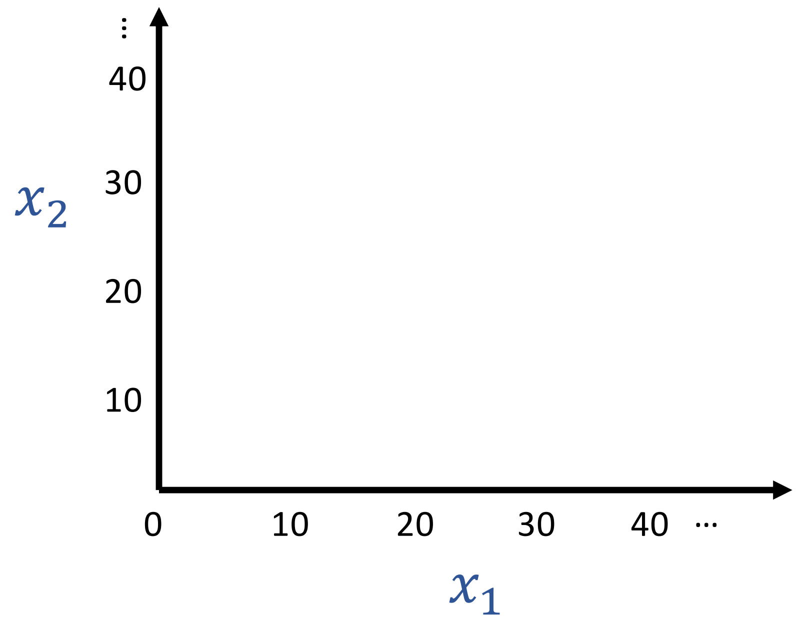
As you come in…

Draw a graph with x_1 as the x-axis and x_2 as the y-axis.

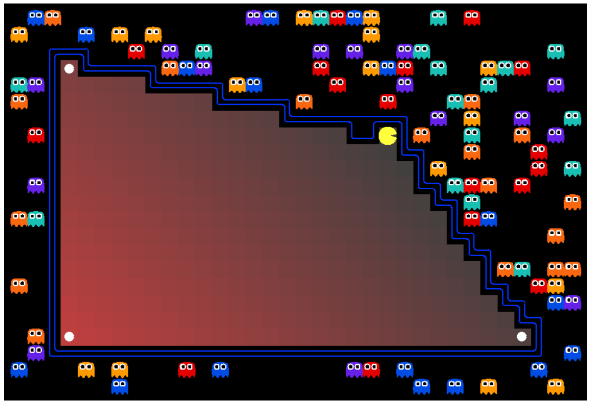
You can restrict attention to $x_1 \geq 0$, $x_2 \geq 0$.

Mark the region where $3\, x_1 + \; 4\, x_2 \leq 100$.



AI: Representation and Problem Solving

Optimization
&
Linear Programming



Instructors: Nihar Shah and Tuomas Sandholm

Slide credits: CMU AI with some drawings from ai.berkeley.edu

Optimization: BIG PICTURE

Oxford Dictionary of English

1817–

optimize, v.

transitive. To render optimal, to make as good as possible; to make the best or most effective use of.

1857–

optimization, n.

The action or process of making the best of something; (also) the action or process of rendering optimal; the state or condition of being optimal.

Optimization

minimize (or maximize)   something

subject to   some constraints

Optimization

"How much **time** to spend on this course?"

maximize   your learning (or your grade)
subject to   also having a life outside of work

Optimization

"How much time to spend on this course?"

objective

maximize time spent on course

how you spend your time

optimization variable(s): this is what you can choose

subject to at least blah time for blah activities

constraint(s)

Optimization

"How much **time** to spend on this course?"

maximize $x_1 + x_2$
x_1, x_2, x_3, x_4

x_1: time spent in 281's lectures
x_2: time spent on 281 outside lectures
x_3: sleeping, eating, …
x_4: spending time with friends, …

subject to $x_1 = 3$, $x_3 \geq$ blah, $x_4 \geq$ blah,

$x_1 + x_2 + x_3 + x_4 = 24*7$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$

Optimization: Many, many applications

- Machine Learning / Natural language processing (including ChatGPT)
- Operations research (e.g., making airline schedules)
- Telecommunications
- Finance
- Power systems
- Healthcare

and many more.

Optimization recipe

- You have a real-world problem to solve
- First write it mathematically as an optimization problem
- There are many optimization “solvers” available online – can use them
  - e.g., Gurobi, scipy.optimize, cvxpy, …
- There are specific representations of optimization problems for which specialized, more efficient algorithms are known.
  - e.g., linear programs, integer programs, …
- Check if your problem has such a representation. If not, check if you can transform your problem to such a representation. If so, use the relevant solver.
  - e.g., scipy.optimize.linprog
  - Otherwise use a generic solver

Linear Programs

A specific representation

Another example: What to eat?

We are staying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

Healthiness goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

We can choose the amount of stir-fry (ounce) and boba (fluid ounces)

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

Variables? Amount of stir-fry x_1 and boba x_2

Objective? Cost $1*x_1 + 0.5*x_2$

Constraints?

Calories min	$100\,x_1 + 50\,x_2 \geq 2000$
Calories max	$100\,x_1 + 50\,x_2 \leq 2500$
Sugar	$3\,x_1 + \;\;4\,x_2 \leq 100$
Calcium	$20\,x_1 + 70\,x_2 \geq 700$
Non-negativity	$x_1 \geq 0, x_2 \geq 0$

We can choose the amount of stir-fry (ounce) and boba (fluid ounces)

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

- $2000 \leq \text{Calories} \leq 2500$
- $\text{Sugar} \leq 100$ g
- $\text{Calcium} \geq 700$ mg

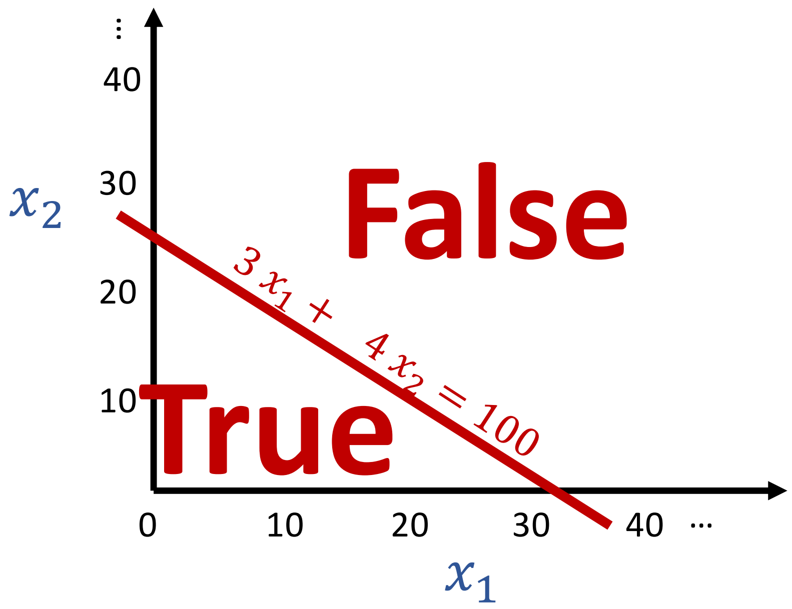
What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

$$
\begin{aligned}
\min_{x_1, x_2}. \quad & 1\, x_1 + 0.5\, x_2 \\
\text{s.t.} \quad & 100\, x_1 + 50\, x_2 \geq 2000 \\
& 100\, x_1 + 50\, x_2 \leq 2500 \\
& 3\, x_1 + \ \ 4\, x_2 \leq 100 \\
& 20\, x_1 + 70\, x_2 \geq 700 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{aligned}
$$

Let's look at any one constraint

$3\,x_1 + \quad 4\,x_2 \leq 100$

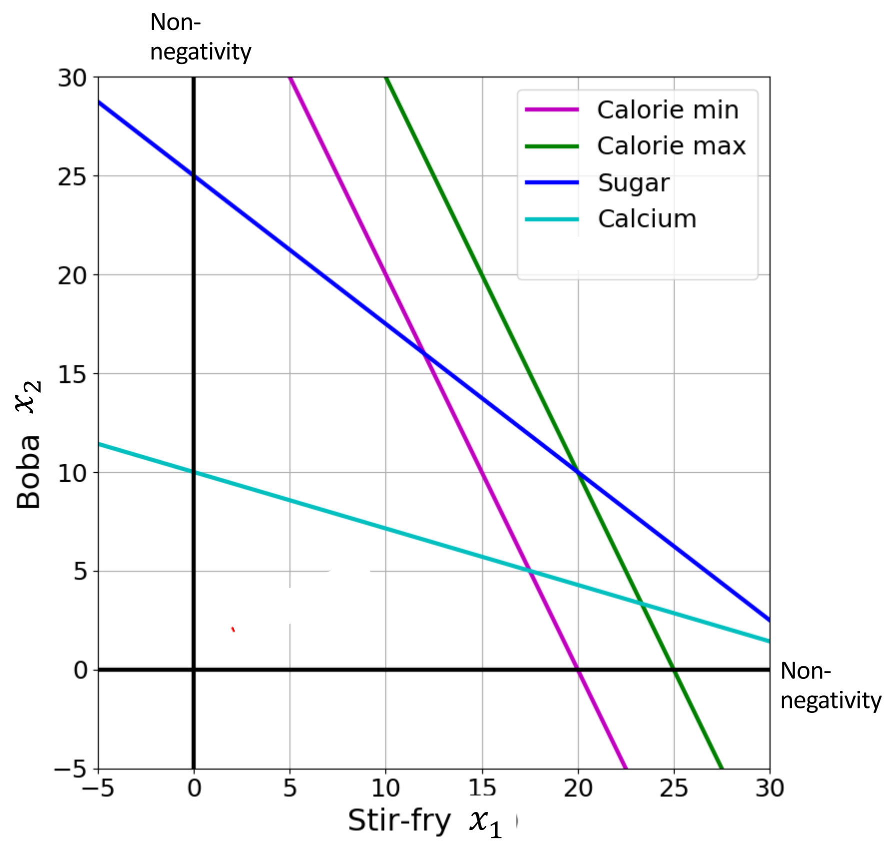


In two dimensions, constraint is simply entire region on one side of a line!

"Linear constraint"

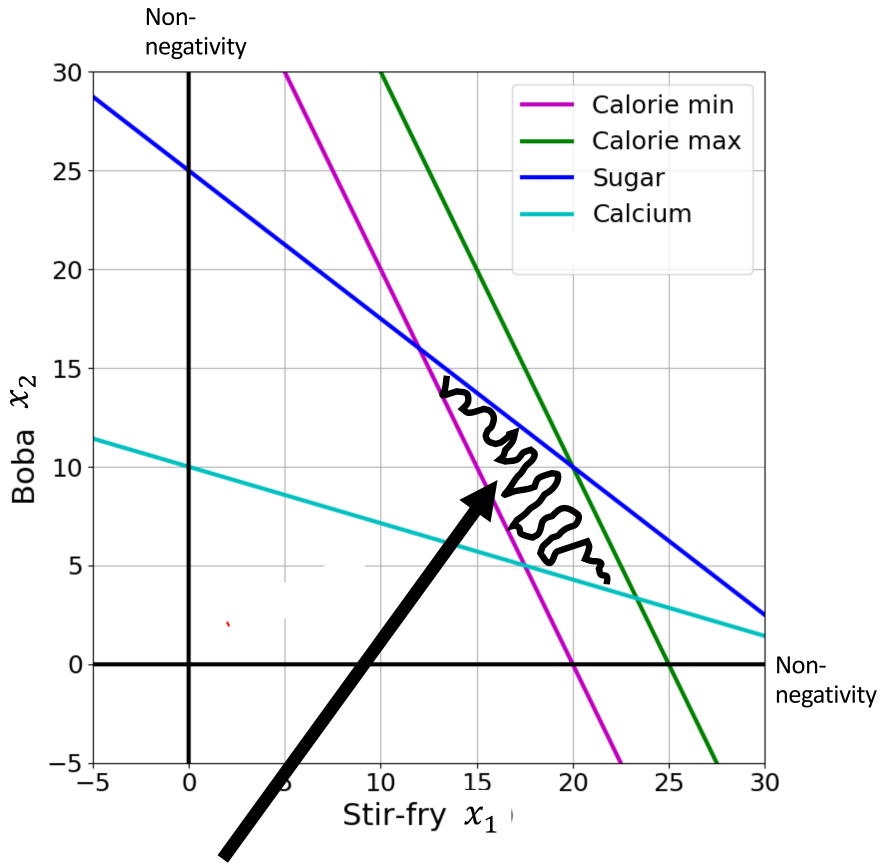
Our constraints are linear

Calories min	$100\,x_1 + 50\,x_2 \geq 2000$
Calories max	$100\,x_1 + 50\,x_2 \leq 2500$
Sugar	$3\,x_1 + 4\,x_2 \leq 100$
Calcium	$20\,x_1 + 70\,x_2 \geq 700$
Non-negativity	$x_1 \geq 0$
	$x_2 \geq 0$



Our constraints are linear

Calories min	$100\,x_1 + 50\,x_2 \geq 2000$
Calories max	$100\,x_1 + 50\,x_2 \leq 2500$
Sugar	$3\,x_1 + 4\,x_2 \leq 100$
Calcium	$20\,x_1 + 70\,x_2 \geq 700$
Non-negativity	$x_1 \geq 0$
	$x_2 \geq 0$



"feasible region": this is where all constraints are satisfied

Mathematical representation of linear constraints

Our problem had 2 variables, and constraints like $3\,x_1 + \quad 4\,x_2 \leq 100$

More generally, consider d variables $x_1, x_2, \ldots, x_d$.
Then a linear constraint is of the form:

$$\text{blah} \; * \; \text{x}_1 + \text{blah} \; * \text{x}_2 + \; \ldots + \text{blah} * \text{xd} \; \leq \text{blah}$$

where each "blah" is a real number

A linear constraint is of the form:

$$\text{blah} * x_1 + \text{blah} * x_2 + \ldots + \text{blah} * xd \leq \text{blah}$$

where each “blah” is a real number.

Are our constraints linear?

Calories min	$100\,x_1 + 50\,x_2 \geq 2000$
Calories max	$100\,x_1 + 50\,x_2 \leq 2500$
Sugar	$3\,x_1 + 4\,x_2 \leq 100$
Calcium	$20\,x_1 + 70\,x_2 \geq 700$
Non-negativity	$x_1 \geq 0$
	$x_2 \geq 0$

First constraint: $100\,x_1 + 50\,x_2 \geq 2000$

Equivalent constraint: $-100\,x_1 - 50\,x_2 \leq -2000$

A linear constraint is of the form:

$$\text{blah} \ * \ \text{x}_1 + \text{blah} \ * \text{x}_2 + \ \ldots + \text{blah} * \text{xd} \ \leq \text{blah}$$

where each “blah” is a real number.

Are our constraints linear?

Calories min	$-100\, x_1 - 50\, x_2 \leq -2000$
Calories max	$100\, x_1 + 50\, x_2 \leq 2500$
Sugar	$3\, x_1 + \ \ 4\, x_2 \leq 100$
Calcium	$-20\, x_1 - 70\, x_2 \leq -700$
Non-negativity	$-x_1 \leq 0$
	$-x_2 \leq 0$

Yes!

Now let's stare at our objective

$1\, x_1 + 0.5\, x_2$

Seems to have a familiar form

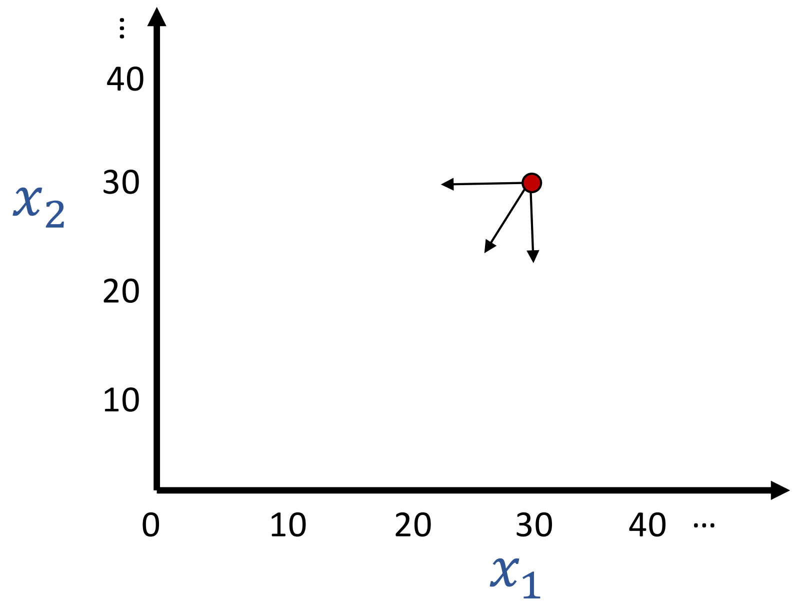
$$\text{blah} * x_1 + \text{blah} * x_2 + \ldots + \text{blah} * xd$$

"Linear objective"

Let's look at it on a graph...

Now let's look at our objective

min. $1\ x_1 + 0.5\ x_2$



Suppose you can move a unit distance starting from this point. In which direction is the cost reduced most?

Simpler question: Which reduces cost more?
- Moving down 1 unit
- Moving left 1 unit
- Moving down by $\frac{1}{\sqrt{1^2+0.5^2}}$ and left by $\frac{0.5}{\sqrt{1^2+0.5^2}}$

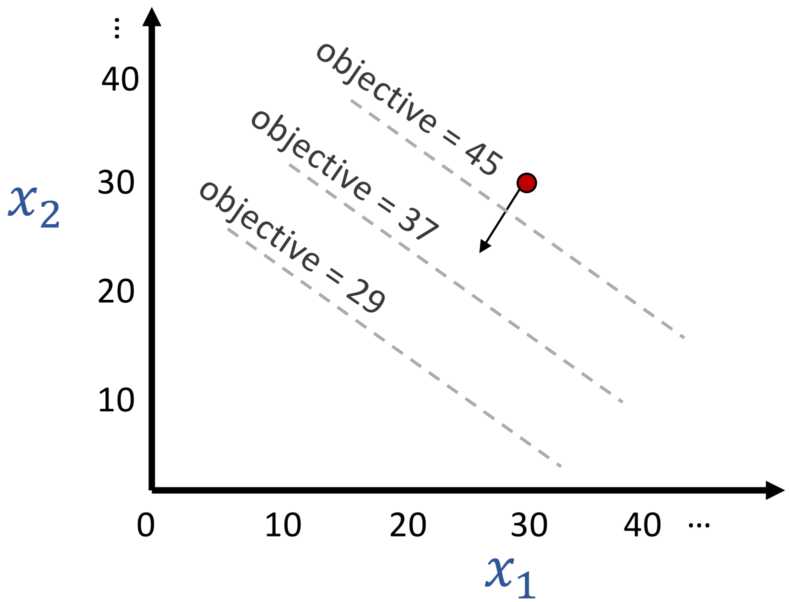
Third option actually results in max decrease

∴Keep going along this line to keep reducing cost

"Linear" objective

Now let's look at our objective

$\min.\ 1\ x_1 + 0.5\ x_2$



- Moving down by $\frac{1}{\sqrt{1^2+0.5^2}}$ and left by $\frac{0.5}{\sqrt{1^2+0.5^2}}$
- Consider direction -[1, 0.5]
- More generally for objective $c^T x$, direction -c
- Contours of objective are perpendicular to it
- Want to find the point in the feasible set that is as far as possible in that direction

Putting it back together

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad 100\,x_1 + 50\,x_2 \geq 2000$$

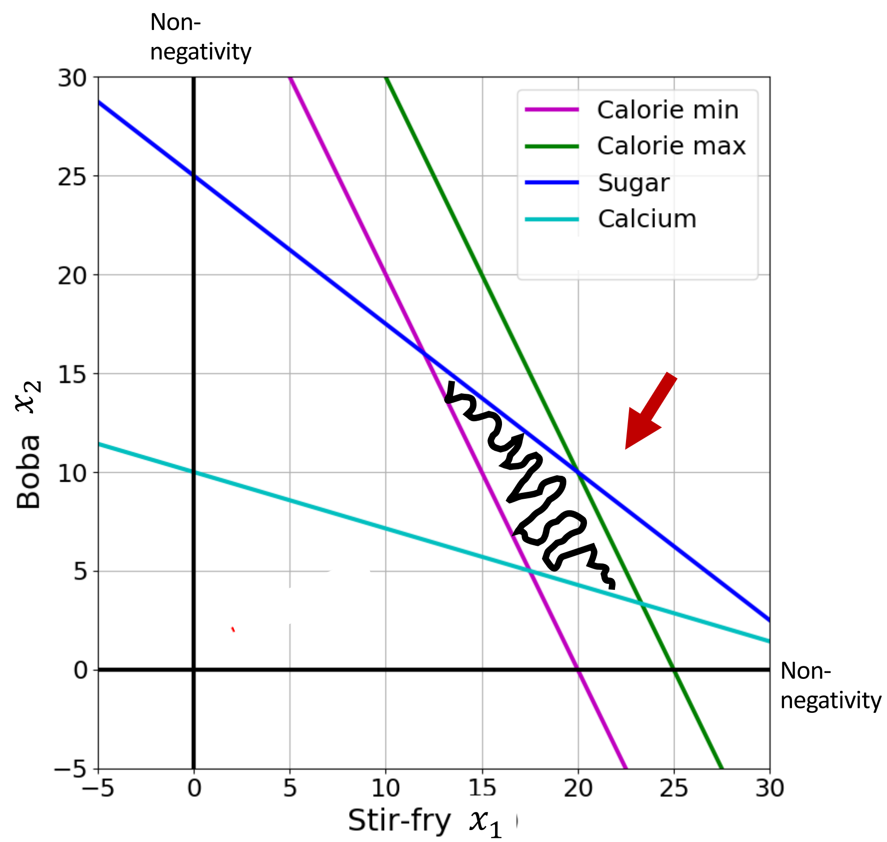
$$100\,x_1 + 50\,x_2 \leq 2500$$

$$3\,x_1 + 4\,x_2 \leq 100$$

$$20\,x_1 + 70\,x_2 \geq 700$$

$$x_1 \geq 0$$

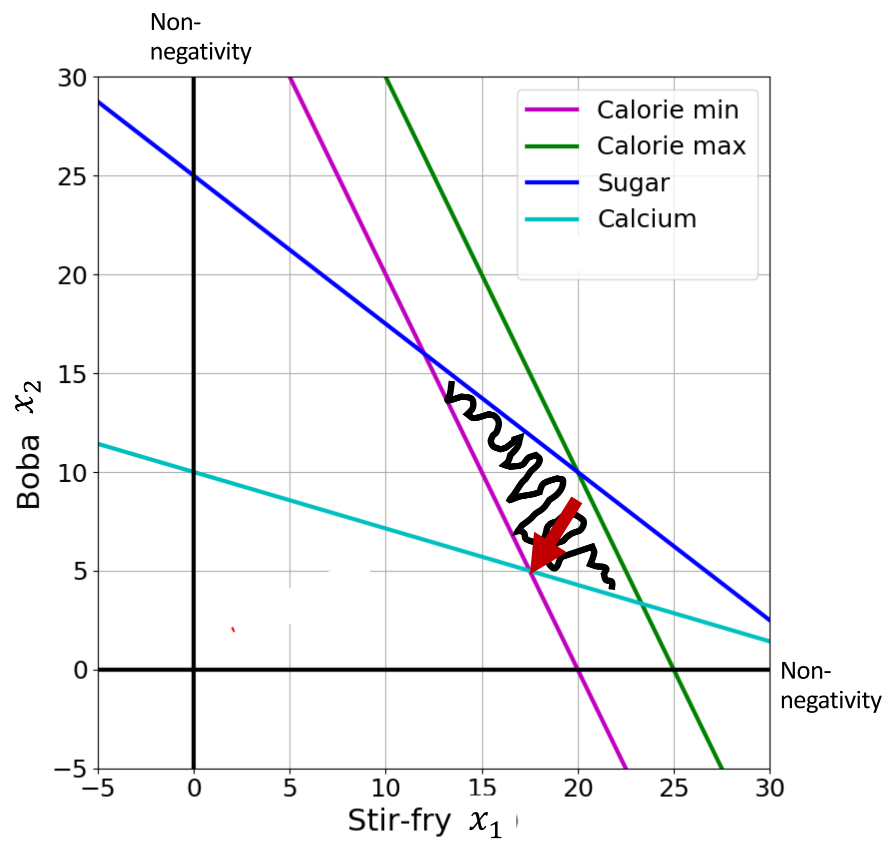
$$x_2 \geq 0$$



More generally

$$\min_{x_1, x_2} \quad c_1\, x_1 + c_2\, x_2$$

$$\begin{aligned}
\text{s.t.} \quad & a_{1,1}\, x_1 + a_{1,2}\, x_2 \le b_1 \\
& a_{2,1}\, x_1 + a_{2,2}\, x_2 \le b_2 \\
& a_{3,1}\, x_1 + a_{3,2}\, x_2 \le b_3 \\
& a_{4,1}\, x_1 + a_{4,2}\, x_2 \le b_4 \\
& a_{5,1}\, x_1 + a_{5,2}\, x_2 \le b_5 \\
& a_{6,1}\, x_1 + a_{6,2}\, x_2 \le b_6
\end{aligned}$$



Even more generally, a linear program is...

$$\min_{\boldsymbol{x}}. \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Stir-fry, Boba

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \\ 0 \\ 0 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium
Non-negativity

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Question 1

What has to increase to add more nutrition constraints?

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \preceq \boldsymbol{b} \end{aligned}$$

Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

C) height A

D) width A

E) length $\boldsymbol{b}$

Question 2

What has to increase to add more menu items?

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

C) height A

D) width A

E) length $\boldsymbol{b}$

Linear Programming

Different representations

Inequality form

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

General form

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}$$
$$\text{s.t.} \quad G\boldsymbol{x} \preceq \boldsymbol{h}$$
$$A\boldsymbol{x} = \boldsymbol{b}$$

Standard form

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t.} \quad \mathrm{A}\boldsymbol{x} = \boldsymbol{b}$$
$$\boldsymbol{x} \succeq 0$$

Can switch between representations!

E.g, $A\boldsymbol{x} = \boldsymbol{b}$ **can be written as** $\begin{bmatrix} \boldsymbol{A} \\ -\boldsymbol{A} \end{bmatrix} \boldsymbol{x} \preceq \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix}$

Optimization: General form

Optimization: General form

Given functions $f: R^d \to R$, $g : R^d \to R^m$

$$\underset{x \in \mathbf{R}^d}{\text{minimize}} \quad f(x)$$

$$\text{subject to} \quad g(x) \preceq 0$$

General form

$$\underset{x \in \mathbf{R}^d}{\text{minimize}} \quad f(x)$$

$$\text{subject to} \quad g(x) \preceq 0$$

Linear program

$f(x) = c^T x$ for some $c \in \mathbf{R}^d$

$g(x) = Ax - b$ for some matrix A and vector b

$$\underset{x \in \mathbf{R}^d}{\text{minimize}} \quad c^T x$$

$$\text{subject to} \quad Ax \preceq b$$

Special case: Constraint Satisfaction Problems

Is there any x which satisfies the constraints?

E.g., map coloring problem

Find any $\boldsymbol{x}$ s.t. $\boldsymbol{x}$ satisfies constraints

$$\underset{x \in \mathbf{R}^d}{\text{minimize}} \quad 1$$

$$\text{subject to} \quad g(x) \preceq 0$$

Poll

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \preceq \boldsymbol{b}$$

If $A \in \mathbb{R}^{M \times N}$, which of the following also equals N?

Select all that apply

A) length $\boldsymbol{x}$

B) length $\boldsymbol{c}$

C) length $\boldsymbol{b}$