Example Simplex Algorithm Run

Example linear program:

$$
\begin{array}{rrl}
x_1 & +x_2 & \leq 3 \\
-x_1 & +3x_2 & \leq 1 \\
 & +x_2 & \leq 3 \\
\hline
x_1 & +x_2 & = z
\end{array}
$$

The last line is the *objective* function we are trying to **maximize**.

We assume:

- all the constraints are $\leq$, and
- all the values of the variables must be ≥ 0.

Slack variables

We re-write into a system of equations by introducing non-negative *slack variables*:

$$
\begin{array}{rrrl}
x_1 & +x_2 & +x_3 & = 3 \\
-x_1 & +3x_2 & +x_4 & = 1 \\
 & +x_2 & +x_5 & = 3 \\
\hline
x_1 & +x_2 & & = z
\end{array}
$$

There is an easy solution to this system of equations:

$$x_3 = 3, x_4 = 1, x_5 = 3 \textit{ and all the rest of the variables} = 0$$

This gives us an objective of 0.

We now proceed with a series of transformations that seek to increase the objective.

Tableau

Re-write to put the non-zero values on the left-hand side:

$$\begin{array}{rrrr} x_3 & = 3 & -x_1 & -x_2 \\ x_4 & = 1 & +x_1 & -3x_2 \\ x_5 & = 3 & & -x_2 \\ \hline z & = 0 & +x_1 & +x_2 \end{array}$$

This is called a *tableau*: Right-hand side variables are all 0, left hand side may be non-zero.

The left-hand side variables are called *basic variables*.

Which variables are candidates for increasing to increase z?

Those with positive coefficients in the objective. Pick one: say x_2.

Entering variable

x_2 is called the *entering* variable (we'll see why in a minute)

How much can we increase x_2 by?

So long as none of the basic variables become negative.

We choose the amount to increase based on the strictest equation:

$$3 - x_2 \geq 0 \implies x_2 \leq 3 \quad (1)$$
$$1 - 3x_2 \geq 0 \implies x_2 \leq 1/3 \quad (2)$$
$$3 - x_2 \geq 0 \implies x_2 \leq 3 \quad (3)$$

Constraint (2) is the strictest, so we set $x_2 = 1/3$.

The leaving variable

Look at the strictest constraint now:

$$x_4 = 1 + x_1 - 3x_2 \implies x_2 = 1/3 + (1/3)x_1 - (1/3)x_4$$

If we increase $x_2 = 1/3$, then x_4 becomes 0.

We re-write constraint (2) to put x_2 on the left-hand side, and substitute in for x_2 in all the remaining equations:

$$\begin{array}{lllll} x_3 & = 8/3 & -(4/3)x_1 & +(1/3)x_4 \\ x_2 & = 1/3 & +(1/3)x_1 & -(1/3)x_4 \\ x_5 & = 8/3 & -(1/3)x_1 & +(1/3)x_4 \\ \hline z & = 1/3 & +(4/3)x_1 & -(1/3)x_4 \end{array}$$

This system of equations is equivalent to what we started with, but now if we set the rhs variables to 0, we get an objective value of 1/3. Progress!

Pivots

x_4 was the *leaving* variable.

Now which variable can we increase?

x_1 is the only variable with a non-negative coefficient in our objective. So, we select x_1 as the entering variable, and see how much we can increase it:

$$x_1 \leq (8/3)/(4/3) = 2 \tag{4}$$
$$x_1 \geq (1/3)/(-1/3) = -1 \tag{5}$$
$$x_1 \leq (8/3)/(1/3) = 8 \tag{6}$$

Notice only those constraints where x_1 has a negative coefficient provide a constraint!

Constraint (4) is the strictest. So we increase x_1 to 2 in the same way as before.

Increasing x_1 to 2

Look at the strictest constraint:

$$x_3 = 8/3 - (4/3)x_1 + (1/3)x_4 \implies x_1 = 2 - (3/4)x_3 - (1/4)x_4$$

Rewrite that constraint in terms of x_1, and substitute in for x_1 everywhere:

$$\begin{array}{lllll} x_1 & = 2 & -(3/4)x_3 & +(1/4)x_4 \\ x_2 & = 1 & -(1/4)x_3 & -(1/4)x_4 \\ x_5 & = 2 & +(1/4)x_3 & +(1/4)x_4 \\ \hline z & = 3 & -x_3 & \end{array}$$

Now: all the coefficients in the objective function are ≤ 0, so we're done. The optimal value is 3 with $x_1 = 2$ and $x_2 = 1$.

Check that these values satisfy the original constraints!

Note

- Note that increasing x_1 *also* increased x_2. This happened when we substituted in for x_1 in the second constraint.

The material beyond this point is optional, and will not appear on homeworks or exams.

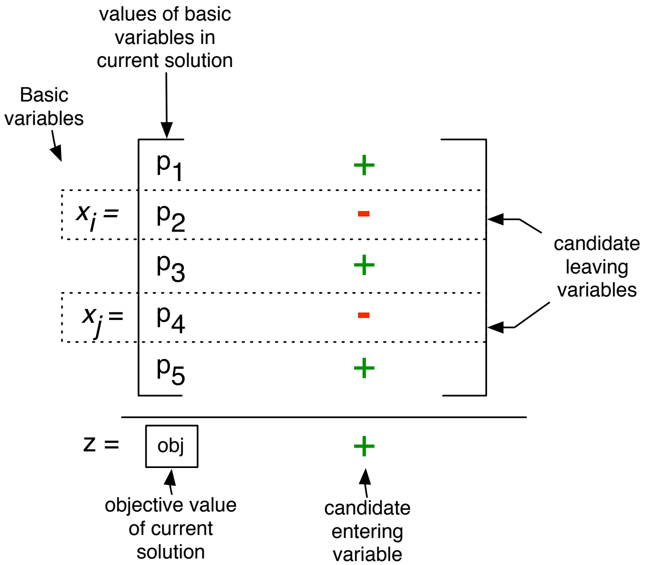
Simplex Algorithm In General

1. Write LP with slack variables (slack vars = initial solution)
2. Choose a variable v in the objective with a positive coefficient to increase
3. Among the equations in which v has a negative coefficient q_{iv}, choose the strictest one

   This is the one that minimizes $-p_i/q_{iv}$ because the equations are all of the form $x_i = p_i + q_{iv}x_v$.

4. Re-write the strictest equation to put v on the left-hand side, and substitute for v everywhere else.
5. If all the coefficients are ≤ 0 in objective, we're done; otherwise, jump back to step 2.

Schematic of a pivot



Special cases

- What if no constraint provides an upper bound on entering variable v?

  $\implies$ problem is unbounded, and has an ∞ maximum.

- What if the strictest upper bound is 0?

  $\implies$ there is another feasible $\vec{x}$ of equivalent cost.

  You may have to do a pivot that doesn't increase the objective (but doesn't decrease it either) in order to make progress.

What if there are multiple choices for the entering variable?

Choose according to some pivot rule:

- Largest coefficient in the objective — increases the objective as much as possible per unit of variable increase.
- Largest increase — pick the one that increases the objective the most.
- Random — choose one at random.
- Steepest edge — choose the variable to maximize:

$$\frac{\vec{c}^T(\vec{x}_{\text{new}} - \vec{x}_{\text{old}})}{||\vec{x}_{\text{new}} - \vec{x}_{\text{old}}||}$$

- Bland's rule — choose the entering variable with the lowest index (and the corresponding leaving variable with the lowest index as well). [Important theoretically, but not used much in practice.]

How do we know this process will terminate?

In general, this process might not terminate!

If you can always increase the objective function (non-zero upper bound on the entering variable), then it must eventually stop.

It might be that you repeatedly have to choose an entering variable with a 0 upper bound.

Bland's rule ensures that you can't cycle forever.

How do we know it's optimal?

Consider a final objective equation, say (a different objective than the example above):

$$z = 24 - 5x_1 - 3x_3$$

This expression is equivalent to whatever objective we started with.

Since all the variables have to be ≥ 0, we must have the optimal $z \leq 24$. Since we've achieved 24, we know it must be optimal.

Initialization

We made one implicit assumption in the discussion above: that initially setting the non-slack variables to 0 would give us a feasible solution.

This is not the case if the b_i values are < 0:

$$\begin{array}{lrrr} x_3 & = 3 & -x_1 & -x_2 \\ x_4 & = -1 & +x_1 & -3x_2 \\ x_5 & = 3 & & -x_2 \\ \hline z & = 0 & +x_1 & +x_2 \end{array}$$

Here, we'd get a "solution" with $x_4 < 0$, which is not allowed.

We solve an *auxiliary* LP problem to find the initial solution.

Auxiliary Problem

Suppose we have LP:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i & i = 1, \ldots, m \\
& x_j \ge 0 & j = 1, \ldots, n
\end{aligned}
$$

Its auxiliary problem is:

$$
\begin{aligned}
\text{maximize} \quad & -x_0 \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j - x_0 \le b_i & i = 1, \ldots, m \\
& x_j \ge 0 & j = 0, \ldots, n
\end{aligned}
$$

where x_0 is a new variable we introduce.

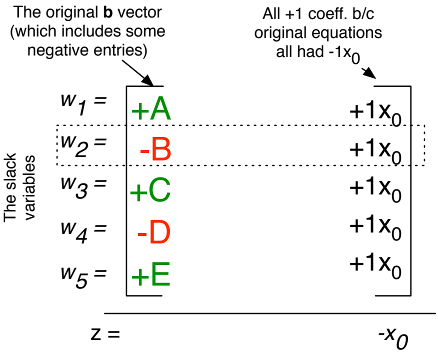
$$\begin{aligned} \text{maximize} \quad & -x_0 \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j - x_0 \le b_i & i = 1, \dots, m \\ & x_j \ge 0 & j = 0, \dots, n \end{aligned}$$

Setting $x_j = 0$ for $j \ge 1$ and x_0 really big will give us a feasible solution to this problem.

The original problem had a solution $\iff$ its auxiliary problem has an optimal solution of objective $= 0$.

Write the tableau for this problem as usual (introducing slack variables) — but it still won't be feasible.

Auxiliary Tableau



Suppose $-B$ is the most negative entry. Rewrite that equation in terms of x_0:

$$x_0 = +B + \text{terms involving variables}$$

Now substitute the red part in for x_0 in every other equation.

This will add B to every constant term $\implies$ all the constant terms become positive (since $-B$ was the most negative).

Initialization Summary

1. Construct the auxiliary tableau.
2. Pivot once with
   - entering variable $= x_0$
   - leaving variable $=$ most negative constant term
3. Solve the auxiliary problem from this starting point using the normal simplex method.
4. If original problem was feasible, will find solution with $x_0 = 0$ for auxiliary problem.
5. Drop the x_0 equation and the variables x_0 from the other equations (ok since they are 0).
6. Put back the original objective function.
7. Continue to apply simplex method.

Summary

- Simplex method widely used in practice.
- Often great performance, fairly simple linear algebra manipulations.
- In some settings, a linear $O(m)$ number of pivots is observed ($m =$ number of constraints).
- But: might run for exponential number of steps, or even forever if a bad pivot rule is chosen.
- Main idea: swap variables in and out of the set of basic variables.