Bug Catching: Automated Program Verification
15414/15614 Spring 2024
Lecture 23: Review

Ruben Martins

April 16, 2024

Resolution

Resolution rule

$$\frac{p \vee C \quad \neg p \vee D}{C \vee D} \bowtie_p$$

Exercise: Resolution

Show the following CNF formula is unsatisfiable by using resolution to derive the empty clause:

$\neg x_1 \lor \neg x_2$	C_1
$x_1 \lor x_2$	C_2
$\neg x_1 \lor \neg x_3$	C_3
$x_1 \lor x_3$	C_4
$\neg x_2 \lor \neg x_3$	C_5
$x_2 \lor x_3$	C_6

Saturation

Saturation

- When we are not able to reach a contradiction with resolution, then by necessity the sequence of clauses must reach *saturation*, that is, any further application of resolution will only lead to clauses already in the sequence.
- If we reach saturation without deducing a contradiction and we conclude the initial theory is satisfiable.

Exercise: Saturation

Show that the following CNF formula is satisfiable by saturation

$$\begin{array}{ll} \neg p \vee \neg q \vee r & C_0 \\ p & C_1 \\ \neg r & C_2 \end{array}$$

Reconstructing a Satisfying Assignment

Robinson's algorithm

Choose an ordering of the variables $p_0, p_1, \ldots, p_{n-1}$

$$
\begin{array}{lcll}
M_0 & = & \{\} & \\
M_{i+1} & = & M_i \cup \{p_i\} & \text{provided there is no } C \in S \text{ s.t. } C \subseteq \overline{M_i \cup \{p_i\}} \\
M_{i+1} & = & M_i \cup \{\neg p_i\} & \text{provided there is a } C \in S \text{ s.t. } C \subseteq \overline{M_i \cup \{p_i\}} \\
 & & & \\
M & = & M_n &
\end{array}
$$

Then $M \models S$, as proved by Robinson.

Exercise: Robinson's Algorithm

Reconstruct a satisfying assignment using Robinson's algorithm

$\neg p \lor \neg q \lor r$	C_0
p	C_1
$\neg r$	C_2
$\neg q \lor r$	$C_3 = C_1 \bowtie_p C_0$
$\neg q$	$C_4 = C_3 \bowtie_r C_2$
$\neg p \lor \neg q$	$C_5 = C_0 \bowtie_r C_2$

Tseitin Encoding

Tseitin Encoding

- Introduce fresh variables to encode subformulas
- Encode the meaning of these fresh variables with clauses
- Guarantees equisatisfiability with a linear increase in the size of the formula

Exercise: Tseitin Encoding

Use the Tseitin Encoding to transform the following propositional formula to CNF:
$\phi = (x \land \neg y) \lor (z \lor (x \land \neg w))$

Unary and Binary Representations

Unary and binary representations

1. Unary representation: a Boolean variable for each possible value
2. Binary representation: binary representation of an integer

Exercise: Representing Integer Variables in SAT

Suppose we want to encode the domain of an integer variable $X = 1, 2, 3$
Encode the domain of this integer variables using:

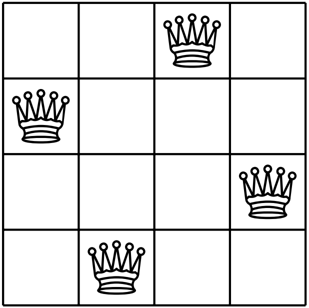
- Unary representation
- Binary representation

SAT Encodings

Variables and Constraints

1. Define the meaning of variables
2. Encode the constraints of the problem into CNF

Exercise: N-Queens



- There should be 4 queens on the board.
- Two queens should never be on the same line.
- Two queens should never be on the same column.
- Two queens should never be on the same diagonal.

Status of a Clause under Partial Interpretation

Status of a clause

Given a partial interpretation I, a clause is:

- Satisfied, if one or more of its literals is satisfied
- Conflicting, if all of its literals are assigned but not satisfied
- Unit, if it is not satisfied and all but one of its literals are assigned
- Unresolved, otherwise

Exercise: Status of Clauses

Given the partial interpretation $I = \{p_1, \neg p_2, p_4\}$ what is the status of the following clauses:

- $(p_1 \vee p_3 \vee \neg p_4)$
- $(\neg p_1 \vee p_2)$
- $(p_2 \vee \neg p_4 \vee p_3)$
- $(\neg p_1 \vee p_3 \vee p_5)$

Unit Propagation

Unit Propagation

- Identify unit clauses:
  - Unit clauses: clauses that have exactly one unassigned literal
- Satisfy the unassigned literal by assigning true if it is positive (l_i) and false if it is negative ($\neg l_i$)
- Repeat until fix point

DPLL algorithm

```
let rec dpll (f: formula) : bool =
  let fp = bcp f in
  match fp with
  | Some True -> true
  | Some False -> false
  | None ->
    begin
    let p = choose_var f in
    let ft = (subst_var f p true) in
    let ff = (subst_var f p false) in
    dpll ft || dpll ff
    end
```

Exercise: DPLL algorithm

Considering the following propositional formula in CNF:

$$
\begin{array}{ll}
\neg x_1 \lor x_2 \lor \neg x_3 & C_1 \\
\neg x_1 \lor \neg x_2 \lor \neg x_3 & C_2 \\
x_1 \lor \neg x_2 \lor \neg x_3 & C_3 \\
x_1 \lor x_2 \lor \neg x_3 & C_4 \\
\neg x_1 \lor x_2 \lor x_3 & C_5 \\
x_1 \lor x_2 \lor x_3 & C_6 \\
\neg x_1 \lor \neg x_2 \lor x_3 & C_7 \\
x_1 \lor \neg x_2 \lor x_3 & C_8
\end{array}
$$

Start with the following decision and apply the DPLL algorithm with unit propagation:

1. (1) Decide $\neg x_1$
2. ...

Congruence Closure

Congruence closure

1. Let S_P be the set of all terms, and their subterms (recursively), in P.
2. Initialize $\cong$ by placing each element of S_P in its own congruence class.
3. For every positive literal $s = t$ in P, merge the congruence classes of s and t.
4. While $\cong$ changes, repeat the following:
   4.1 Propagate the congruence axiom, to account for any merged congruence classes from the previous step. For any $s \cong t$, if $f(\ldots, s, \ldots)$ and $f(\ldots, t, \ldots)$ are currently in different congruence classes, then merge them.
5. Check the negative equality literals in P against the computed $\cong$.
   - For any $s \neq t$ appearing in P, if $s \cong t$, then return that P is unsat.
   - Otherwise, $s \ncong t$ for all $s \neq t$ appearing in P, so return that P is sat.

Exercise: Congruence closure

Use the congruence algorithm to determine the satisfiability of the following formula:

$f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \wedge g(f(x)) \neq x$

Nelson-Oppen

Nelson-Oppen

The Nelson-Oppen procedure for a formula φ that combines different theories consists of:

1. **Purification**: Purify φ into $F_1, \ldots, F_n$.
2. Apply the decision procedure for T_i to F_i. If there exists i such that F_i is unsatisfiable in T_i, then φ is unsatisfiable.
3. **Equality propagation**: If there exists i, j such that F_i T_i-implies an equality between variables of φ that is not T_j-implied by F_j, add this equality to F_j and go to step 2.
4. If all equalities have been propagated then the formula is satisfiable.

Exercise: Nelson-Oppen algorithm

Solve the following formula using the Nelson-Oppen algorithm:

$$\phi = f(f(x) - f(y)) \neq f(z) \land x \leq y \land y + z \leq x \land 0 \leq z$$

DPLL(T)

DPLL(T)

The key idea behind this framework is to decompose the SMT problem into parts we can deal with efficiently:

- Use SAT solver to cope with the **Boolean structure** of the formula;
- Use dedicated conjunctive **theory solver** to decide satisfiability in the background theory.

Exercise: DPLL(T)

Use the DPLL(T) algorithm to determine if the following formula is satisfiable:

$$\varphi : g(a) = c \wedge (f(g(a)) \neq f(c) \vee g(a) = d) \wedge c \neq d$$

Bounded Model Checking

Bounded Model Checking

BMC computes an *underapproximation* of a program by assuming that all loops in the program are unrolled to some fixed, pre-determined finite depth k.

Exercise: From programs to SAT

Consider the domain of the numbers to be $\{0, 1, 2\}$ and:

- Precondition: number1 can be 0 or 1
- Precondition: number2 can be 0 or 1
- Postcondition: sum will be the sum of number1 and number2

```
int number1;
int number2;
int sum = number1;
 for(int i = 0; i < number2; i++) {
    sum += 1; // Increment sum by 1, number2 times
}
assert (sum == number1 + number2);
```

Write a CNF formula that corresponds to verifying the postcondition.

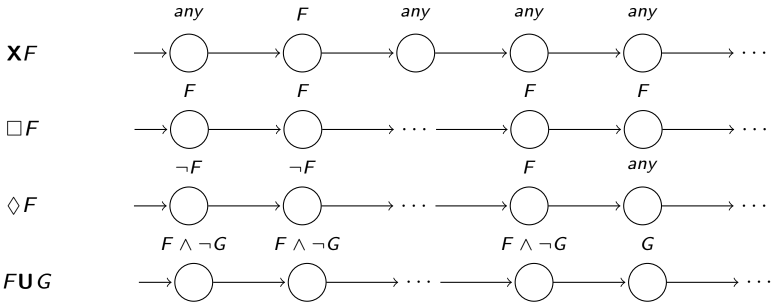
LTL Semantics

LTL Semantics

The truth of LTL formulas in a trace σ is defined inductively as follows:

1. $\sigma \models F$ iff $\sigma_0 \models F$ for a state formula F provided that $\sigma_0 \neq \Lambda$
2. $\sigma \models \neg P$ iff $\sigma \not\models P$, i.e. it is not the case that $\sigma \models P$
3. $\sigma \models P \wedge Q$ iff $\sigma \models P$ and $\sigma \models Q$
4. $\sigma \models \mathbf{X}P$ iff $\sigma^1 \models P$
5. $\sigma \models \Box P$ iff $\sigma^i \models P$ for all $i \geq 0$
6. $\sigma \models \Diamond P$ iff $\sigma^i \models P$ for some $i \geq 0$
7. $\sigma \models P\mathbf{U}Q$ iff there is an $i \geq 0$ such that $\sigma^i \models Q$ and $\sigma^j \models P$ for all $0 \leq j < i$

Examples of traces satisfying LTL formulas



Kripke Structure

Definition (Kripke structure)

- A *Kripke frame* $(W, \curvearrowright)$ consists of a set W with a transition relation $\curvearrowright \subseteq W \times W$
  - $s \curvearrowright t$ indicates that there is a direct transition from s to t in the Kripke frame $(W, \curvearrowright)$
  - The elements $s \in W$ are called states.
- A*Kripke structure* $K = (W, \curvearrowright, v, I)$ is a Kripke frame $(W, \curvearrowright)$ with a mapping $v : W \rightarrow 2^V$
  - 2^V is the powerset of V assigning truth-values to all the propositional atoms in all states.
  - A Kripke structure has a set of initial states $I \subseteq W$.

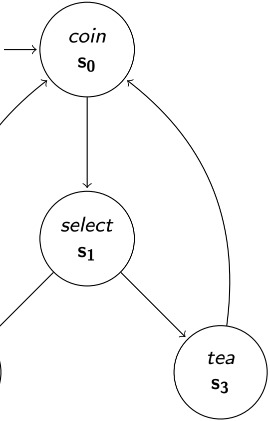
Kripke Structure

Computation Structure

A Kripke structure $K = (W, \curvearrowright, v, I)$ is called a *computation structure* if:

- W is a finite set of states
- every element $s \in W$ has at least one direct successor $t \in W$ with $s \curvearrowright t$.

Exercise: LTL Formulas and Kripke Structures



Which formulas are satisfied by this Kripke structure?

- $\Box \mathit{coffee}$
- $\Box \Diamond (\mathit{coffee} \vee \mathit{tea})$
- $\Box (\mathit{coin} \wedge \mathbf{X}\mathit{select} \rightarrow \Diamond (\mathit{coffee} \vee \mathit{tea}))$

Exercise: LTL Formulas

Show that the following LTL formulas are valid using the semantics of the LTL operators or provide a counterexample if they are incorrect.

1. $\Diamond(P \vee Q) \leftrightarrow \Diamond P \vee \Diamond Q$
2. $\Box(P \vee Q) \leftrightarrow \Box P \vee \Box Q$

CTL Semantics

CTL Semantics

In a fixed computation structure $K = (W, \curvearrowright, v)$, the truth of CTL formulas in state s is defined inductively as follows:

- $s \models \mathbf{AX}P$ iff all successors t with $s \curvearrowright t$ satisfy $t \models P$
- $s \models \mathbf{EX}P$ iff at least one successor t with $s \curvearrowright t$ satisfies $t \models P$
- $s \models \mathbf{A}\Box P$ iff all paths $s_0, s_1, s_2, \ldots$ starting in $s_0 = s$ satisfy $s_i \models P$ for all $i \geq 0$
- $s \models \mathbf{E}\Box P$ iff some path $s_0, s_1, s_2, \ldots$ starting in $s_0 = s$ satisfies $s_i \models P$ for all $i \geq 0$

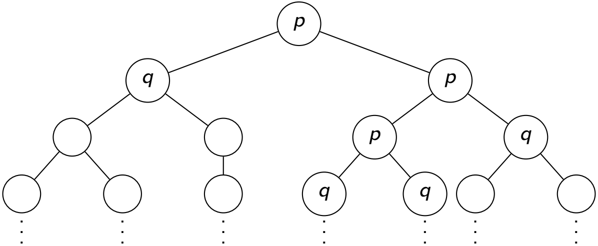
CTL Semantics

CTL Semantics

In a fixed computation structure $K = (W, \curvearrowright, v)$, the truth of CTL formulas in state s is defined inductively as follows:

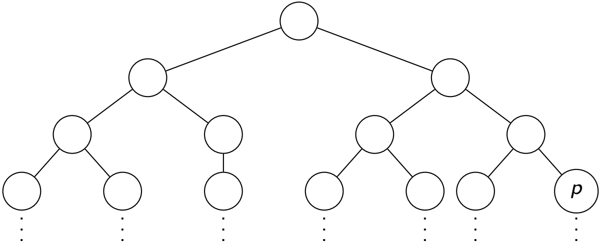
- $s \models \mathbf{A}\Diamond P$ iff all paths $s_0, s_1, s_2, \ldots$ starting in $s_0 = s$ satisfy $s_i \models P$ for some $i \geq 0$
- $s \models \mathbf{E}\Diamond P$ iff some path $s_0, s_1, s_2, \ldots$ starting in $s_0 = s$ satisfies $s_i \models P$ for some $i \geq 0$
- $s \models \mathbf{A}P\mathbf{U}Q$ iff all paths $s_0, s_1, s_2, \ldots$ starting in $s_0 = s$ have some $i \geq 0$ such that $s_i \models Q$ and $s_j \models P$ for all $0 \leq j < i$
- $s \models \mathbf{E}P\mathbf{U}Q$ iff some path $s_0, s_1, s_2, \ldots$ starting in $s_0 = s$ has some $i \geq 0$ such that $s_i \models Q$ and $s_j \models P$ for all $0 \leq j < i$

Example: Visualization of CTL formulas



Visualization of a CTL formula: $\mathbf{A}[P\,\mathbf{U}\,Q]$

Example: Visualization of CTL formulas



Visualization of a CTL formula: $\mathbf{E}\Diamond P$

Exercise: CTL vs. LTL

Show that the following formulas are not equivalent by giving a Kripke structure that satisfies one formula but not the other:

- LTL formula $\Diamond\Box P$
- CTL formula $\mathbf{AFAG}P$