1: Operators of Dynamic Logic for While Programs

DL	Operator	Meaning
$e = \tilde{e}$	equals	true iff values of terms e and $\tilde{e}$ are equal
$e \geq \tilde{e}$	greater-or-equal	true iff value of e greater-or-equal to value of $\tilde{e}$
$p(e_1,\dots,e_k)$	predicate	true iff p holds for the value of $(e_1,\dots,e_k)$
$\neg P$	negation / not	true if P is false
$P \land Q$	conjunction / and	true if both P and Q are true
$P \lor Q$	disjunction / or	true if P is true or if Q is true
$P \to Q$	implication / implies	true if P is false or Q is true
$P \leftrightarrow Q$	bi-implication / equivalent	true if P and Q are both true or both false
$\forall x\, P$	universal quantifier / for all	true if P is true for all real values of variable x
$\exists x\, P$	existential quantifier / exists	true if P is true for some real value of variable x
$[\alpha]P$	$[\cdot]$ modality / box	true if P is true after all runs of HP α
$\langle\alpha\rangle P$	$\langle\cdot\rangle$ modality / diamond	true if P is true after at least one run of HP α

2: Statements and Effects of While Programs

Program	Operation	Effect
$x := e$	assignment	assigns value of term e to variable x
$?Q$	test	check truth of first-order formula Q at current state
$\alpha;\, \beta$	sequential composition	β starts after α finishes
$\texttt{if}(Q)\,\alpha\,\texttt{else}\,\beta$	if-then-else	run α if Q is true at current state else run β
$\texttt{while}(Q)\,\alpha$	while loop	repeats α as long as Q is true
α^*	nondeterministic repetition	repeats α n-times for any $n \in \mathbb{N}$

3: Semantics of Dynamic Logic formula P in interpretation I

$\omega \models e \geq \tilde{e}$	iff $\omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]$	
$\omega \models p(e_1,\dots,e_k)$	iff $(\omega[\![e_1]\!],\dots,\omega[\![e_k]\!]) \in I(p)$	where predicate symbol p is interpreted as relation $I(p)$
$\omega \models \neg P$	iff $\omega \not\models P$ that is, it is not the case that $\omega \models P$	
$\omega \models P \land Q$	iff $\omega \models P$ and $\omega \models Q$	
$\omega \models P \to Q$	iff $\omega \not\models P$ or $\omega \models Q$	
$\omega \models \exists x\, P$	iff $\nu \models P$ for some state ν with $\nu = \omega$ except for the value of x	
$\omega \models \forall x\, P$	iff $\nu \models P$ for all states ν with $\nu = \omega$ except for the value of x	
$\omega \models \langle\alpha\rangle P$	iff $\nu \models P$ for some state ν such that $(\omega,\nu) \in [\![\alpha]\!]$	
$\omega \models [\alpha]P$	iff $\nu \models P$ for all states ν such that $(\omega,\nu) \in [\![\alpha]\!]$	

4: Semantics of While Program α is relation $[\![\alpha]\!] \subseteq \mathcal{S} \times \mathcal{S}$ between initial and final states

$$[\![x := e]\!] = \{(\omega,\nu) : \nu = \omega \text{ except } \nu[\![x]\!] = \omega[\![e]\!]\}$$
$$[\![?Q]\!] = \{(\omega,\omega) : \omega \models Q\}$$
$$[\![\texttt{if}(Q)\,\alpha\,\texttt{else}\,\beta]\!] = \{(\omega,\nu) : (\omega \models Q \text{ and } (\omega,\nu) \in [\![\alpha]\!]) \text{ or } (\omega \not\models Q \text{ and } (\omega,\nu) \in [\![\beta]\!])\}$$
$$[\![\alpha;\beta]\!] = \{(\omega,\nu) : (\omega,\mu) \in [\![\alpha]\!], (\mu,\nu) \in [\![\beta]\!]\}$$
$$[\![\alpha^*]\!] = \{(\omega,\nu) : \text{there are } n{\in}\mathbb{N}, \mu_0{=}\omega, \mu_1, \mu_2, \dots, \mu_n{=}\nu \text{ such that } (\mu_i,\mu_{i+1}) \in [\![\alpha]\!] \text{ for all } 0{\leq}i{<}n\}$$
$$[\![\texttt{while}(Q)\,\alpha]\!] = \big\{(\omega,\nu) : \mu_0{=}\omega, \mu_1, \mu_2, \dots, \mu_n{=}\nu \text{ and } \mu_i \models Q, (\mu_i,\mu_{i+1}) \in [\![\alpha]\!] \text{ and } \mu_n \not\models Q \text{ for all } 0{\leq}i{<}n\big\}$$

5: Dynamic Logic axioms

$\langle\cdot\rangle$ $\langle\alpha\rangle P \leftrightarrow \neg[\alpha]\neg P$

$[:=]$ $[x := e]p(x) \leftrightarrow p(e)$

$[?]$ $[?Q]P \leftrightarrow (Q \to P)$

[if] $[\texttt{if}(Q)\,\alpha\,\texttt{else}\,\beta]P \leftrightarrow (Q \to [\alpha]P) \land (\neg Q \to [\beta]P)$

[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

[unwind] $[\texttt{while}(Q)\,\alpha]P \leftrightarrow [\texttt{if}(Q)\,\{\alpha;\texttt{while}(Q)\,\alpha\}]P$

K $[\alpha](P \to Q) \to ([\alpha]P \to [\alpha]Q)$

I $[\alpha^*]P \leftrightarrow P \land [\alpha^*](P \to [\alpha]P)$

C $[\alpha^*]\forall n{>}0\,(\varphi(n) \to \langle\alpha\rangle\varphi(n-1)) \to \forall n{\geq}0\,(\varphi(n) \to \langle\alpha^*\rangle\varphi(0))$ $(n \notin \alpha)$

V $p \to [\alpha]p$ $(FV(p) \cap BV(\alpha) = \emptyset)$

Det $\langle\alpha\rangle P \to [\alpha]P$ (α is deterministic)

6: First-order axioms

$\forall$i $(\forall x\, p(x)) \to p(e)$ (arbitrary term e)

$\forall\!\to$ $\forall x\,(P \to Q) \to (\forall x\, P \to \forall x\, Q)$

$V_\forall$ $p \to \forall x\, p$ $(x \notin FV(p))$

7: Dynamic Logic sequent calculus proof rules

loop $\dfrac{\Gamma \vdash J, \Delta \quad J \vdash [\alpha]J \quad J \vdash P}{\Gamma \vdash [\alpha^*]P, \Delta}$

while $\dfrac{\Gamma \vdash J, \Delta \quad J, Q \vdash [\alpha]J \quad J, \neg Q \vdash P}{\Gamma \vdash [\texttt{while}(Q)\,\alpha]P, \Delta}$

con $\dfrac{\Gamma \vdash \exists n{\geq}0\,\varphi(n), \Delta \quad \varphi(n), n{>}0 \vdash \langle\alpha\rangle\varphi(n-1) \quad \varphi(0) \vdash P}{\Gamma \vdash \langle\alpha^*\rangle P, \Delta}$ (n new)

var $\dfrac{\Gamma \vdash J, \Delta \quad J, Q, \varphi = n \vdash \langle\alpha\rangle(J \land \varphi < n) \quad J, Q \vdash \varphi \geq 0 \quad J, \neg Q \vdash P}{\Gamma \vdash \langle\texttt{while}(Q)\,\alpha\rangle P, \Delta}$ (n new)

$M_{[\cdot]}R$ $\dfrac{\Gamma \vdash [\alpha]Q, \Delta \quad Q \vdash P}{\Gamma \vdash [\alpha]P, \Delta}$

$M_{\langle\cdot\rangle}R$ $\dfrac{\Gamma \vdash \langle\alpha\rangle Q, \Delta \quad Q \vdash P}{\Gamma \vdash \langle\alpha\rangle P, \Delta}$

$M_{[\cdot]}L$ $\dfrac{\Gamma, [\alpha]Q \vdash \Delta \quad P \vdash Q}{\Gamma, [\alpha]P \vdash \Delta}$

$M_{\langle\cdot\rangle}L$ $\dfrac{\Gamma, \langle\alpha\rangle Q \vdash \Delta \quad P \vdash Q}{\Gamma, \langle\alpha\rangle P \vdash \Delta}$

G $\dfrac{P}{\Gamma \vdash [\alpha]P, \Delta}$

8: Propositional sequent calculus proof rules

$$\neg R\ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta} \qquad \wedge R\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee R\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \rightarrow R\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta}$$

$$\neg L\ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta} \qquad \wedge L\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee L\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \qquad \rightarrow L\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta}$$

$$\text{id}\ \frac{}{\Gamma, P \vdash P, \Delta} \qquad \top R\ \frac{}{\Gamma \vdash true, \Delta} \qquad WR\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash P, \Delta}$$

$$\text{cut}\ \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta} \qquad \bot L\ \frac{}{\Gamma, false \vdash \Delta} \qquad WL\ \frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta}$$

9: Quantifier sequent calculus proof rules

$$\forall R\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta}\ (y \notin \Gamma, \Delta, \forall x\, p(x)) \qquad \exists R\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}\ (\text{arbitrary term } e)$$

$$\forall L\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta}\ (\text{arbitrary term } e) \qquad \exists L\ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta}\ (y \notin \Gamma, \Delta, \exists x\, p(x))$$

$$\mathbb{Z}\ \frac{}{\Gamma \vdash \Delta}\quad (\textstyle\bigwedge_{A \in \Gamma} A \ \rightarrow\ \bigvee_{B \in \Delta} B \text{ is a valid formula of (linear) integer arithmetic})$$

10: Derived axioms

$[]\wedge$ $[\alpha](P \wedge Q) \leftrightarrow [\alpha]P \wedge [\alpha]Q$

[unfold] $[\mathtt{while}(Q)\,\alpha]P \leftrightarrow (Q \rightarrow [\alpha][\mathtt{while}(Q)\,\alpha]P) \wedge (\neg Q \rightarrow P)$

$K_{\langle\cdot\rangle}$ $[\alpha](P \rightarrow Q) \rightarrow (\langle\alpha\rangle P \rightarrow \langle\alpha\rangle Q)$

inv2var $[\alpha]J \rightarrow (\langle\alpha\rangle(J \rightarrow \varphi) \rightarrow \langle\alpha\rangle\varphi)$

11: Derived rules

$$[:=]_=\ \frac{\Gamma, y = e \vdash p(y), \Delta}{\Gamma \vdash [x := e]p(x), \Delta}\ (y \text{ new}) \qquad =R\ \frac{\Gamma, x = e \vdash p(e), \Delta}{\Gamma, x = e \vdash p(x), \Delta} \qquad =L\ \frac{\Gamma, x = e, p(e) \vdash \Delta}{\Gamma, x = e, p(x) \vdash \Delta}$$