1 Definitions

1. Conditional Probability: $P(A \mid B) =$
2. Product Rule: $P(A, B) =$
3. Bayes' Theorem: $P(A \mid B) =$
4. Normalization: $P(A \mid B) =$
5. Chain Rule: $P(A, B, C) =$
6. Law of Total Probability: [using only $P(B)$ and $P(A \mid B)$] $P(A) =$
7. Independence: A, B independent, $P(A, B) =$
8. Conditional Independence: If A and B are conditionally independent given C, then $P(A, B \mid C) =$

2 Warm Up

(a) State the two ways to write the chain rule (conditional probability decomposition) for $P(A, B)$

(b) Rearrange the above equation to find $P(A \mid B)$

(c) Find $P(a)$ in terms of the joint $P(a, b)$ for any $a \in A, b \in B$ (Hint: these are specific values, answer should include a sum)

(d) Find $P(b \mid a)$ in terms of the joint $P(a, b)$ for any $a \in A, b \in B$

(e) Find $P(b \mid a)$ in terms of the distributions $P(b)$, $P(a \mid b)$, for any $a \in A, b \in B$

(f) Assume we had some fixed a and wanted to find each element of $P(b \mid a)$ (i.e. wanted to find $P(B \mid a)$). Would the numerator of the fraction in the previous question change for each value of b? What about the denominator? How could you use this to do the calculation with less steps?

(g) Assume A is a random variable that can take 3 values, B is a random variable that can take 2 values, and C is a random variable that can take 1 value. What do the following probability tables sum to?

   (a) $P(A \mid b)$
   (b) $P(A \mid C)$
   (c) $P(C \mid B)$
   (d) $P(B \mid a)$
   (e) $P(B \mid A)$

3 Cake

Consider the above cake with 12 slices. Let s_1 indicate a slice with no sprinkles and s_2 be a slice with sprinkles. Let c_1 indicate a slice with no candles and c_2 be a slice with candles. Let S be a random variable indicating sprinkles and C be a random variable indicating candles. Calculate the following probabilities.

1. $P(C = c_1)$
2. $P(S = s_1, C = c_2)$
3. $P(C = c_2 \mid S = s_1)$
4. $\sum_{s\in\{s_1,s_2\}} \sum_{c\in\{c_1,c_2\}} P(s, c)$
5. $\sum_{c\in\{c_1,c_2\}} \sum_{s\in\{s_1,s_2\}} P(s \mid c)$
6. $\sum_{s\in\{s_1,s_2\}} \sum_{c\in\{c_1,c_2\}} P(c \mid s)$

4 Queries on a Large Joint Distribution

Consider binary (two outcomes) random variables A, B, C, D, and the following joint distribution table of all four variables.

A	B	C	D	$P(A, B, C, D)$
$+a$	$+b$	$+c$	$+d$	12/64
$+a$	$+b$	$+c$	$-d$	4/64
$+a$	$+b$	$-c$	$+d$	2/64
$+a$	$+b$	$-c$	$-d$	2/64
$+a$	$-b$	$+c$	$+d$	8/64
$+a$	$-b$	$+c$	$-d$	4/64
$+a$	$-b$	$-c$	$+d$	2/64
$+a$	$-b$	$-c$	$-d$	4/64
$-a$	$+b$	$+c$	$+d$	6/64
$-a$	$+b$	$+c$	$-d$	3/64
$-a$	$+b$	$-c$	$+d$	4/64
$-a$	$+b$	$-c$	$-d$	6/64
$-a$	$-b$	$+c$	$+d$	2/64
$-a$	$-b$	$+c$	$-d$	1/64
$-a$	$-b$	$-c$	$+d$	3/64
$-a$	$-b$	$-c$	$-d$	1/64

1. Calculate the following probabilities:
   (a) $P(+c)$
   (b) $P(+a, -b)$
   (c) $P(-b \mid +a)$
   (d) $P(-a, +b, -d)$
   (e) $P(+c \mid -a, +b, -d)$
   (f) $P(+c, +d \mid +a, +b)$
2. What value do the following probability tables sum to?
   (a) $P(B)$
   (b) $P(+b \mid C, +d)$
   (c) $P(C, D \mid +a, +b)$