)$ with $0 < p, q < 1$ is a NE, what are the necessary conditions for p and q?

$$u_A(F, s_B) = u_A(C, s_B) \qquad u_B(s_A, F) = u_B(s_A, C)$$

Find Mixed Strategy Nash Equilibrium

How to find mixed strategy NE (after iterative removal)?

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

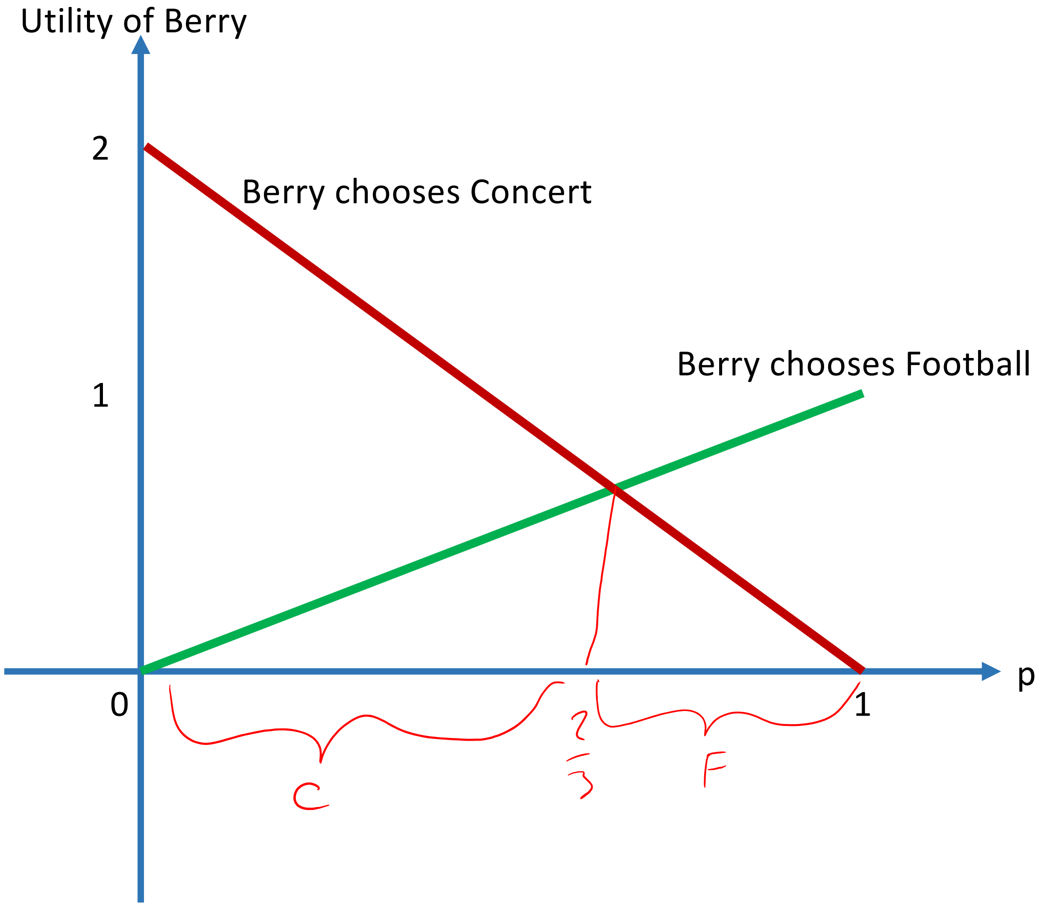
If $s_A = (p, 1 - p)$ and $s_B = (q, 1 - q)$ with $0 < p, q < 1$ is a NE, what are the necessary conditions for p and q?

$$u_A(F, s_B) = u_A(C, s_B) \qquad\qquad u_B(s_A, F) = u_B(s_A, C)$$

Why? Remember Theorem 1: A mixed strategy is BR iff all actions in the support are BR.

So…if $s_A \in BR(s_B)$, then $F \in BR(s_B)$and $C \in BR(s_B)$

Visualizing



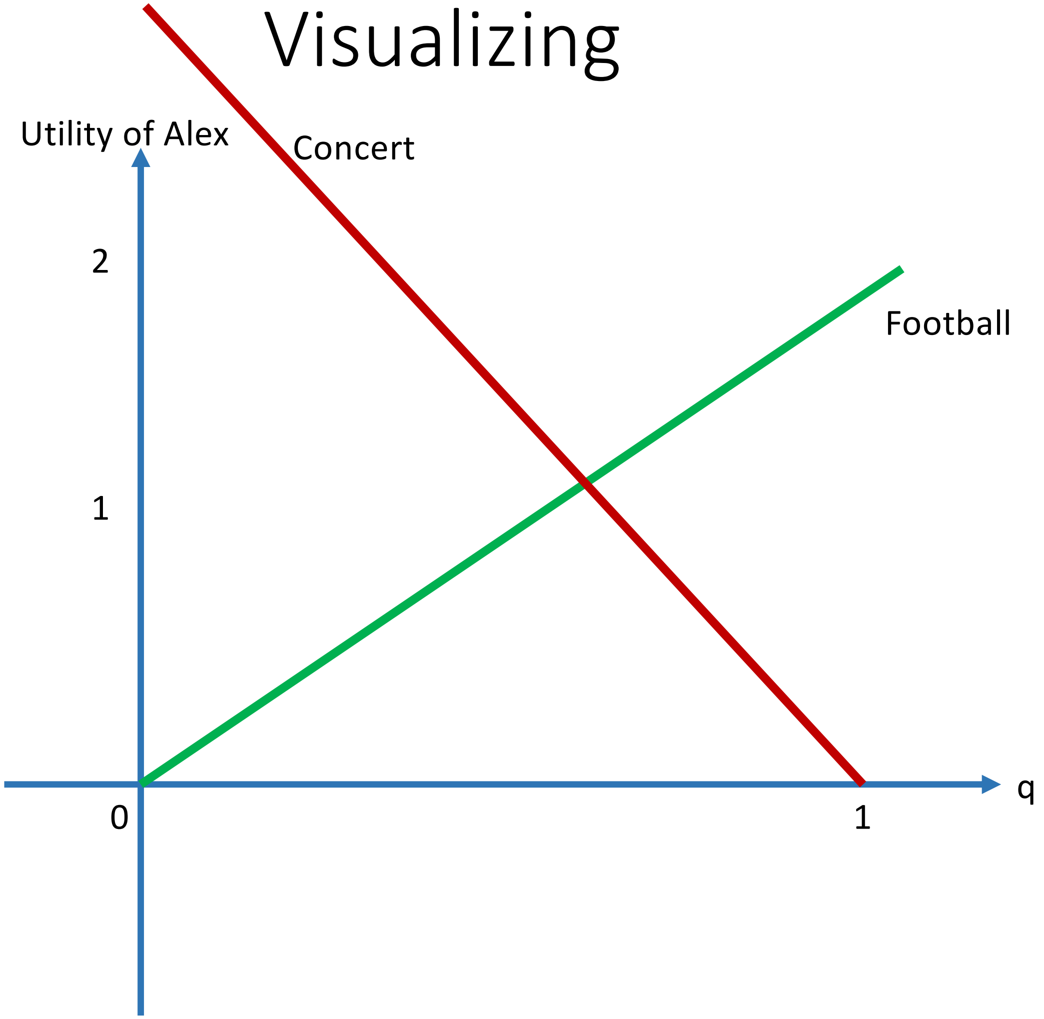
Alex wants to choose p such that Berry doesn’t want to deviate from his strategy

Berry wants the most reward he can get, so he will deviate if one strategy has more utility than another

The only way these two conditions is met is if we choose the p such that any strategy Berry picks will yield equal utility for Berry

			Berry	
			Football	Concert
p	Alex	Football	2,1	0,0
1-p		Concert	0,0	3,2

Visualizing



Berry wants to choose q such that Alex doesn’t want to deviate from his strategy

Alex wants the most reward he can get, so he will deviate if one strategy has more utility than another

The only way these two conditions is met is if we choose the q such that any strategy Alex picks will yield equal utility for Alex

		Berry	
		q	1 - q
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	3,2

Solution Concepts in Games

How should one player play and what should we expect all the players to play?

- Dominant strategy and dominant strategy equilibrium
- Nash Equilibrium
- Stackelberg Equilibrium

Power of Commitment

What are the PSNEs in this game, and the players' utilities?

What action should player 2 choose if player 1 commits to playing b? What is player 1's utility?

What action should player 2 choose if player 1 commits to playing a and b uniformly randomly? What is player 1's expected utility?

Player 2

Player 1

	c	d
a	2,1	4,0
b	1,0	3,2

Stackelberg Equilibrium

Stackelberg Game

- Leader commits to a strategy first
- Follower responds after observing the leader’s strategy

Stackelberg Equilibrium

- Follower best responds to leader’s strategy
- Leader commits to a strategy that maximizes her utility assuming follower best responds

Player 2

		c	d
Player 1	a	2,1	4,0
	b	1,0	3,2

Stackelberg Equilibrium

If the leader can only commit to a pure strategy, or you know that the leader’s strategy in equilibrium is a pure strategy, the equilibrium can be found by enumerating the leader’s pure strategies

If ties for the follower are broken by the follower such that the leader benefits, the leader can exploit this. This is the strong Stackelberg equilibrium (SSE)

In general, the leader can commit to a mixed strategy and in that case, for the leader: $u^{SSE} \geq u^{NE}$ (first-mover advantage)! & solvable by linear programming!
[Conitzer & Sandholm 2006; von Stengel and Zamir 2010; see also Prof. Fei Fang’s work here at CMU]

Berry

		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

Player 2

		c	d
Player 1	a	2,1	4,0
	b	1,0	3,2

In-Class Activity

What is the Mixed Strategy Nash Equilibrium for this new problem?

Berry

Alex

	Football	Concert
Football	4,1	0,0
Concert	0,0	3,3

Post-Lecture Poll

Does iterative elimination of strictly dominated strategies always lead to a pure-strategy Nash equilibrium?

a) Yes

b) No