Theorem proving in

Cayden Codel
15-414 Bug Catching
Prof. Ruben Martins

April 25, 2024

Mathematical proof is hard.

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Can we get computers to do it?

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Answer: it depends.

Who does the proving?

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Computers Humans

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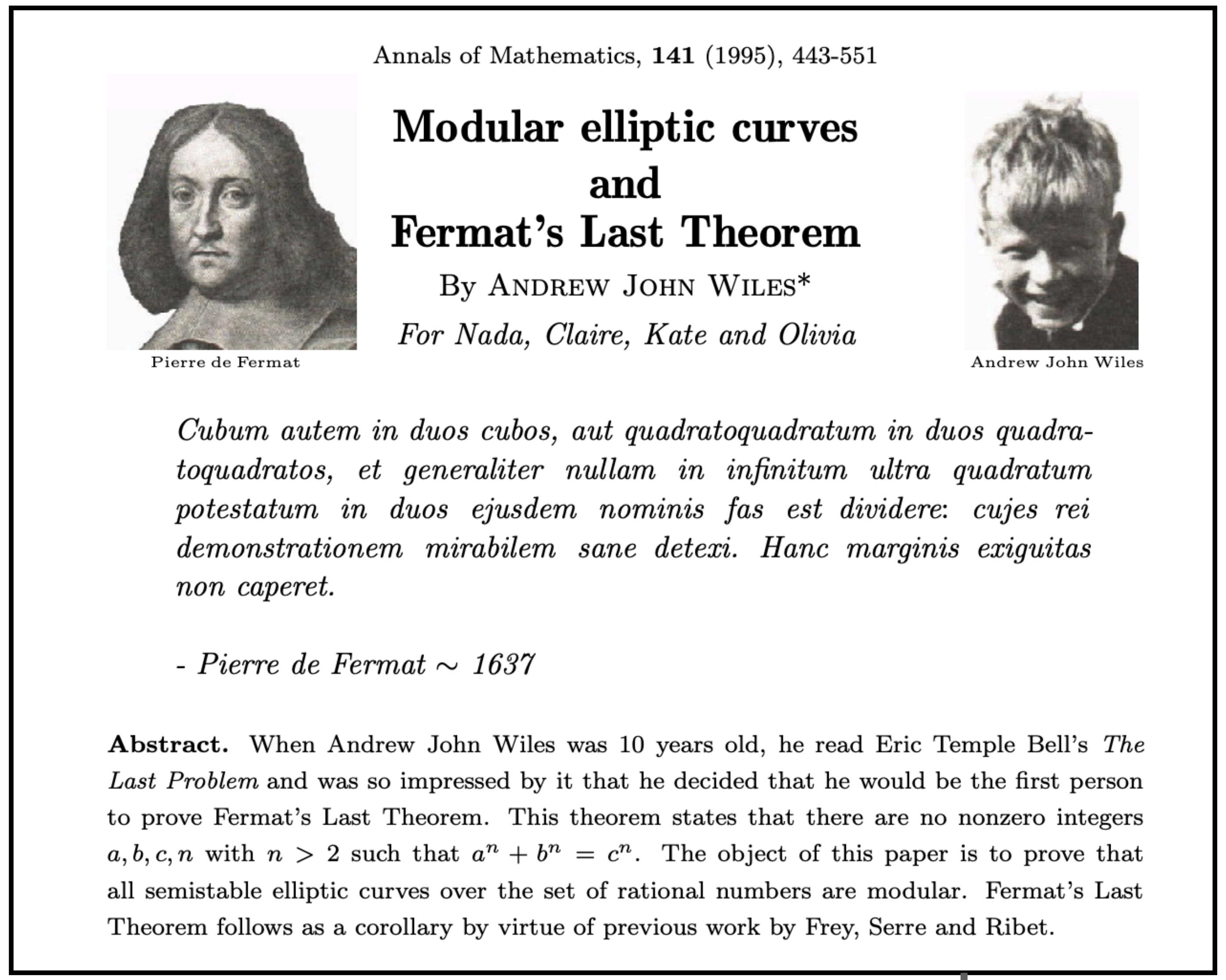
Annals of Mathematics, **141** (1995), 443-551

Modular elliptic curves
and
Fermat's Last Theorem

By Andrew John Wiles*

For Nada, Claire, Kate and Olivia

Pierre de Fermat

Andrew John Wiles

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatum in duos ejusdem nominis fas est dividere: cujes rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

- Pierre de Fermat $\sim$ *1637*

Abstract. When Andrew John Wiles was 10 years old, he read Eric Temple Bell's *The Last Problem* and was so impressed by it that he decided that he would be the first person to prove Fermat's Last Theorem. This theorem states that there are no nonzero integers a, b, c, n with $n > 2$ such that $a^n + b^n = c^n$. The object of this paper is to prove that all semistable elliptic curves over the set of rational numbers are modular. Fermat's Last Theorem follows as a corollary by virtue of previous work by Frey, Serre and Ribet.

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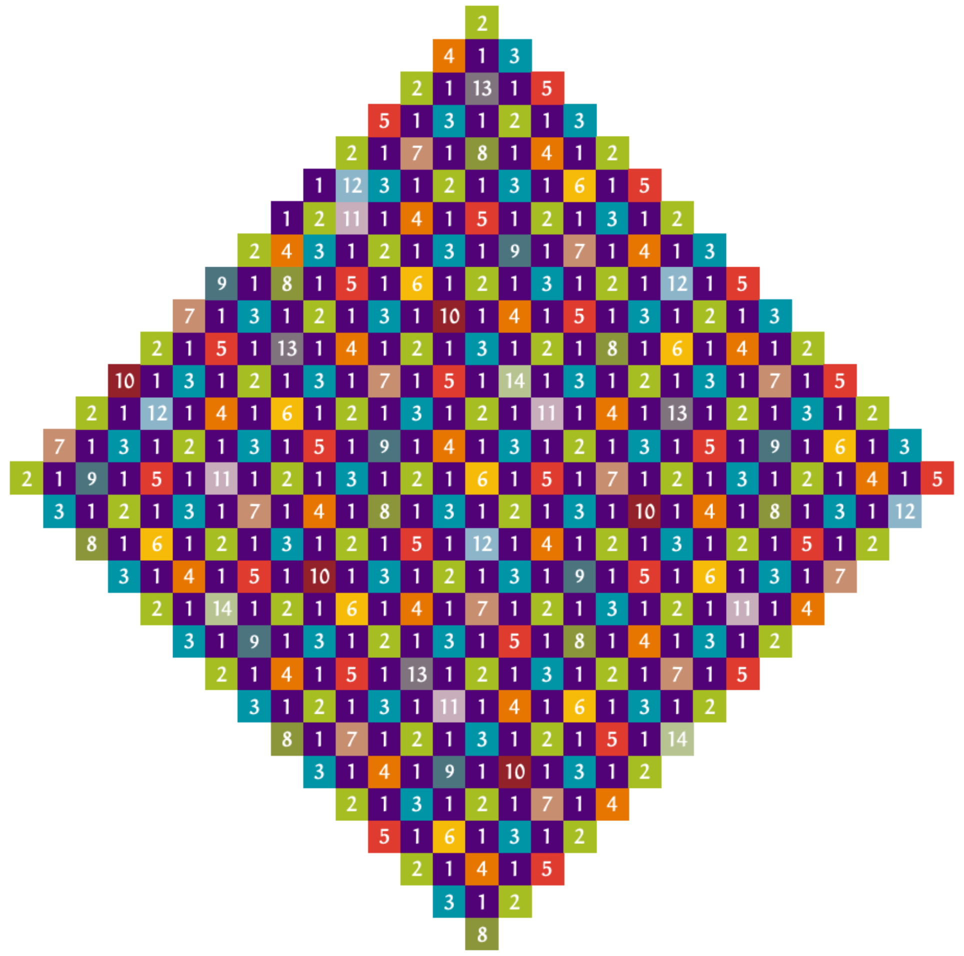

The Packing Chromatic Number of the Infinite Square Grid is 15

Bernardo Subercaseaux ✉ and Marijn J.H. Heule

Carnegie Mellon University, Pittsburgh PA 15203, USA
{bsuberca, mheule}@cs.cmu.edu

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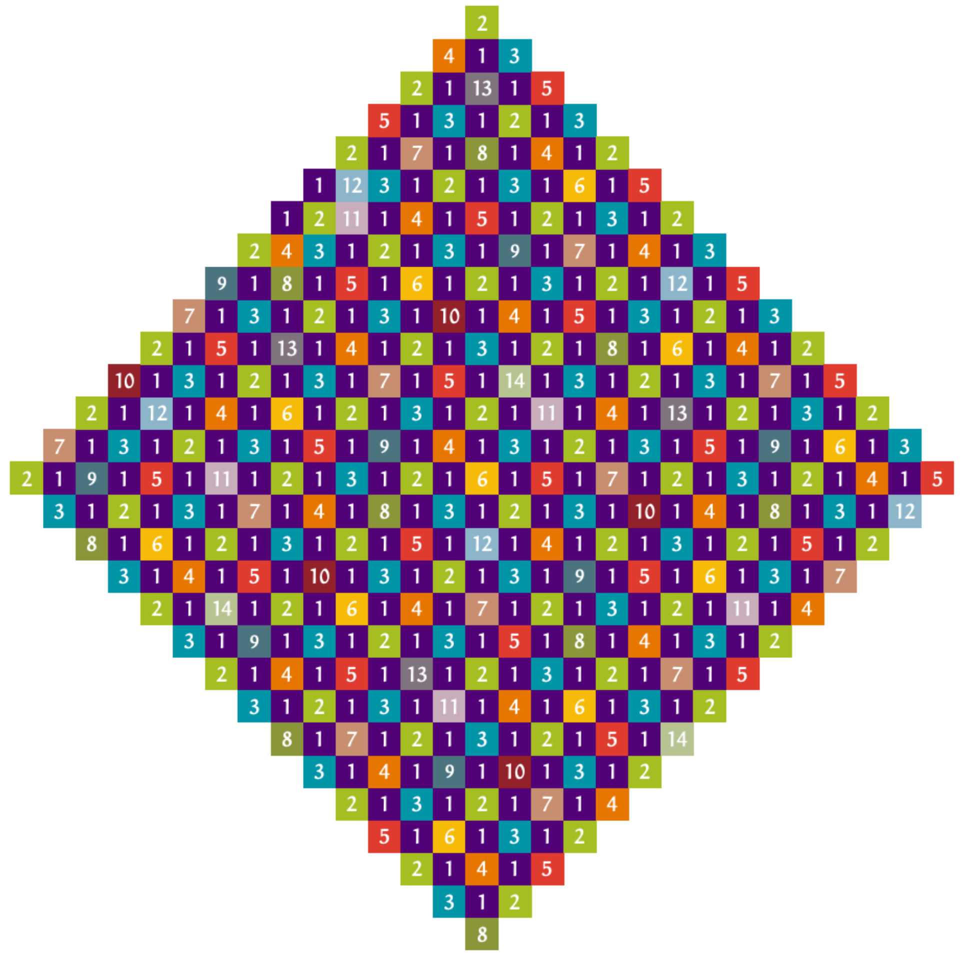

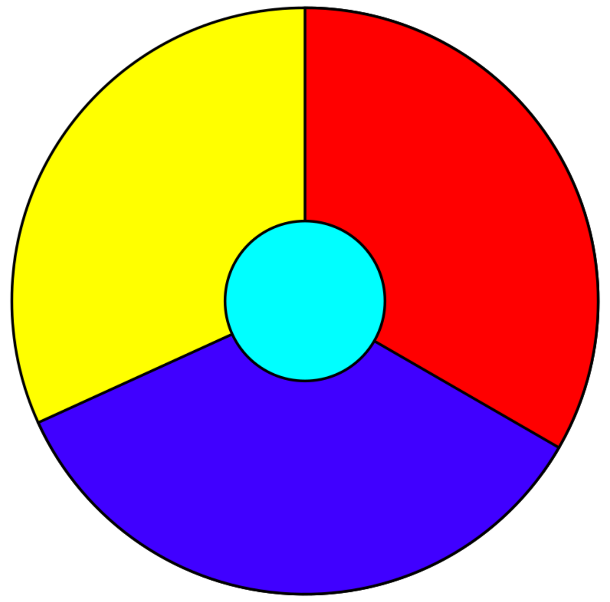

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A computer-checked proof of the Four Colour Theorem

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This report gives an account of a successful formalization of the proof of the Four Colour Theorem, which was fully checked by the Coq v7.3.1 proof assistant [13].

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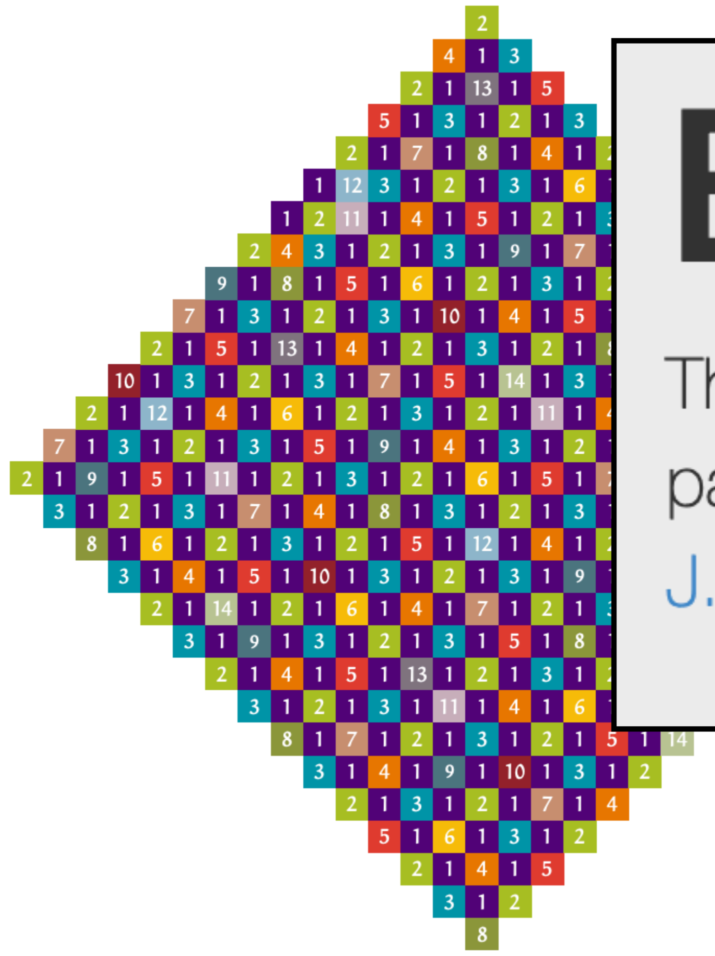

Everything's Bigger in Texas

This page provides access to results, proofs and tools on "the largest math proof ever" presented in the SAT 2016 paper **Solving and Verifying the boolean Pythagorean Triples problem via Cube-and-Conquer** by Marijn J.H. Heule, Oliver Kullmann, and Victor Marek (best paper award). A preprint is available on arXiv.

Fortunately there comes SAT solving to the rescue, which actually is r
approach solved the problem and resulted into a 200 terabytes proof

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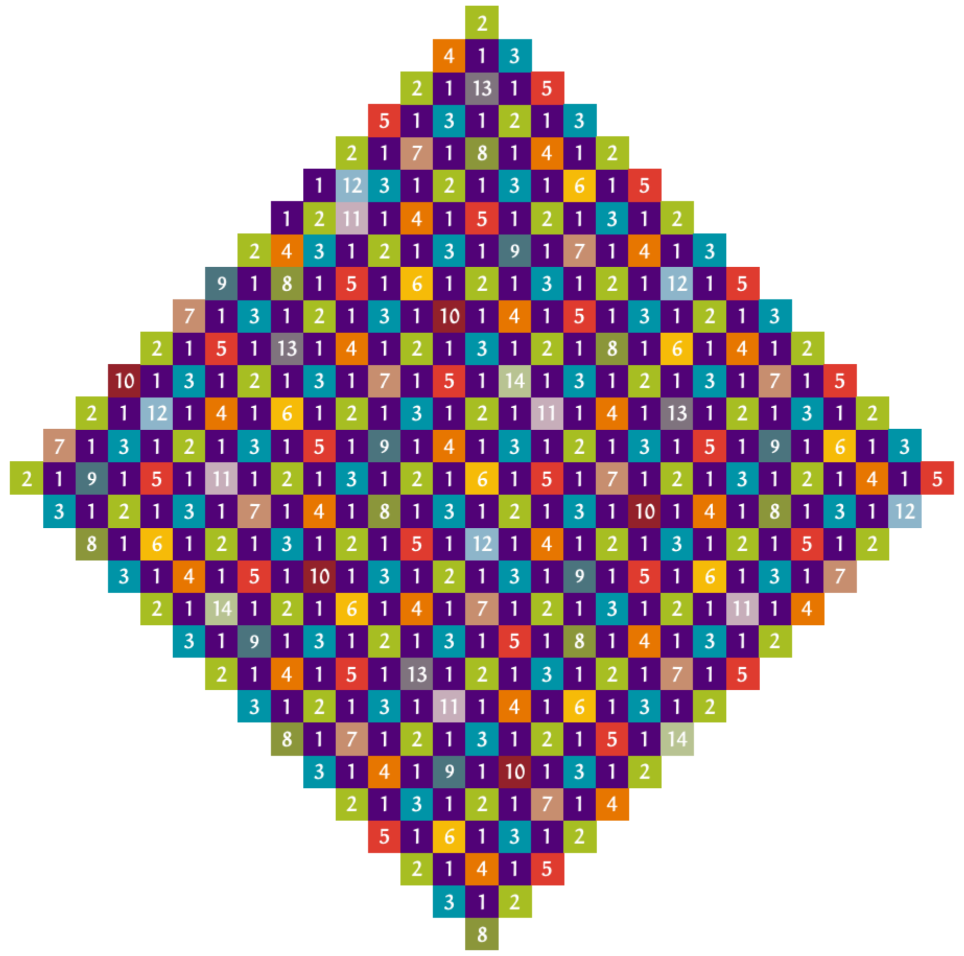

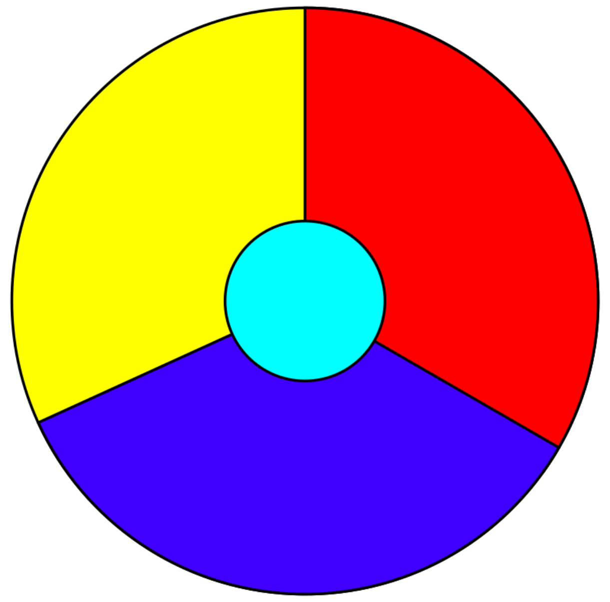

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A human-computer compromise

- Write code (functional, with inductive types)
- Write theorems
- Prove theorems with tactics
- Libraries of mathematical proofs

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- Critical for several formalization efforts

Formal Verification of the Empty Hexagon Number

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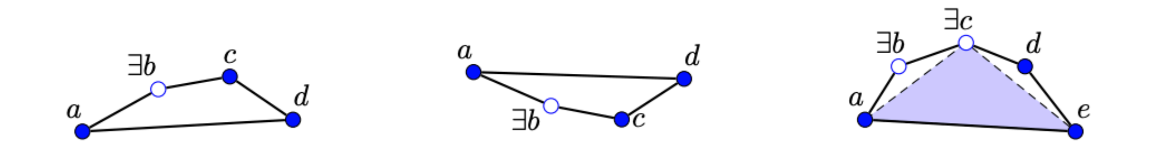
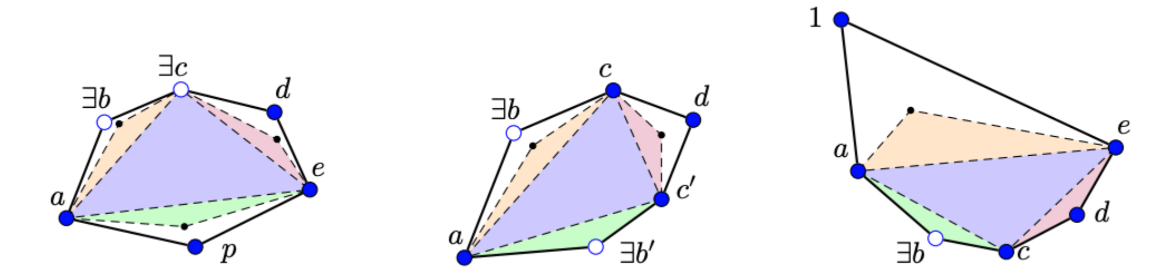


(a) $u^4_{a,c,d}$ (b) $v^4_{a,c,d}$ (c) $u^5_{a,d,e}$

Figure 7 Illustration of the 4-cap (7a), 4-cup (7b), and 5-cap (7c) variables. The highlighted region denotes an empty triangle.



(a) $(u^5_{a,d,e} \wedge o_{a,p,e})$ (b) $(u^4_{a,c,d} \wedge v^4_{a,c',d} \wedge h_{a,c,c'})$ (c) $(v^4_{a,c,d} \wedge o_{c,d,e} \wedge h_{a,c,e})$

Figure 8 Illustration of some *forbidden configurations* that imply 6-holes. Figure 8a corresponds to the configuration forbidden by clause (13), Figure 8b to the one forbidden by clause (15), and Figure 8c to clause (17). All highlighted regions denote empty triangles.

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- A way for computer scientists to verify code

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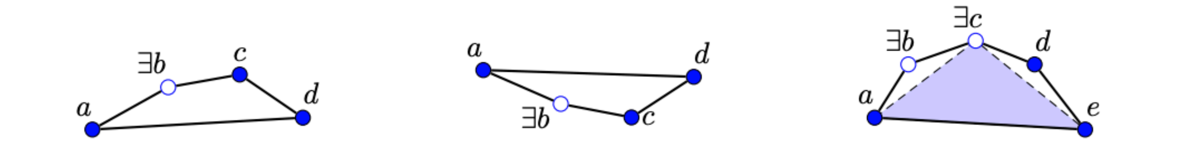

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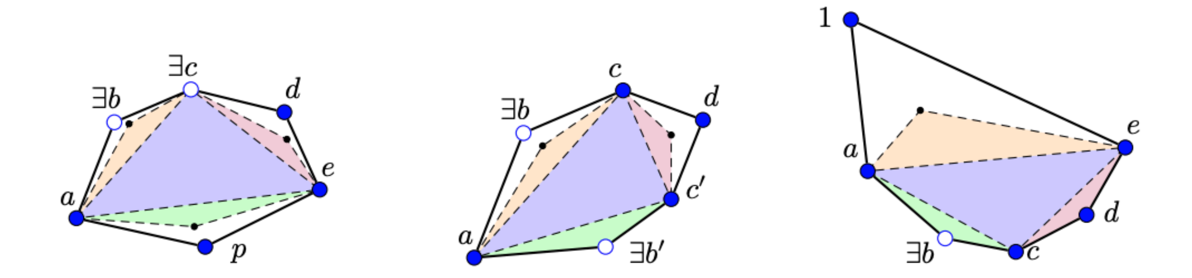

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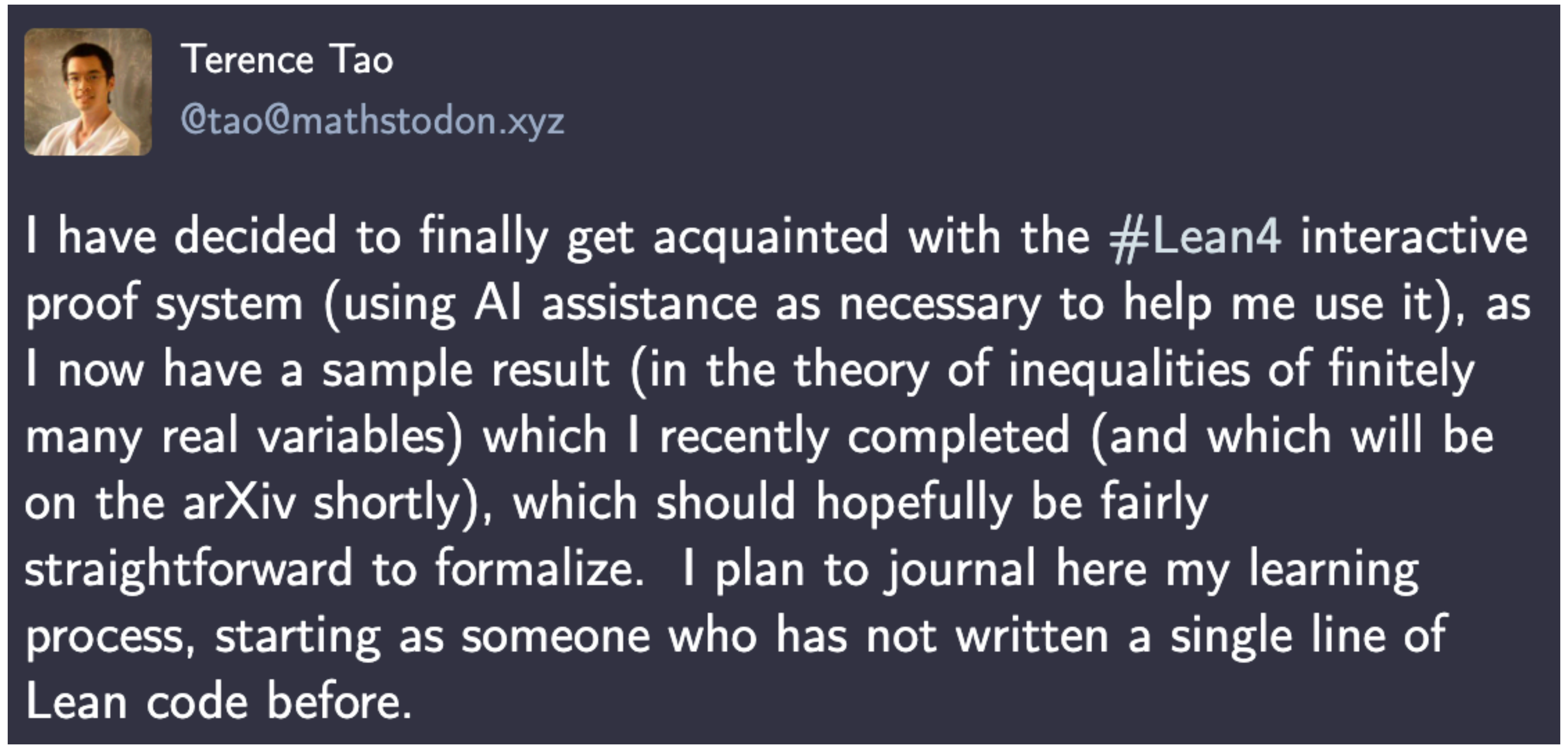
Terence Tao
@tao@mathstodon.xyz

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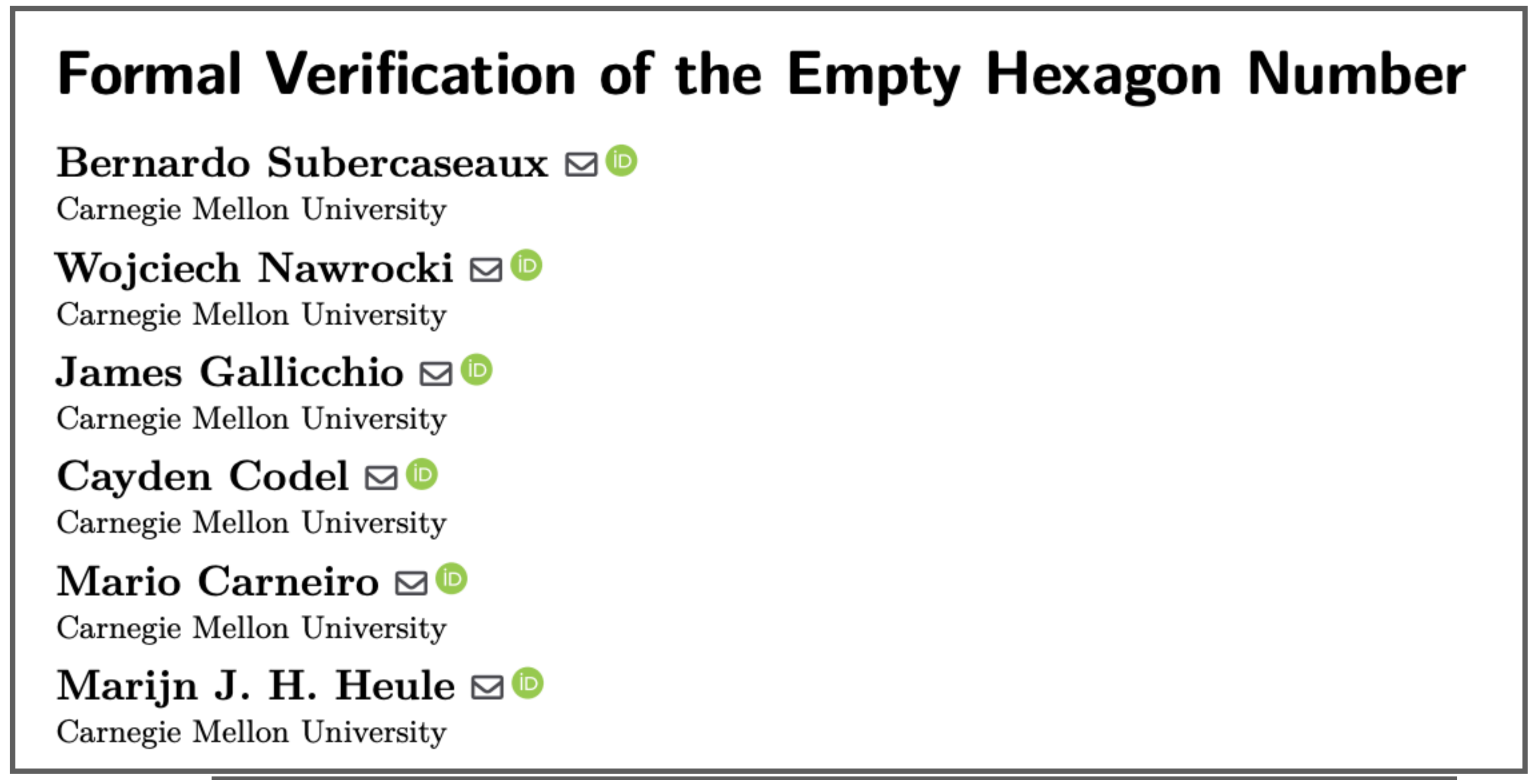
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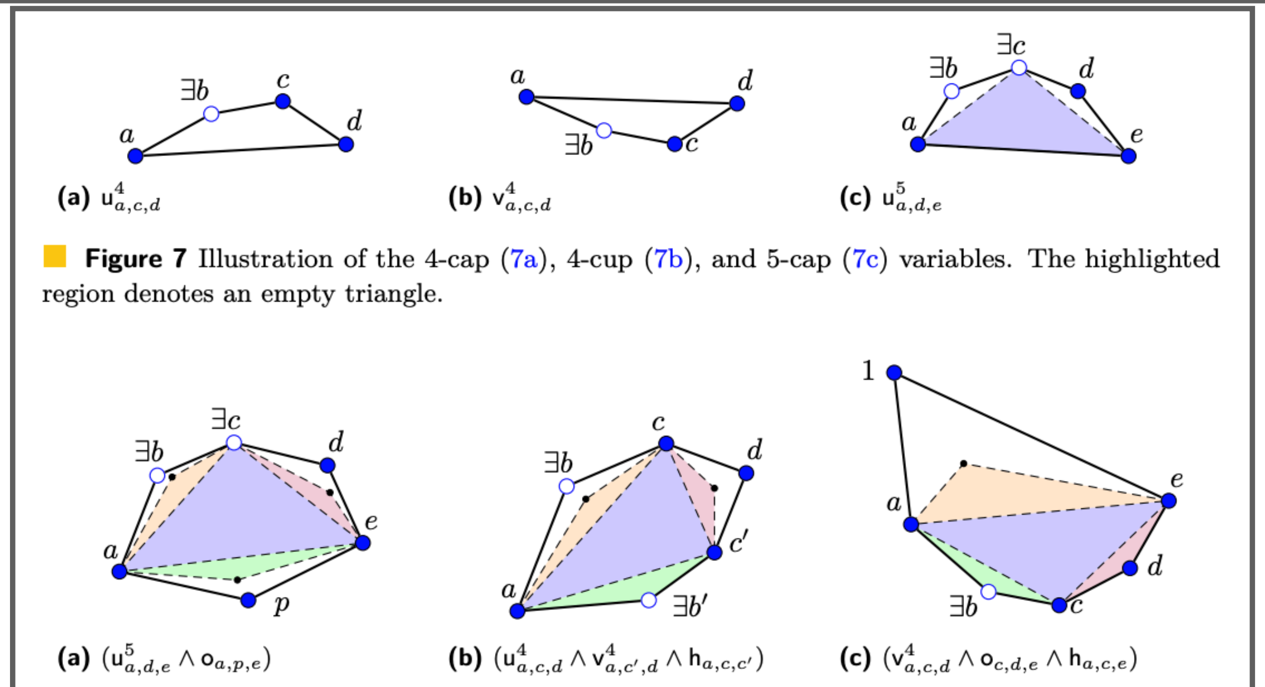

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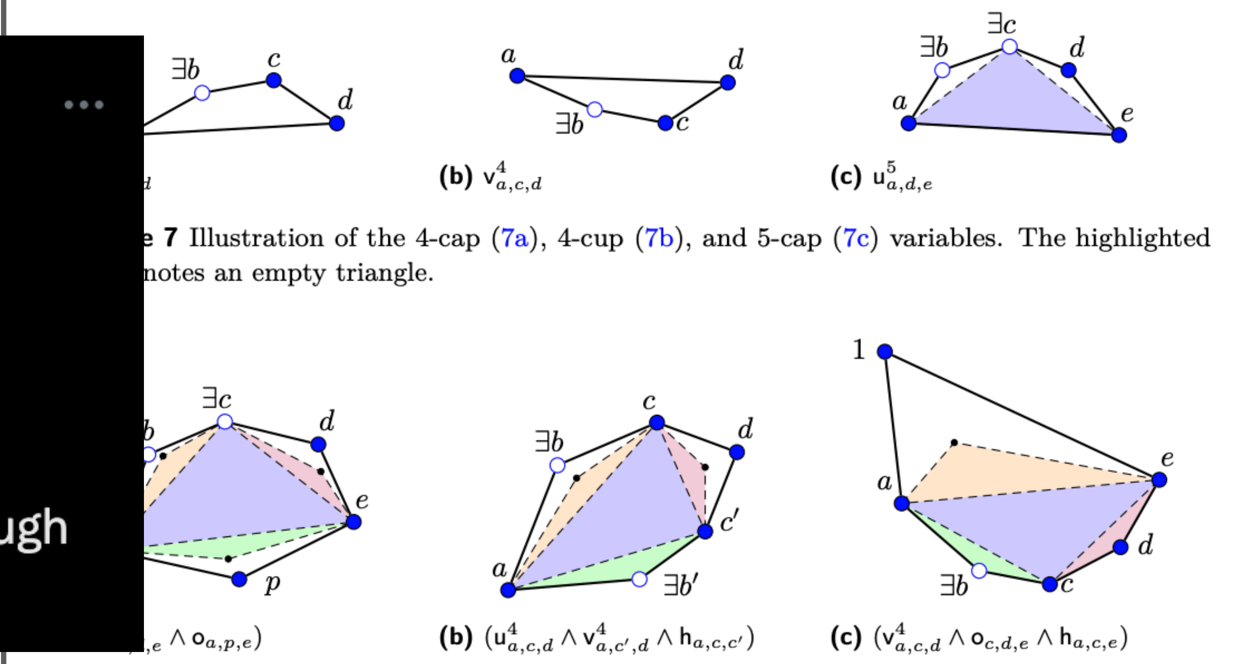

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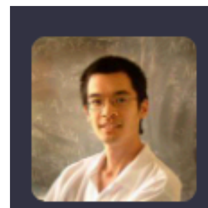

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Kevin Buzzard
@XenaProject

I got a research grant to begin the proof of formalising Fermat's Last Theorem in Lean! gow.epsrc.ukri.org/NGBOViewGrant....

The research buys out my teaching and admin for 5 years, which I suspect will not be enough to get it done, but it will certainly be enough to make a big dent in it.

Let’s play around with Lean!