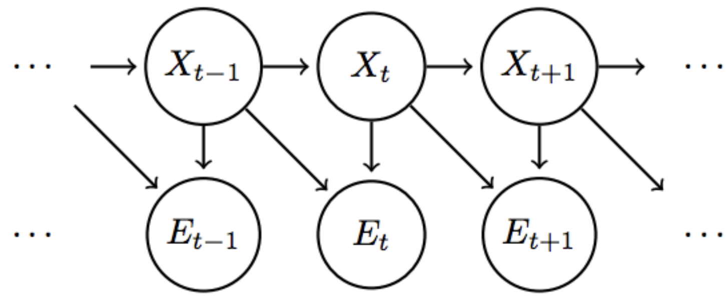
1 HMMs: Warmup

1. What are the three components of a hidden markov model? What makes it ”hidden”?

2. Write an expression for the joint distribution of a hidden markov model consisting of states $X_0, \ldots, X_n$ and evidence variables $E_1, \ldots, E_N$. How does the expression reflect the underlying structure of the model?

3. For each of the following descriptions in English of an inference task, write the corresponding probability expression:
   - Draw conclusions about our current underlying state given evidence up to the current time step
   - Draw conclusions about our future underlying state given evidence up to the current time step
   - Draw conclusions about a past underlying state given evidence up to the current time step
   - Draw conclusions about the sequence of underlying states given evidence up to the current time step
   - Draw conclusions about the most likely sequence of underlying states given evidence up to the current time step

4. Hidden Markov Models can be extended in a number of ways to incorporate additional relations. Since the independence assumptions are different in these extended Hidden Markov Models, the forward algorithm updates will also be different. What is the forward algorithm updates for the extended Hidden Markov Models specified by the following Bayes net?



2 HMMs: Tracking a Jabberwock

You have been put in charge of a Jabberwock for your friend Lewis. The Jabberwock is kept in a large tugley wood which is conveniently divided into an $N \times N$ grid. It wanders freely around the N^2 possible cells. At each time step $t = 1, 2, 3, \ldots$, the Jabberwock is in some cell $X_t \in \{1, \ldots, N\}^2$, and it moves to cell X_{t+1} randomly as follows: with probability $1 - \epsilon$, it chooses one of the (up to 4) valid neighboring cells uniformly at random; with probability ϵ, it uses its magical powers to teleport to a random cell uniformly at random among the N^2 possibilities (it might teleport to the same cell). Suppose $\epsilon = \frac{1}{2}$, $N = 10$ and that the Jabberwock always starts in $X_1 = (1, 1)$.

(a) Compute the probability that the Jabberwock will be in $X_2 = (2, 1)$ at time step 2. What about $P(X_2 = (4, 4))$?

At each time step t, you don't see X_t but see E_t, which is the row that the Jabberwock is in; that is, if $X_t = (r, c)$, then $E_t = r$. You still know that $X_1 = (1, 1)$.

(b) Suppose we see that $E_1 = 1$, $E_2 = 2$. Fill in the following table with the distribution over X_t after each time step, taking into consideration the evidence. Your answer should be concise. Hint: you should not need to do any heavy calculations.

<table>
<tr><th>t</th><th>$P(X_t \mid e_{1:t-1}, X_1 = (1,1))$</th><th>$P(X_t \mid e_{1:t}, X_1 = (1,1))$</th></tr>
<tr><td>1</td><td><table><tr><th>$\boldsymbol{X_1}$</th><th>$\boldsymbol{P(X_1)}$</th></tr><tr><td>(1, 1)</td><td></td></tr><tr><td>all other values</td><td></td></tr></table></td><td><table><tr><th>$\boldsymbol{X_1}$</th><th>$\boldsymbol{P(X_1)}$</th></tr><tr><td>(1, 1)</td><td></td></tr><tr><td>all other values</td><td></td></tr></table></td></tr>
<tr><td>2</td><td><table><tr><th>$\boldsymbol{X_2}$</th><th>$\boldsymbol{P(X_2 \mid e_1, X_1 = (1,1))}$</th></tr><tr><td>(1, 2)</td><td></td></tr><tr><td>(2, 1)</td><td></td></tr><tr><td>all other values</td><td></td></tr></table></td><td><table><tr><th>$\boldsymbol{X_2}$</th><th>$\boldsymbol{P(X_2 \mid e_{1:2}, X_1 = (1,1))}$</th></tr><tr><td>(2, 1)</td><td></td></tr><tr><td>$(2, a)$ $(\forall a, a > 1)$</td><td></td></tr><tr><td>all other values</td><td></td></tr></table></td></tr>
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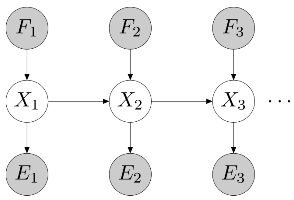
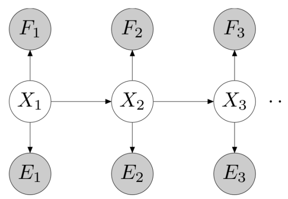
You are a bit unsatisfied that you can't pinpoint the Jabberwock exactly. But then you remembered Lewis told you that the Jabberwock teleports only because it is frumious on that time step, and it becomes frumious independently of anything else. Let us introduce a variable $F_t \in \{0, 1\}$ to denote whether it will teleport at time t. We want to to add these frumious variables to the HMM. Consider the two candidates:



(A)



(B)

(A)	(B)
$X_1 \perp\!\!\!\perp X_3 \mid X_2$	$X_1 \perp\!\!\!\perp X_3 \mid X_2$
$X_1 \perp\!\!\!\perp E_2 \mid X_2$	$X_1 \perp\!\!\!\perp E_2 \mid X_2$
$X_1 \perp\!\!\!\perp F_2 \mid X_2$	$X_1 \perp\!\!\!\perp F_2 \mid X_2$
$X_1 \perp\!\!\!\perp E_4 \mid X_2$	$X_1 \perp\!\!\!\perp E_4 \mid X_2$
$X_1 \perp\!\!\!\perp F_4 \mid X_2$	$X_1 \perp\!\!\!\perp F_4 \mid X_2$
$E_3 \perp\!\!\!\perp F_3 \mid X_3$	$E_3 \perp\!\!\!\perp F_3 \mid X_3$
$E_1 \perp\!\!\!\perp F_2 \mid X_2$	$E_1 \perp\!\!\!\perp F_2 \mid X_2$
$E_1 \perp\!\!\!\perp F_2 \mid E_2$	$E_1 \perp\!\!\!\perp F_2 \mid E_2$

(c) For each model, circle the conditional independence assumptions above which are true in that model.

(d) Which Bayes net is more appropriate for the problem domain here, (A) or (B)? Justify your answer.

For the following questions, your answers should be fully general for models of the structure shown above, not specific to the teleporting Jabberwock.

(e) For (A), express $P(X_{t+1}, e_{1:t+1}, f_{1:t+1})$ in terms of $P(X_t, e_{1:t}, f_{1:t})$ and the conditional probability tables used to define the network. Assume the E and F nodes are all observed.

(f) For (B), express $P(X_{t+1}, e_{1:t+1}, f_{1:t+1})$ in terms of $P(X_t, e_{1:t}, f_{1:t})$ and the CPTs used to define the network. Assume the E and F nodes are all observed.

Suppose that we don't actually observe the F_ts.

(g) For (A), express $P(X_{t+1}, e_{1:t+1})$ in terms of $P(X_t, e_{1:t})$ and the CPTs used to define the network.

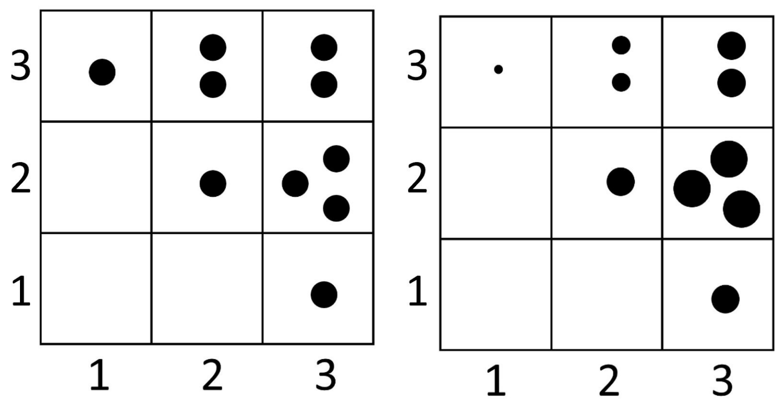
(h) For (B), express $P(X_{t+1}, e_{1:t+1})$ in terms of $P(X_t, e_{1:t})$ and the CPTs used to define the network.

1 Particle Filtering: Warmup

(a) **True / False:** The particle filtering algorithm is consistent since it gives correct probabilities as the number of samples N tends to infinity.

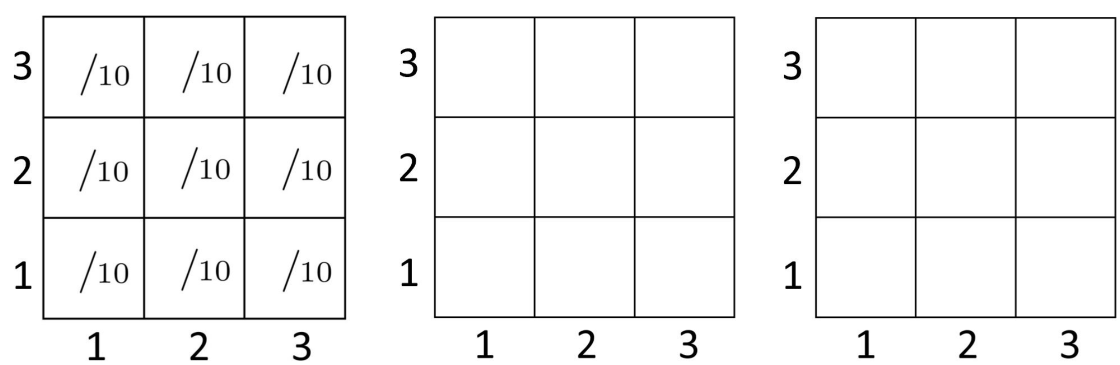
(b) **True / False:** The number of samples we use in the particle filtering algorithm increases from one time step to the next.

(c) The following state space contains 10 particles. The left grid shows the prior belief distribution of the particles at time t, while the grid on the right shows the particles weighted by the observations $P(e_t|S_t)$.



State	Weight
(1,3)	0.1
(2,2)	0.4
(2,3)	0.2
(3,1)	0.4
(3,2)	0.9
(3,3)	0.4

Fill in the following grids to update the belief distribution. Each square in the "Belief" grid should correspond to $\hat{P}(S_t|e_{1:t-1})$, the estimated probability of a particle being in state S at time t. Each square in the "Unnormalized" grid should correspond to the probability $P(S_t, e_t|e_{1:t-1})$. The "Normalized" grid should contain our updated belief distribution $\hat{P}(S_t|e_t, e_{1:t-1})$.



Belief Unnormalized Normalized

2 Tracking the Jabberwock

Lewis' Jabberwock is in the wild: its position is in a two-dimensional discrete grid, but this time the grid is not bounded. In other words, the position of the Jabberwock is a pair of integers $z = (x, y) \in \mathbb{Z}^2 = \{\dots, -2, -1, 0, 1, 2, \dots\} \times \{\cdots, -2, -1, 0, 1, 2, \cdots\}$. At each time step $t = 1, 2, 3, \dots$, the Jabberwock is in some cell $Z_t = z \in \mathbb{Z}^2$, and it moves to cell Z_{t+1} randomly as follows: with probability 1/2, it stays where it is; otherwise, it chooses one of its four neighboring cells uniformly at random (fortunately, no teleportation is allowed this week!).

(a) Write a function for the transition probability $P(Z_{t+1} = (x', y') | Z_t = (x, y))$.

We will use the particle filtering algorithm to track the Jabberwock. As a source of randomness use values in order from the following sequence $\{a_i\}_{1 \leq i \leq 14}$. Use these values to sample from any discrete distribution of the form $P(X)$ where X takes values in $\{1, 2, \dots, N\}$. Given $a_i \sim U[0, 1]$, return j such that $\sum_{k=1}^{j-1} P(X = k) \leq a_i < \sum_{k=1}^{j} P(X = k)$. You have to fix an ordering of the elements for this procedure to make sense.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
0.142	0.522	0.916	0.792	0.703	0.231	0.036	0.859	0.677	0.221	0.156	0.249

At each time step t you get an observation of the x coordinate R_t in which the Jabberwock sits, but it is a noisy observation. Given the true position $Z_t = (x, y)$, you observe the correct value according to the following probability:

$$P(R_t = r | Z_t = (x, y)) \propto (0.5)^{|x-r|}$$

(b) Suppose that you know that half of the time, the Jabberwock starts at $z_1 = (0, 0)$, and the other half, at $z_1 = (1, 1)$. You get the following observations: $R_1 = 1, R_2 = 0, R_3 = 1$. Fill out the table for each time step using a particle filter with 2 particles to compute an approximation to $P(Z_1, Z_2, Z_3 | r_1, r_2, r_3)$. Sample transitions from the table below using the a_i's as our source of randomness. The a_i's you should use for each step habe been indicated in the last row of each table. Note that going "left" decrements the x-coordinate by 1, and going "down" decrements the y-coordinate by 1.

[0; 0.5)	Stay
[0.5; 0.625)	Up
[0.625; 0.75)	Left
[0.75; 0.875)	Right
[0.875; 1)	Down

Initial	Belief $\hat{P}(z_1)$	Weights $P(r_1\|z_1)$	Unnormalized $\hat{P}(z_1, r_1)$	Normalized $\hat{P}(z_1\|r_1)$	Resampling
$p_1 =$ (0,0) $p_2 =$ (1,1) a_1, a_2	1/2 1/2				$p_1 =$ (,) $p_2 =$ (,) a_3, a_4

Transition $P(z_2\|z_1)$	Belief $\hat{P}(z_2\|r_1)$	Weights $P(r_2\|z_2)$	Unnormalized $\hat{P}(z_2, r_2\|r_1)$	Normalized $\hat{P}(z_2\|r_1, r_2)$	Resampling
$p_2 =$ (,) $p_2 =$ (,) a_5, a_6					$p_1 =$ (,) $p_2 =$ (,) a_7, a_8

Transition $P(z_3\|z_2)$	Belief $\hat{P}(z_3\|r_1, r_2)$	Weights $P(r_3\|z_3)$	Unnormalized $\hat{P}(z_3, r_3\|r_1, r_2)$	Normalized $\hat{P}(z_3\|r_1, r_2, r_3)$	Resampling
$p_1 =$ (,) $p_2 =$ (,) a_9, a_{10}					$p_1 =$ (,) $p_2 =$ (,) a_{11}, a_{12}

(d) Use your samples (the unweighted particles in the last step) to evaluate the posterior probability that the x-coordinate of Z_3 is different than the column of Z_3, i.e. $X_3 \neq Y_3$.

(e) What is the problem of using the elimination algorithm instead of a particle filter for tracking Jabberwock?