1 CSP as IP

Alice, Bob, and Charles want to study in Gates, which has 9 floors. To maximize productivity, we need to assign each of them to a separate floor without violating the following constraints:

- Alice only has access to floors 4 through 8
- Bob only has access to floors 3 through 7
- Charles must be at least 2 floors higher than Bob
- Alice must be at least 1 floor higher than Bob
- Alice must be at least 1 floor lower than Charles

Our goal is to assign Alice, Bob, and Charles to the highest possible floors.

1. Formulate the problem as a CSP.

   Variables: X_A, X_B, X_C (where X_A is Alice's floor number, X_B is Bob's floor number, and X_C is Charles's floor number)

   Domains:

   - $D_A \in \{4, 5, 6, 7, 8\}$
   - $D_B \in \{3, 4, 5, 6, 7\}$
   - $D_C \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

   Constraints:

   - $X_C \geq 2 + X_B$ (Charles must be at least 2 floors higher than Bob)
   - $X_A \geq 1 + X_B$ (Alice must be at least 1 floor higher than Bob)
   - $X_A \leq X_C - 1$ (Alice must be at least 1 floor lower than Charles)

2. Formulate the problem as an IP problem.

   Variables: X_A, X_B, X_C (where X_A is Alice's floor number, X_B is Bob's floor number, and X_C is Charles's floor number).

   Goal: We want to find the $max_{X_A,X_B,X_C} X_A + X_B + X_C$. This is equivalent to saying we want to find $min_{X_A,X_B,X_C}(-X_A - X_B - X_C)$

   Constraints:

   - Alice only has access to floors 4 through 8

   $$X_A \geq 4, \text{ which is equivalent to } -X_A \leq -4$$
   $$X_A \leq 8$$

   - Bob only has access to floors 3 through 7

   $$X_B \geq 3, \text{ which is equivalent to } -X_B \leq -3$$
   $$X_B \leq 7$$

- Charles only has access to floors 1 through 9 (Note: this constraint was not explicitly given, but Gates has 9 floors, and we need to make sure that Charles is assigned to a floor between 1 and 9, inclusive)

$$X_C \geq 1, \text{ which is equivalent to } -X_C \leq -1$$
$$X_C \leq 9$$

- Charles must be at least 2 floors higher than Bob

$$X_C \geq 2 + X_B, \text{ which is equivalent to } X_B - X_C \leq -2$$

- Alice must be at least 1 floor higher than Bob

$$X_A \geq 1 + X_B, \text{ which is equivalent to } X_B - X_A \leq -1$$

- Alice must be at least 1 floor lower than Charles

$$X_A \leq X_C - 1, \text{ which is equivalent to } X_A - X_C \leq -1$$

Therefore, we want to find $min_{X_A,X_B,X_C}(-X_A - X_B - X_C)$ such that

$$-X_A \leq -4$$
$$X_A \leq 8$$

$$-X_B \leq -3$$
$$X_B \leq 7$$

$$-X_C \leq -1$$
$$X_C \leq 9$$

$$X_B - X_C \leq -2$$
$$X_B - X_A \leq -1$$
$$X_A - X_C \leq -1$$

2 Baymax's Factory

Baymax and the 281 TAs have opened a factory to produce special medicine and bandages. These are really difficult to produce and require the collaboration of robots and humans.

To produce an ounce of medicine, it takes 0.2 hours of human labor and 4 hours of robot labor. To produce an inch of bandage, it takes 0.5 hours of human labor and 2 hours of robot labor. An ounce of medicine sells for $30 and an inch of bandages sells for $30. Medicine and bandages can be sold in fractions of an ounce or inch.

We want to maximize our profit so we can buy gifts for all the students. However, the TAs are really busy so they can only devote 90 human hours. In addition, Baymax can only devote 800 robot hours because he has other obligations to tend to. How can we maximize our profit?

1. Is this a linear, mixed or integer programming problem? Formulate and solve it.

   It is a linear programming problem, as the medicine and bandages can be sold a fraction of a unit. Let x be the ounces of medicine and y be the inches of bandages produced.

   Objective: Maximize total profit:

   $$\min_{x,y} -30x - 30y$$

   Constraints:

   $$0.2x + 0.5y \leq 90$$
   $$4x + 2y \leq 800$$
   $$x \geq 0, y \geq 0$$

   Putting this problem in inequality form we have:

   $$\min_{\mathbf{x}} \mathbf{c}^\top \mathbf{x} \; s.t. \; A\mathbf{x} \leq \mathbf{b}$$

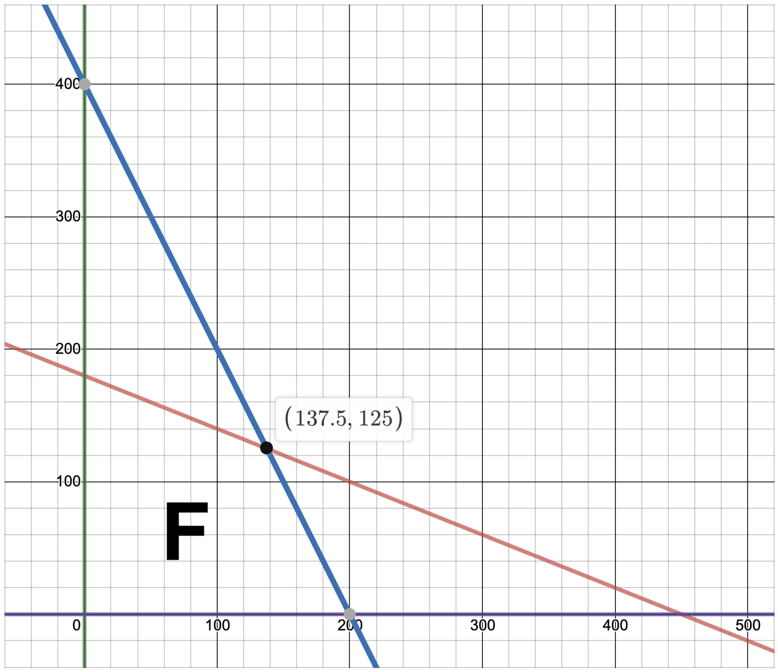
   where

   $$\mathbf{c} = [-30 \;\; -30]^\top$$
   $$\mathbf{x} = [x \; y]^\top$$
   $$\mathbf{b} = [0 \; 0 \; 90 \; 800]$$
   $$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0.2 & 0.5 \\ 4 & 2 \end{bmatrix}$$

   Given the constraints, we can solve for x and y:

   $$y = 180 - 0.4x$$
   $$y = 400 - 2x$$

   That gives us the following graph:



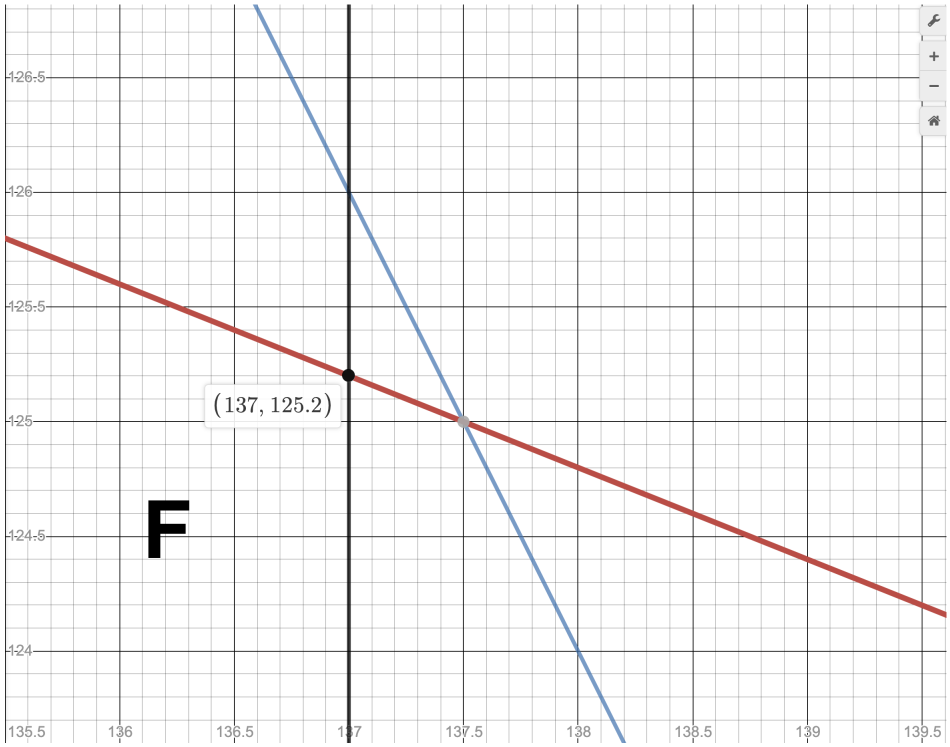
Since we want to maximize profit, we want to choose the furthest point, giving us $x = 137.5$ and $y = 125$.

2. Now suppose the items can only be sold in whole units (by ounce/inch). Is this a linear, mixed, or integer programming problem? Perform branch and bound for one branch level. You do not have to evaluate; writing out the constraints will suffice.

   This is an integer programming problem, and the formulation is identical to part (a). However, the domains of x and y are reduced to integers. We can solve the problem with branch and bound.

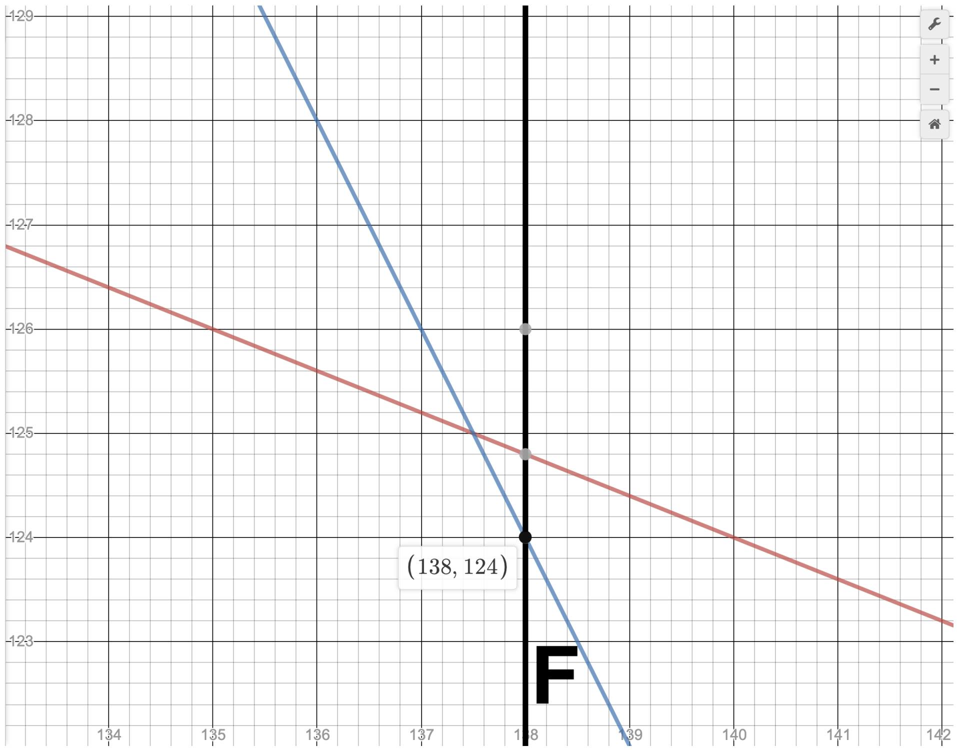
   We first use linear programming to find the optimal point of (137.5, 125), as we did in (1a). Since $x = 137.5$ is not an integer, we branch on it by adding the constraints that $x \leq 137$ or $x \geq 138$.

   Left branch:
   $x \leq 137$
   $0.2x + 0.5y \leq 90$
   $4x + 2y \leq 800$
   $x \geq 0, y \geq 0$



The linear programming solution is (137, 125.2), so we add the left branch to the pq with value -7866. Next we consider the right branch.

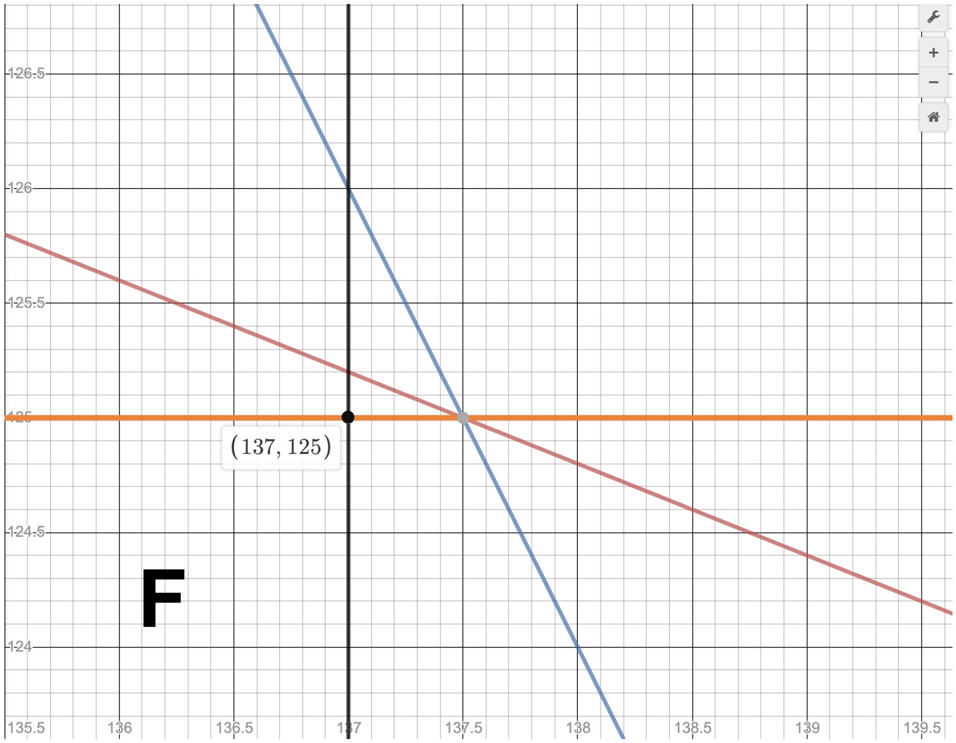
Right branch:
$x \geq 138$
$0.2x + 0.5y \leq 90$
$4x + 2y \leq 800$
$x \geq 0, y \geq 0$



the linear programming solution (138, 124) has a value of -7860 and we similarly add it to the pq.

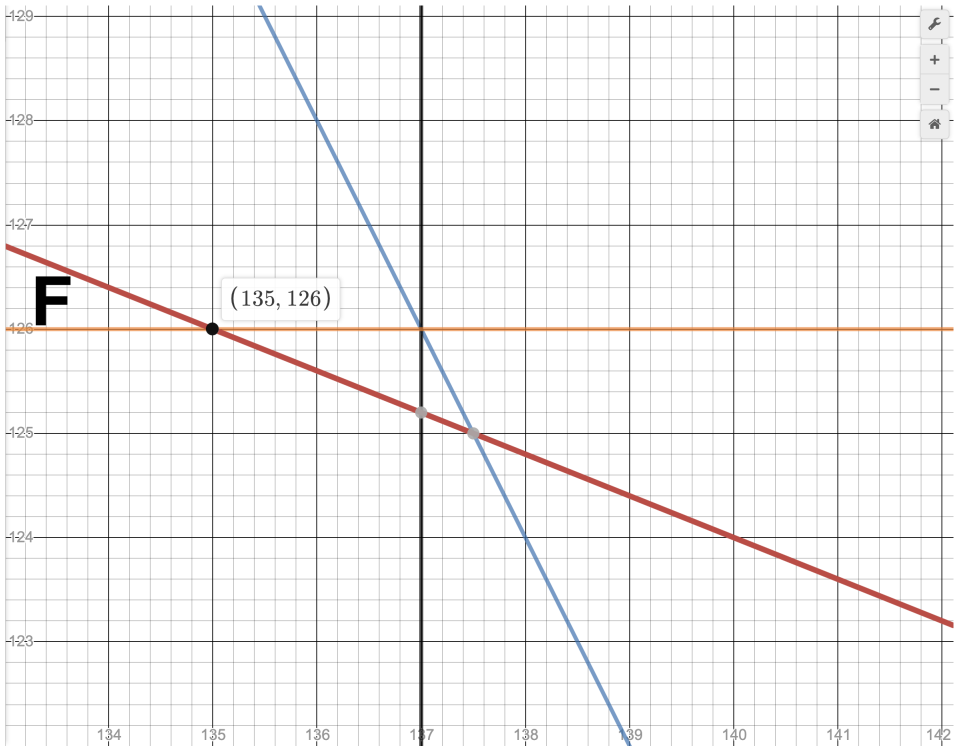
The left branch has a better value so it is popped of the priority queue first and then since the left branch solution is not an integer value the left-right and left-left branches are considered. Specifically we now need to branch on the y value by adding the constraints that $y \leq 125$ or $y \geq 126$.

Left-Left branch:
$x \leq 137$
$0.2x + 0.5y \leq 90$
$4x + 2y \leq 800$
$x \geq 0, y \geq 0$
$y \leq 125$



The linear programming solution is (137, 125) and it is added to the priority queue along with the value -7860.

Left-Right branch:
$x \leq 137$
$0.2x + 0.5y \leq 90$
$4x + 2y \leq 800$
$x \geq 0, y \geq 0$
$y \geq 126$



The linear programming solution is (135, 126) and it is added to the pq along with the value -7830.

Next on the pq there is a tie between right and left-left branches. Both solutions are integer so the one that is popped of the queue next will be returned as our solution with objective -7860 and point (138, 124) or (137, 125).

3. Now assume medicine can be sold in fractions but bandages can only be sold in whole units. What kind of a programming problem would this be, and how would our evaluation process differ from the problem type in part b?

   This will be a mixed integer linear programming problem. We will evaluate by only branching and bounding on the number of bandages.

4. How many optimal solutions can a LP have? How about IP?

   Both LP and IP can have an infinite number of optimal solutions. Imagine a cost vector that's perpendicular to a constraint boundary. Then, we could have that constraint boundary cross infinitely many integers/real numbers (i.e. the line x = 0).

3 4-Queens

Recall the 4-Queens problem. The goal is to place 4 chess queens on a 4x4 chess board such at no two queens are in the same row, column and diagonal.

Formulate the 4-Queens problem as an integer programming problem.

Let our variables be x_{ij} for $0 \leq i \leq 3$, $0 \leq j \leq 3$, representing whether there is a queen in row i, column j. We want to find $\max_x \sum_i \sum_j x_{ij}$ such that $x_{ij} \in \{0, 1\}$.

Check: only one queen in each row - fix i and iterate over each column, ensuring they sum up to ≤ 1.

$$\sum_j x_{ij} = 1 \forall i \in \{0, 3\}$$

Check: only one queen in each column: fix j and iterate over each row, ensuring they sum up to ≤ 1.

$$\sum_i x_{ij} = 1 \forall j \in \{0, 3\}$$

Check: at most one queen in positive-slope diagonals (stretching from top left to bottom right):

$$\sum_{i,j:i+j=k} x_{ij} \leq 1, \forall k \in \{0, 1, 2, ..., 6\}$$

(k=0: (0, 0) | k=1: (0,1), (1,0) | k=2: (0,2), (1,1), (2,0) | k=3 ...)

Check: at most one queen in negative-slope diagonals (stretching from bottom left to top right):

$$\sum_{i,j:i-j=k} x_{ij} \leq 1, \forall k \in \{-3, -2, -1, ..., 3\}$$

Note that the equalities should all be represented as inequalities ≤ 1 and the negation of it ≤ -1. Putting this problem in standard form we have:

$$\min_{\mathbf{x}} 0 \ s.t. \ A\mathbf{x} = \mathbf{b}$$

where

$$\mathbf{x} = [x_{00} \ x_{01} \ x_{02} \ x_{03} \ x_{10} \ x_{11} \ x_{12} \ x_{13} \ x_{20} \ x_{21} \ x_{22} \ x_{23} \ x_{30} \ x_{31} \ x_{32} \ x_{33}]^\top$$

$$\mathbf{b} = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]$$

$$A = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Where the first 4 rows of A are row restrictions, the next 4 are column restrictions, and the last 7 are diagonal restrictions.