15-451/651: Design & Analysis of Algorithms September 1, 2022
Lecture #2: Concrete models and tight upper/lower bounds last changed: January 18, 2023

In this lecture, we will examine some simple, concrete models of computation, each with a precise definition of what counts as a step, and try to get tight upper and lower bounds for a number of problems. Specific models and problems examined in this lecture include:

- The number of comparisons needed to sort an array.
- The number of exchanges needed to sort an array.
- The number of comparisons needed to find the largest and second-largest elements in an array.

Objectives of this lecture

In this lecture, we want to:

- Understand the concept of a model of computation, with several examples (the comparison model, the exchange model)
- Understand the definition of a lower bound in a specific model
- See some examples of how to prove lower bounds in specific models, specifically for sorting and selection problems

Recommended study resources

- CLRS, *Introduction to Algorithms*, Chapter 8.1, Lower bounds for sorting
- DPV, *Algorithms*, Chapter 2.3, Mergesort (Page 59)

1 Terminology: Upper Bounds and Lower Bounds

In this lecture, we will look at (worst-case) upper and lower bounds for a number of problems in several different concrete models. Each model will specify exactly what operations may be performed on the input, and how much they cost. Each model will have some operations that cost a certain amount (like performing a comparison, or swapping a pair of elements), some that are free, and some that are not allowed at all.

Definition: Upper bound

By an *upper bound* of U_n for some problem and some length n, we mean that *there exists an algorithm A that for every input x of length n costs at most U_n*.

A lower bound for some problem and some length n, is obtained by the negation of an upper bound for that n. It says that some upper bound is not possible (for that value of n). If we take the above statement (in italics) and negate it, we get the following. *for every algorithm A there exists an input x of length n such that A costs more than U_n on input x.* Rephrasing slightly:

Definition: Lower bound

By a *lower bound* of L_n for some problem and some length n, we mean that *for any algorithm A there exists an input x of length n on which A costs at least L_n steps.*

These were definitions for a single value of n. Now a function $f : \mathbb{N} \to \mathbb{R}$ is an upper bound for a problem if $f(n)$ is an upper bound for this problem for every $n \in \mathbb{N}$. And a function $g(\cdot)$ is an lower bound for a problem if $g(n)$ is a lower bound for this problem for every n.

The reason for this terminology is that if we think of our goal as being to understand the "true complexity" of each problem, measured in terms of the best possible worst-case guarantee achievable by any algorithm, then an upper bound of $f(n)$ and lower bound of $g(n)$ means that the true complexity is somewhere between $g(n)$ and $f(n)$.

Finally, what is the *cost* of an algorithm? As we said before, that depends on the particular model of computation we're using. We will consider different models below, and show each has their own upper and lower bounds. When you're writing programs, you typically use rules-of-thumb to calculate the cost of opetations (adding two integers costs 1, copying strings of length ℓ should cost $\approx \ell$, etc) — we make this precise using certain cost models.

2 Sorting in the comparison model

One natural model for examining problems like sorting is what is known as the comparison model.

Definition: Comparison Model

In the *comparison model*, we have an input consisting of n items (typically in some initial order). An algorithm may compare two items (asking is $a_i < a_j$?) at a cost of 1. Moving the items around is free. No other operations on the items are allowed (using them as indices, adding them, etc).

For the problem of *sorting* in the comparison model, the input is an array $a = [a_1, a_2, \ldots, a_n]$, and the output is a permutation of the input $\pi(a) = [a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(n)}]$ in which the elements are in increasing order. We begin this lecture by showing the following lower bound for comparison-based sorting.

Theorem 1: Lower bound for sorting in the comparison model

Any deterministic comparison-based sorting algorithm must perform at least $\lg(n!)$ comparisons to sort n elements in the worst case.[a] Specifically, for any deterministic comparison-based sorting algorithm $\mathcal{A}$, for all $n \geq 2$ there exists an input I of size n such that $\mathcal{A}$ makes at least $\lg(n!) = \Omega(n \log n)$ comparisons to sort I.

[a] As is common in CS, we will use "lg" to mean "$\log_2$".

To prove this theorem, we cannot assume the sorting algorithm is going to necessarily choose a pivot as in Quicksort, or split the input as in Mergesort — we need to somehow analyze *any possible* (comparison-based) algorithm that might exist. We now present the proof, which uses a very nice information-theoretic argument. (This proof is deceptively short: it's worth thinking through each line and each assertion.)

Proof of Theorem 1. Let us begin with a simple general claim. Suppose you have some problem where there are M possible different outputs the algorithm could require: e.g., for sorting by comparisons where the output can be viewed as a specific permutation of the input, each possible permutation of the input is possible, and hence $M = n!$. Suppose furthermore that for each of these outputs, there exists some input under which it is the only correct answer. This is true for sorting. Then, we have a worst-case lower bound of $\lg M$. The reason is that the algorithm needs to find out which of these M outputs is the right one, and each YES/NO question could be answered in away that removes at most half of the possibilities remaining from consideration. (Why half? Because each comparison has a binary output, so it breaks the set of possibilities

into two parts. At least one of these parts contain at least half the possibilities.) So, in the worst case, it takes at least $\lg M = \lg n!$ steps to find the right answer. □

The above is often called an "information theoretic" argument because we are in essence saying that we need at least $\lg(M) = \lg(n!)$ bits of information about the input before we can correctly decide what output we need to produce.

What does $\lg(n!)$ look like? We have: $\lg(n!) = \lg(n) + \lg(n-1) + \lg(n-2) + ... + \lg(1) < n\lg(n) = O(n \log n)$ and $\lg(n!) = \lg(n) + \lg(n-1) + \lg(n-2) + ... + \lg(1) > (n/2)\lg(n/2) = \Omega(n \log n)$. So, $\lg(n!) = \Theta(n \log n)$.

However, since today's theme is tight bounds, let's be a little more precise. We can in particular use the fact that $n! \in [(n/e)^n, n^n]$ to get:

$$\begin{array}{rcccl} n\lg n - n\lg e & < & \lg(n!) & < & n\lg n \\ n\lg n - 1.443n & < & \lg(n!) & < & n\lg n. \end{array}$$

Since $1.433n$ is a low-order term, sometimes people will write this fact this as: $\lg(n!) = (n\lg n)(1 - o(1))$, meaning that the ratio between $\lg(n!)$ and $n\lg n$ goes to 1 as n goes to infinity.

How Tight is this Bound? Assume n is a power of 2 — in fact, let's assume this for the entire rest of today's lecture. Can you think of an algorithm that makes at most $n\lg n$ comparisons, and so is tight in the leading term? In fact, there are several algorithms, including:

Binary insertion sort If we perform insertion-sort, using binary search to insert each new element, then the number of comparisons made is at most $\sum_{k=2}^{n} \lceil \lg k \rceil \leq n \lg n$. Note that insertion-sort spends a lot in moving items in the array to make room for each new element, and so is not especially efficient if we count movement cost as well, but it does well in terms of comparisons.

Mergesort Merging two lists of $n/2$ elements each requires at most $n-1$ comparisons. So, unrolling the recurrence, we get $(n-1) + 2(n/2-1) + 4(n/4-1) + \ldots + n/2(2-1) = n\lg n - (n-1) < n \lg n$.

3 A tree-based view: Decision trees

A handy way to visualize and analyze algorithms in the comparison model is to consider the *decision tree* of the algorithm. A decision tree is a binary tree that is defined as follows.

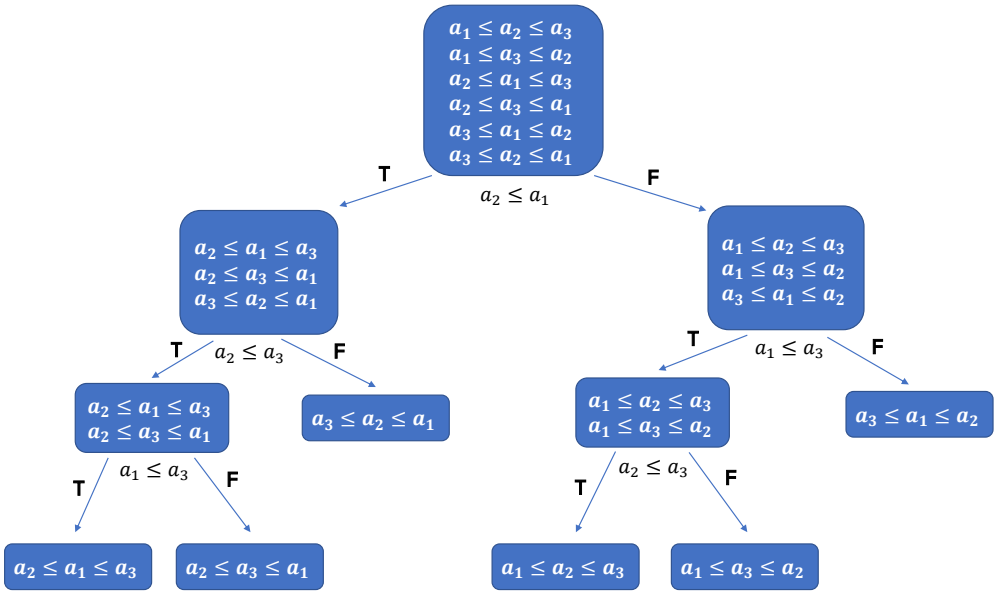
Definition: Decision Tree

The decision tree is a way of representing an algorithm in the comparison model. It is a binary tree such that:

- At the root node, it contains a list of all the possible valid outputs for the problem. For example, for sorting, the root node contains all $n!$ permutations
- Each internal node corresponds to a particular comparison operation, e.g., $a_5 < a_6$. The left child of the node corresponds to the result of the comparison being true, and the right node corresponds to it being false
- Each internal node consists of a list of possible valid outputs given the results of all of the comparison operations of its ancestors
- Leaf nodes correspond to a single valid output

Note that very importantly, a decision tree corresponds to a **specific algorithm**, but does not depend on the input to the algorithm (except that it may depend on the **size** of the input n). The decision tree encodes the exact behavior of the algorithm, so given a decision tree, you could simulate the algorithm on any given

input by starting at the root, performing comparisons, and traveling down a path in the tree until you reach a leaf, which is the output of the algorithm. You can think of it like a flowchart that represents the behavior of the algorithm. Here is an example decision tree for some hypothetical sorting algorithm for $n = 3$.



It turns out that decision trees can be a useful tool for analyzing lower bounds. We must keep in mind that a decision tree always represents a particular algorithm, so to prove a lower bound, we must argue about the structure of *any possible decision tree* for the problem. Observe that the worst-case number of comparisons for the algorithm corresponds exactly to the longest root-to-leaf path, i.e., the height of the tree. Therefore, if we can argue about the height of any possible decision tree, we have an argument for a lower bound! Let's see the sorting lower bound theorem from before proved using decision trees.

Proof of Theorem 1 using Decision Trees. There are $n!$ possible sorted orders, each of which must appear as a leaf in the tree. Let the height of the tree be denoted by h, and then note that a binary tree of height h can not have more than 2^h leaves. Therefore $n! \le 2^h$ and so taking the log of both sides gives us $h \ge \lg n!$ which implies that $\lg n!$ is a lower bound for the problem. □

4 Selection in the comparison model

4.1 Finding the maximum of n elements

How many comparisons are necessary and sufficient to find the maximum of n elements, in the comparison model of computation?

Claim: Upper bound on select-max in the comparison model

$n - 1$ comparisons are sufficient to find the maximum of n elements.

Proof. Just scan left to right, keeping track of the largest element so far. This makes at most $n - 1$ comparisons. □

Now, let's try for a lower bound. One simple lower bound is that since there are n possible answers for the location of the maximum element, our previous argument gives a lower bound of $\lg n$. But clearly this is not at all tight. Also, we have to look at all the elements (else the one not looked at may be larger than all the ones we look at). But looking at all n elements could be done using $n/2$ comparisons; not tight either. In fact, we can give a better lower bound of $n - 1$.

Claim: Lower bound on select-max in the comparison model

$n - 1$ comparisons are needed in the worst-case to find the maximum of n elements.

Proof. Suppose some algorithm $\mathcal{A}$ claims to find the maximum of n elements using less than $n-1$ comparisons. Consider an arbitrary input of n distinct elements, and construct a graph in which we join two elements by an edge if they are compared by $\mathcal{A}$. If fewer than $n - 1$ comparisons are made, then this graph must have at least two components. Suppose now that algorithm $\mathcal{A}$ outputs some element u as the maximum, where u is in some component C_1. In that case, pick a different component C_2 and add a large positive number (e.g., the value of u) to every element in C_2. This process does not change the result of any comparison made by $\mathcal{A}$, so on this new set of elements, algorithm $\mathcal{A}$ would still output u. Yet this now ensures that u is not the maximum, so $\mathcal{A}$ must be incorrect. □

Since the upper and lower bounds are equal, the bound of $n - 1$ is tight. Note that this argument was different from the "information theoretic" bound we used for sorting. Here we showed that if the algorithm makes "too few" comparisons on some input In and outputs *out*, we can give another input In' where the algorithms would do the same comparisons and receive the same answers to them, and hence also output *out*, but *out* is the incorrect output for input In'.

4.2 An Adversary Argument

A slightly different lower bound argument comes from showing that if an algorithm makes "too few" comparisons, then an adversary can fool it into giving the incorrect answer. Here is a little example. We want to show that any deterministic sorting algorithm on 3 elements must perform at least 3 comparisons in the worst case. (This result follows from the information theoretic lower bound of $\lceil \lg 3! \rceil = 3$, but let's give a different proof.)

If the algorithm does fewer than two comparisons, some element has not been looked at, and the algorithm must be incorrect. So after the first comparison, the three elements are w the winner of the first query, l the loser, and z the other guy. If the second query is between w and z, the adversary replies $w > z$; if it is between l and z, the adversary replies $l < z$. Note that in either case, the algorithm must perform a third query to be able to sort correctly.

In this kind of argument the goal is to construct an adversary Ada who will answer the algorithm's comparisons in such a way that (a) all Ada's answers are consistent with some input In, and (b) her answers make the algorithm perform "many" comparisons.

4.3 Finding the second-largest of n elements

How many comparisons are necessary (lower bound) and sufficient (upper bound) to find the second largest of n elements? Again, let us assume that all elements are distinct.

Claim: Lower bound on select-second-max in the comparison model

$n - 1$ comparisons are needed in the worst-case to find the second-largest of n elements.

Proof. The same argument used in the lower bound for finding the maximum still holds. □

Let us now work on finding an upper bound. Here is a simple one to start with.

Claim: Upper bound #1 on select-second-max in the comparison model

$2n - 3$ comparisons are sufficient to find the second-largest of n elements.

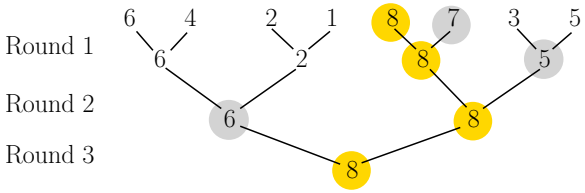
Proof. Just find the largest using $n - 1$ comparisons, and then the largest of the remainder using $n - 2$ comparisons, for a total of $2n - 3$ comparisons. □

We now have a gap: $n - 1$ versus $2n - 3$. It is not a huge gap: both are $\Theta(n)$, but remember today's theme is tight bounds. So, which do you think is closer to the truth? It turns out, we can reduce the upper bound quite a bit:

Claim: Upper bound #2 on select-second-max in the comparison model

$n + \lg n - 2$ comparisons are sufficient to find the second-largest of n elements.

Proof. As a first step, let's find the maximum element using $n - 1$ comparisons, but in a tennis-tournament or playoff structure. That is, we group elements into pairs, finding the maximum in each pair, and recurse on the maxima. E.g.,



Now, given just what we know from comparisons so far, what can we say about possible locations for the second-highest number (i.e., the second-best player)? The answer is that the second-best must have been directly compared to the best, and lost.[1] This means there are only $\lg n$ possibilities for the second-highest number, and we can find the maximum of them making only $\lg(n) - 1$ more comparisons. □

At this point, we have a lower bound of $n - 1$ and an upper bound of $n + \lg(n) - 2$, so they are nearly tight. It turns out that, in fact, the lower bound can be improved to exactly meet the upper bound.[2]

5 Sorting in the exchange model

Consider a shelf containing n unordered books to be arranged alphabetically. In each step, we can swap any pair of books we like. How many swaps do we need to sort all the books? Formally, we are considering the problem of *sorting* in the *exchange model*.

[1] Apparently the first person to have pointed this out was Charles Dodgson (better known as Lewis Carroll!), writing about the proper way to award prizes in lawn tennis tournaments.

[2] First shown by Kislitsyn (1964).

Definition: The Exchange Model

In the **exchange model**, an input consists of an array of n items, and the only operation allowed on the items is to swap a pair of them at a cost of 1 step. All other (planning) work is free: in particular, the items can be examined and compared to each other at no cost.

Question: how many exchanges are necessary (lower bound) and sufficient (upper bound) in the exchange model to sort an array of n items in the worst case?

Claim: Upper bound on sorting in the exchange model

$n-1$ exchanges is sufficient.

Proof. For this we just need to give an algorithm. For instance, consider the algorithm that in step 1 puts the smallest item in location 1, swapping it with whatever was originally there. Then in step 2 it swaps the second-smallest item with whatever is currently in location 2, and so on (if in step k, the kth-smallest item is already in the correct position then we just do a no-op). No step ever undoes any of the previous work, so after $n-1$ steps, the first $n-1$ items are in the correct position. This means the nth item must be in the correct position too. □

But are $n-1$ exchanges necessary in the worst-case? If n is even, and no book is in its correct location, then $n/2$ exchanges are clearly necessary to "touch" all books. But can we show a better lower bound than that?

Claim: Lower bound on sorting in the exchange model

In fact, $n-1$ exchanges are necessary, in the worst case.

Proof. Here is how we can see it. Create a graph in which a directed edge (i, j) means that that the book in location i must end up at location j. An example is given in Figure 1.

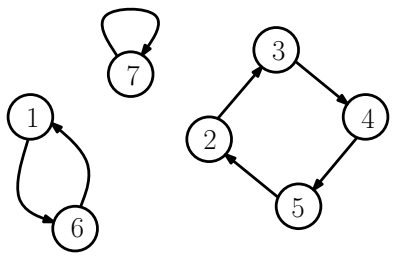


Figure 1: Graph for input `[f c d e b a g]`

This is a special kind of directed graph: it is a permutation — a set of cycles. In particular, every book points to *some* location, perhaps its own location, and every location is pointed to by exactly one book. Now consider the following points:

1. What is the effect of exchanging any two elements (books) that are in the same cycle?

   Answer: Suppose the graph had edges (i_1, j_1) and (i_2, j_2) and we swap the elements in locations i_1 and i_2. Then this causes those two edges to be replaced by edges (i_2, j_1) and (i_1, j_2) because now it is the element in location i_2 that needs to go to j_1 and the element in i_1 that needs to go to j_2. This means that if i_1 and i_2 were in the same cycle, that cycle now becomes two disjoint cycles.

2. What is the effect of exchanging any two elements that are in different cycles?

   Answer: If we swap elements i_1 and i_2 that are in different cycles, then the same argument as above shows that this merges those two cycles into one cycle.

3. How many cycles are in the final sorted array?

   Answer: The final sorted array has n cycles.

Putting the above 3 points together, suppose we begin with an array consisting of a single cycle, such as $[n, 1, 2, 3, 4, \ldots, n-1]$. Each operation at best increases the number of cycles by 1 and in the end we need to have n cycles. So, this input requires $n-1$ operations. □

6 Query models, and the evasiveness of connectivity

Optional content — Will not appear on the homeworks or the exams

To finish with something totally different, let's look at the query complexity of determining if a graph is connected. Assume we are given the adjacency matrix G for some n-node graph. That is, $G[i,j] = 1$ if there is an edge between i and j, and $G[i,j] = 0$ otherwise. We consider a model in which we can *query* any element of the matrix G in 1 step. All other computation is free. That is, imagine the graph matrix has values written on little slips of paper, face down. In one step we can turn over any slip of paper. How many slips of paper do we need to turn over to tell if G is connected?

Claim: Easy upper bound

$n(n-1)/2$ queries are sufficient to determine if G is connected.

Proof. This just corresponds to querying every pair (i,j). Once we have done that, we know the entire graph and can just compute for free to see if it is connected. □

Interestingly, it turns out the simple upper-bound of querying every edge is a lower bound too. Because of this, connectivity is called an "evasive" property of graphs.

Claim: Lower bound

$n(n-1)/2 = \binom{n}{2}$ queries are necessary to determine connectivity in the worst case.

Here are two proofs of this theorem.

Proof 1. We think of this as a game between two players: the *evader* and the *querier*. The querier asks a sequence of questions of the form "Is there an edge between vertices x and y?". For each question the evader must answer the question. The goal of the evader is to force the querier to ask as many questions as possible. We'll give a strategy for the evader which forces the querier to ask $\binom{n}{2}$ questions to determine if the graph is connected.

The evader's strategy will be to maintain the following invariant.

> At any point in time the edges that have been declared to exist form a forest of trees involving all the vertices of the graph. For each tree T all queries among the vertices of the T have already been asked. And for every pair of trees T and T' in the forest there exists a pair (x,y) with $x \in T$ and $y \in T'$ that has not been queried.

Note that until there is just one tree, the querier does not know if the graph is connected or not.

Initially each tree is just one vertex, and the invariant trivially holds. The evader maintains the invariant as follows. If the query is for two vertices in the same tree, then the evader just gives the answer it gave before to this query. Nothing changes and the invariant still holds.

Say the query is for vertices x and y in two different trees T_x and T_y. Then the evader determines if all the other possible edges between T_x and T_y have already been queried. If that is the case, then the evader answers 1 for "yes", otherwise it answers 0 for "no". Note that if it answers yes, (and joins the two trees) then, by induction, all the edges within the set $T_x \bigcup T_y$ have already been queried, and the invariant holds. (If it answers "no" the invariant trivially still holds.)

At any point in time before there is just one tree it is not known if the graph is connected or not. Finally when there is just one tree the graph is connected, but by that point all of the $\binom{n}{2}$ the edges have been queried. □

Proof 2. Here is the strategy for the adversary: when the algorithm asks us to flip over a slip of paper, we return the answer 0 *unless* that would force the graph to be disconnected, in which case we answer 1. (It is not important to the argument, but we can figure this out by imagining that all un-turned slips of paper are 1 and seeing if that graph is connected.) Now, here is the key claim:

> *Claim:* we maintain the invariant that for any un-asked pair (u, v), the graph revealed so far has no path from u to v.
>
> *Proof of claim:* If there was, consider the last edge (u', v') revealed on that path. We could have answered 0 for that and kept the same connectivity in the graph by having an edge (u, v). So, that contradicts the definition of our adversary strategy.

Now, to finish the proof: Suppose an algorithm halts without examining every pair. Consider some unasked pair (u, v). If the algorithm says "connected," we reveal all-zeros for the remaining unasked edges and then there is no path from u to v (by the key claim) so the algorithm is wrong. If the algorithm says "disconnected," we reveal all-ones for the remaining edges, and the algorithm is wrong by definition of our adversary strategy. So, the algorithm must ask for all edges. □

Ending Notes: Regarding Models of Computation

Most of real-world software programs are ran on Von Neumann architectures, which reflect the computers we use today. Using complexity theoretic terms, this is called the Random Access Machine (RAM) model. This is also the default model of computation that we use for analyzing algorithms. In the RAM model, the primary metric we often care about is the running time of the algorithm. We also care about the space consumption.

There are also other models of computation. For example, hardware circuitry use the "circuit" model of computation. Interestingly, modern cryptographic techniques such as "computation on encrypted data" also uses circuit to model the underlying computation. In the circuit model of computation, we care about the circuit's size and depth.

So why do we care about bounding the cost in the concrete models seen in class today, such as number of comparisons and number of queries? Isn't the RAM model the primary model of computation we care about? There are many reasons why we care:

- Proving a lower bound in the concrete model often tells us the limit of a class of algorithms, e.g., comparison-based algorithms cannot sort an array in $o(nlogn)$ cost, so if we want $o(nlogn)$, we have to use a non-comparison-based algorithm. This is very useful in algorithm design as a general guide as well as sanity check.
- Lower bounds in a concrete model often tells us information-theoretic limits. For example, the query-model lower bound for graphs tells us if the graph is stored on a server, and we want to determine whether it's connected, in the worst case we have to download all pairs of the adjacency matrix.
- Often times, not all operations are of the same cost. In these cases, we often care about optimizing the cost of the most heavy-weight operation or building block. This approach is used a lot in analyzing and improving the performance of practical systems and architectures. In cryptography, we often adopt this approach too, e.g., public-key operations are more expensive than secret-key operations, so we often focus on optimizing public-key operations.