15-451/651 Algorithm Design & Analysis, Spring 2024

Recitation #6

Objectives

- Practice reducing problems to network flows, where the max-flow/min-cut yields the optimal solution to the original problem.

Review

Residual capacity: $c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \end{cases}$

Decrease the capacity of the edges you've push flow through and then draw edges representing the reverse of the flow we've pushed (to track it so that we could "undo" it).

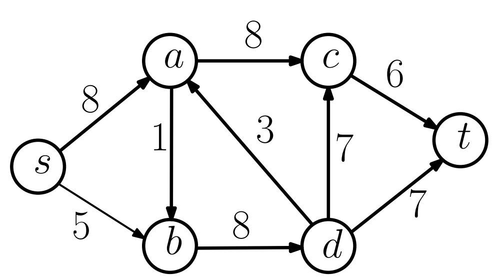
Residual graph: Given a flow f in graph G, the residual graph G_f is the directed graph with all edges of positive residual capacity (including reverse edges in the original graph G).

Ford Fulkerson: while there exists an $s \to t$ path P of positive residual capacity, push the maximum possible flow along P (so we saturate this path with flow).

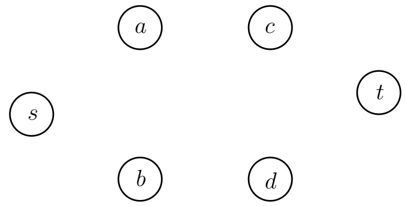
Min cut / max flow: In any flow network G, for any two vertices s and t, the maximum flow from s to t equals the capacity of the minimum (s,t)-cut. An (s,t)-cut is a disjoint partition of the vertices into (S,T) such that $s \in S$ and $t \in T$. The capacity of this cut is the sum of the (non-residual) capacities of the directed edges crossing the cut from S to T.

Recitation Problems

1. **(Example of running Ford-Fulkerson)** Consider the graph below.



Run Ford-Fulkerson, drawing the **residual network** at every step, and show the final residual network. What is the maximum flow? What is the capacity of the minimum cut? Partition the vertices into two sets (S,T) such that $s \in S$, $t \in T$, and (S,T) is a minimum cut.


a
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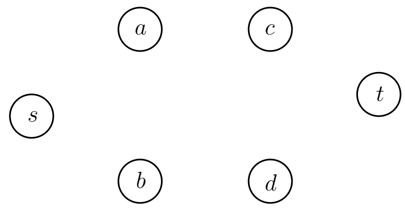
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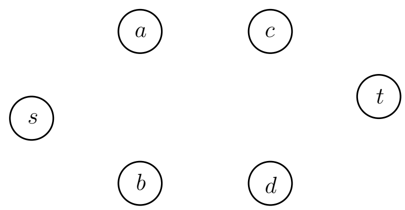
a
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2. **(FOX Tiles)** There is an n by n grid of the letters F, O, and X. The goal is to form as many disjoint copies of the word FOX as possible in the given grid. To form FOX you start at any F, move to a neighboring O, then move to a neighboring X, (you can move up, down, left, or right to get to a neighbor). The following figure shows one such problem on the left, along with three possible FOXs on the right.

X	F	F	O
O	F	O	X
F	F	X	O
X	F	O	F

X	F	F	O
O	F	O	X
F	F	X	O
X	F	O	F

X	F	F	O
O	F	O	X
F	F	X	O
X	F	O	F

Give a polynomial-time algorithm to find the solution with the largest number of FOX's. Your algorithm should consist of reducing the problem to an instance of max flow.

3. **(Cut into Teams)** CMU intramural sport season is coming up, and Maximus Flow and Minnie Cut want to form teams to compete in the intramural football game. They have a group of n students that they want to split into two teams with Maximus captaining team A and Minnie captaining team B. Each student has some preference toward being on Maximus' or Minnie's team however. In particular, Max and Min need to pay \$$A_i$ for student i to join team A and \$$B_i$ for student i to join team B. To further complicate things, some pairs of students are friends. For each friend pair, they would prefer to be on the same team. For each friend pair (i, j) that ends up on different teams, Max and Min will have to pay an extra \$$K_{i,j}$ for making them unhappy. Being college students with no money and no free time, Maximus and Minnie want to form these teams with the lowest price possible and in polynomial time. Note that the two teams do not have to be of the same size.

    (a) Reduce this to constructing a flow network where a given cut (partition of the vertices into two sets (A, B)) corresponds to assigning students to either team A or B.

(b) Now suppose each friend pair (i, j) refused to be separated and will quit if they do. How can you adjust your flow network without deleting/merging any nodes to guarantee that no such friend pair will be put on different teams?

4. **(Two Types of Cuts - Optional)** Cuts can be defined as subsets of the edges of the graph instead of partitions of the vertices. We will prove that these two definitions are *almost* equivalent.

   A subset X of directed edges separates s and t if every directed path from s to t contains at least one directed edge in X. For any subset S of vertices, let σS denote the set of directed edges leaving S (i.e. $\sigma S := \{u \to v | u \in S, v \notin S\}$).

   (a) Prove that if (S, T) is an (s, t) cut, then σS separates s and t.

   (b) Let X be an arbitrary subset of edges that separates s and t. Prove that there is an (s, t) cut (S, T) such that $\sigma S \subseteq X$.

   (c) Let X be a minimal subset of edges that separates s and t. Prove that there is an (s, t) cut (S, T) such that $\sigma S = X$.