Warm-up

Good morning!

Design an algorithm to determine the winner of three candidates a, b, c given the ranking provided by n individual voters, described by a $3 \times n$ matrix M

function voting(M)
Input: M where $M_{ij} \in$ {a, b, c} is the candidate at rank j for voter i Output: $x \in$ {a, b, c} describes the winner
Return x

Example Matrix M

	Voter 1	Voter 2	Voter 3	Voter 4
Rank 1	a	c	b	a
Rank 2	b	b	c	b
Rank 3	c	a	a	c

Announcements

Feedback (please don't forget!):

- www.cmu.edu/hub/fce
- https://www.ugrad.cs.cmu.edu/ta/S24/feedback/

Final Exam:

- All material is fair game
  - Will focus disproportionately on material not yet covered on midterm exams
- Look at post on Piazza with instructions

AI: Representation and Problem Solving

Social Choice

Instructors: Tuomas Sandholm and Nihar Shah

Slide credits: CMU AI

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Social Choice Theory

A mathematical theory that deal with aggregation of individual preferences

Wide applications in economics, public policy, etc.

Origins in Ancient Greece

18th century

- Formal foundations by Condorcet and Borda

19th Century

- Charles Dodgson

20th Century

- Nobel Prize in Economics

20th Century – Winners of Nobel Memorial Prize in Economic Sciences

Kenneth Arrow

Amartya Kumar Sen

Voting Model

Model

- Set of voters $N = \{1..n\}$
- Set of alternatives A ($|A| = m$)
  - These can be presidents, task allocations, resource allocations, etc.
- Each voter has a ranking over the alternatives
- Preference profile: collection of all voters' rankings

Voter ID	1	2	3	4
Ranking	a	c	b	a
	b	b	c	b
	c	a	a	c

Voting Rules

Voting rule: function that maps preference profiles to alternatives that specifies the winner of the election

function voting(M)
Input: M where $M_{ij} \in \{a, b, c\}$ is the candidate at rank j for voter i Output: $x \in \{a, b, c\}$ describes the winner
Return x

Example Matrix M

a	c	b	a
b	b	c	b
c	a	a	c

Voting Rules

Plurality (used in many political elections)

- Each voter gives one point to top alternative
- Alternative with most points wins

Who's the winner? a

Voter ID	1	2	3	4
Ranking	a	c	b	a
	b	b	c	b
	c	a	a	c

Voting Rules

Borda count (used for national election in Slovenia)

- Each voter awards $m - k$ points to alternative ranked k^{th}
- Alternative with most points wins

Who's the winner? b

Voter ID	1	2	3	4	$m - k$
Ranking	a	c	b	a	2
	b	b	c	b	1
	c	a	a	c	0

a: 2+0+0+2 = 4
b: 1+1+2+1 = 5
c: 0+2+1+0 = 3

Pairwise Election

Alternative x beats y in pairwise election if majority of voters prefer x to y

Who beats whom in pairwise election?

b beats c

Voter ID	1	2	3	4
Ranking	a	c	b	a
	b	b	c	b
	c	a	a	c

Voting Rules

Plurality with runoff

- First round: two alternatives with highest plurality scores survive
- Second round: pairwise election between the two

x beats y if majority of voters prefer x to y

Who's the winner?

Voter ID	1	2	3	4	5
Ranking	a	c	~~b~~	a	c
	~~b~~	~~b~~	c	~~b~~	~~b~~
	c	a	a	c	a

Voting Rules

Plurality with runoff

- First round: two alternatives with highest plurality scores survive
- Second round: pairwise election between the two

x beats y if majority of voters prefer x to y

Who's the winner? a and c survive, and then c beats a

Voter ID	1	2	3	4	5
Ranking	a	c	b	a	c
	b	b	c	b	b
	c	a	a	c	a

Voting Rules

Single Transferable Vote (STV)

- (Used in Ireland, Australia, New Zealand, Maine, San Francisco, Cambridge)
- $m-1$ rounds: In each round, alternative with least plurality votes is eliminated
- Alternative left is the winner

Who’s the winner? c is eliminated, then d, then a, leaving b as the winner.

Voter ID	1	2	3	4	5
Ranking	a	d	b	a	b
	b	b	c	b	d
	d	c	a	d	a
	c	a	d	c	c

Note: When d is eliminated, the vote from voter 2 is effectively transferred to b

Representation of Preference Profile

Identity of voters does not matter

Only record ***how many*** voters has a preference

33 voters	16 voters	3 voters	8 voters	18 voters	22 voters
a	b	c	c	d	e
b	d	d	e	e	c
c	c	b	b	c	b
d	e	a	d	b	d
e	a	e	a	a	a

Tie Breaking

Commonly used tie breaking rules include

- Borda count
- Having the most votes in the first round
- …

Social Choice Axioms

How do we choose among different voting rules?

What are the desirable properties?

Majority consistency

Majority consistency: If a majority of voters (> 50% of voters) rank alternative x first, then x should be the final winner

Poll 1

Which rules are NOT majority consistent?

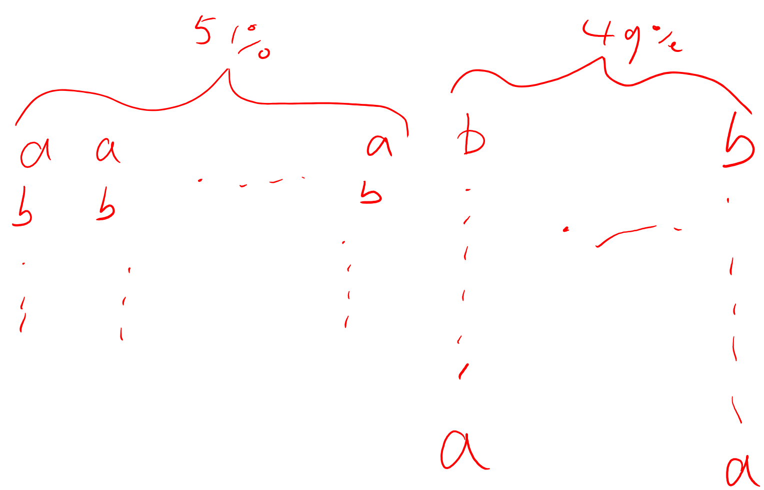
A. Plurality: Each voter give one point to top alternative

B. Borda count: Each voter awards $m - k$ points to alternative ranked k^{th}

C. Plurality with runoff: Pairwise election between two alternatives with highest plurality scores

D. STV: In each round, alternative with least plurality votes is eliminated

E. None



Condorcet Consistency

Recall: x beats y in a pairwise election if majority of voters prefer x to y

Condorcet winner is an alternative that beats every other alternative in pairwise election

Does a Condorcet winner always exist?

Condorcet paradox = cycle in majority preferences

Voter ID	**1**	**2**	**3**
Ranking over alternatives (first row is the most preferred)	a	c	b
	b	a	c
	c	b	a

Condorcet Consistency: a Condorcet winner (if one exists) should always win

If a rule satisfies majority consistency, does it satisfy Condorcet consistency?

Vice versa?

Poll 2

Which rules ARE Condorcet consistent?

A. Plurality: Each voter give one point to top alternative
B. Borda count: Each voter awards $m - k$ points to alternative ranked k^{th}
C. Plurality with runoff: Pairwise election between two alternatives with highest plurality scores
D. STV: In each round, alternative with least plurality votes is eliminated
E. None

Condorcet Consistency

Winner under different voting rules in this example

- Plurality:
- Borda:
- Plurality with runoff:
- STV:
- Condorcet winner:

33 voters	16 voters	3 voter	8 voters	18 voters	22 voters
a	b	c	c	d	e
b	d	d	e	e	c
c	c	b	b	c	b
d	e	a	d	b	d
e	a	e	a	a	a

Strategy-Proofness

Consider Borda Count

Who is the winner?

Voter ID	1	2	3	$m-k$
Ranking over alternatives (first row is the most preferred)	b	b	a	3
	a	a	b	2
	c	c	c	1
	d	d	d	0

Who is the winner now?

Voter ID	1	2	3	$m-k$
Ranking over alternatives (first row is the most preferred)	b	b	a	3
	a	a	c	2
	c	c	d	1
	d	d	b	0

Strategy-Proofness

A single voter can manipulate the outcome!

Voter ID	1	2	3	$m-k$
Ranking over alternatives (first row is the most preferred)	b	b	a	3
	a	a	b	2
	c	c	c	1
	d	d	d	0

b: 2*3+1*2=8
a: 2*2+1*3=7

b is the winner

Voter ID	1	2	3	$m-k$
Ranking over alternatives (first row is the most preferred)	b	b	a	3
	a	a	c	2
	c	c	d	1
	d	d	b	0

b: 2*3+1*0=6
a: 2*2+1*3=7

a is the winner

Strategy-Proofness

A voting rule is strategyproof (SP) if a voter can never benefit from lying about his preferences (regardless of what other voters do)

- Benefit: a more preferred alternative is selected as winner

Do not lie: b is the winner

Voter ID	1	2	3
Ranking	b	b	a
	a	a	b
	c	c	c
	d	d	d

Lie: a is the winner

Voter ID	1	2	3
Ranking	b	b	a
	a	a	c
	c	c	d
	d	d	b

If a voter's preference is a>b>c, c will be selected w/o lying, and b will be selected w/ lying, then the voter still benefits

Poll 3

Which of the introduced voting rules are strategyproof?

A. Plurality: Each voter give one point to top alternative

B. Borda count: Each voter awards $m - k$ points to alternative ranked k^{th}

C. Plurality with runoff: Pairwise election between two alternatives with highest plurality scores

D. STV: In each round, alternative with least plurality votes is eliminated

E. None

Greedy Algorithm for f −Manipulation

Given voting rule f and preference profile of $n - 1$ voters, how can the last voter report preference to let a specific alternative y *uniquely* win (no tie breaking)?

Greedy algorithm for f −Manipulation

Rank y in the first place
While there are unranked alternatives
    If $\exists x$ that can be placed in the next spot without preventing y from winning,
        place this alternative in the next spot
    else
        return false
return true (with final ranking)

Correctness proved (Bartholdi et al., 1989)

Greedy Algorithm for f −Manipulation

Example with Borda count voting rule

Voter ID	1	2	3
Ranking over alternatives (first row is the most preferred)	b	b	a
	a	a	
	c	c	
	d	d	

Greedy Algorithm for f −Manipulation

Example with Borda count voting rule

Voter ID	1	2	3	$m-k$
Ranking over alternatives (first row is the most preferred)	b	b	a	3
	a	a	b	2
	c	c		1
	d	d		0

b: 2*3+1*2=8
a: 2*2+1*3=7

Cannot put b here

Voter ID	1	2	3	$m-k$
Ranking over alternatives (first row is the most preferred)	b	b	a	3
	a	a	c	2
	c	c	d	1
	d	d	b	0

c: 2*1+1*2=4
a: 2*2+1*3=7
c can be placed second

b: 2*3+1*1=7
b cannot be placed third

d: 2*0+1*1=1
d can be placed third

Other Properties

A voting rule is dictatorial if there is a voter who always gets their most preferred alternative

A voting rule is constant if the same alternative is always chosen (regardless of the stated preferences)

A voting rule is onto if any alternative can win, for some set of stated preferences

Which of the introduced voting rules (Plurality, Borda count, Plurality with runoff, STV) are dictatorial, constant, or onto?

Results in Social Choice Theory

Constant functions and dictatorships are SP Why?

Theorem (Gibbard-Satterthwaite): If $m \geq 3$, then any voting rule that is SP and onto is dictatorial

- Any voting rule that is onto and nondictatorial is manipulable
- It is impossible to have a voting rule that is strategyproof, onto, and nondictatorial

Activity: Favorite topics of 15281 (by approval voting)

Learning Objectives

Understand the voting model

Find the winner under the following voting rules

- Plurality, Borda count, Plurality with runoff, Single Transferable Vote

Describe the following concepts, axioms, and properties of voting rules

- Pairwise election, Condorcet winner
- Majority consistency, Condorcet consistency, Strategyproofness
- Dictatorial, constant, onto

Understand the possibility of satisfying multiple properties

Describe the greedy algorithm for voting rule manipulation

Post-Lecture Poll

Consider the following ***randomized*** voting rule.

- With probability p, select a dictator at random
- Otherwise (i.e., with probability 1-p), select two candidates at random (possibly with unequal probabilities), and conduct a plurality election among the two

Is this voting rule strategyproof?

a) Yes

b) No