15451 Spring 2023

Mechanism Design

a.k.a **inverse game theory**

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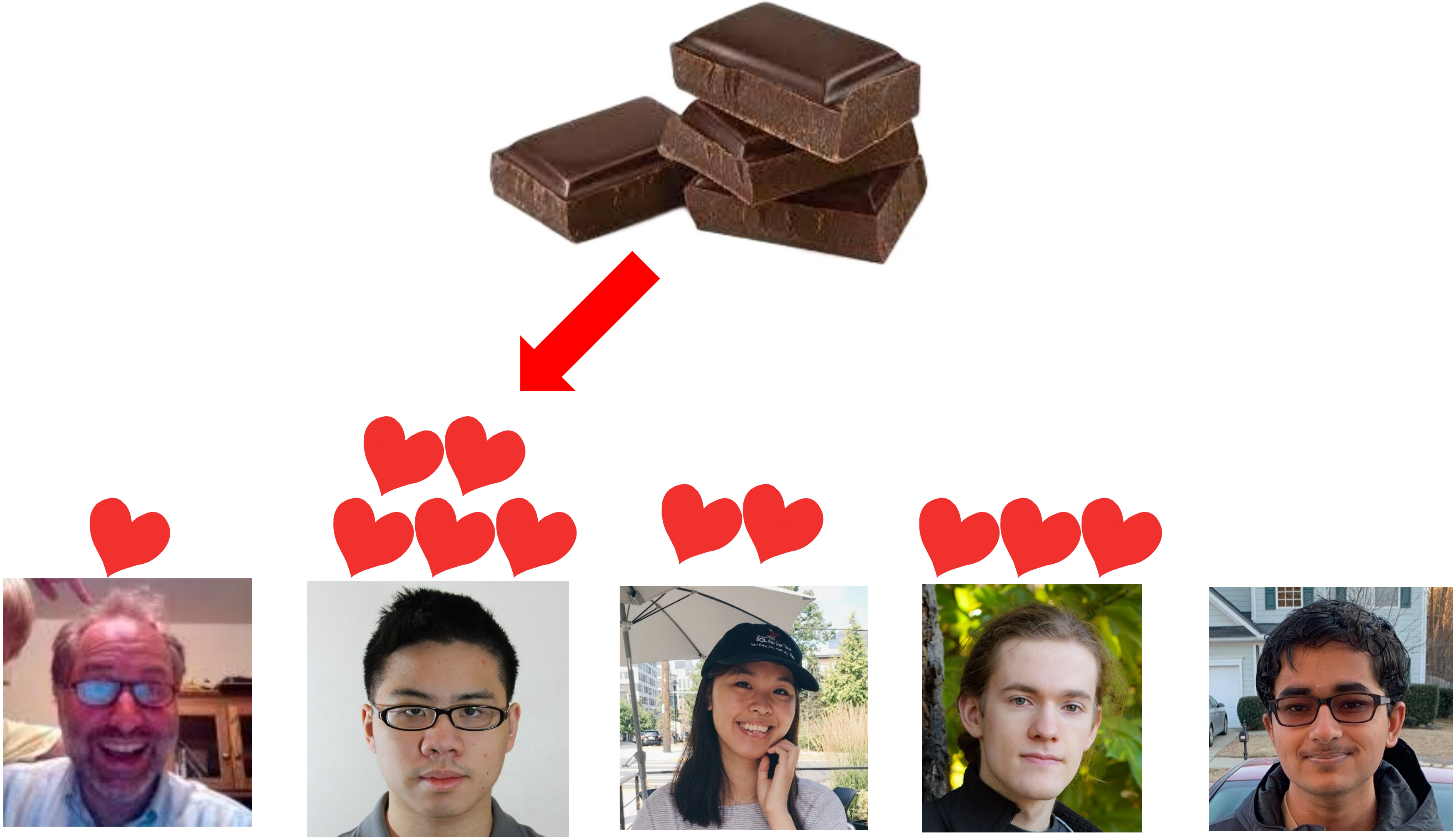
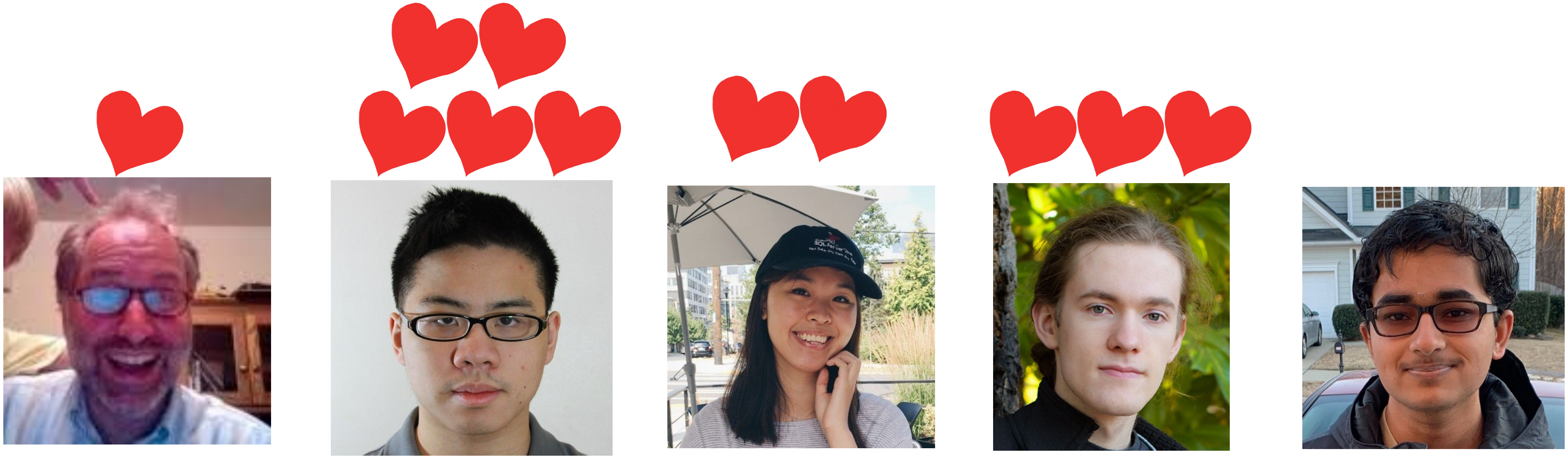
to give away



Maximizes **social welfare**

= sum of happiness



Let's do this in class

On a scale of **0-5** how much do you like

Problem:

Everyone is incentivized to report 5

i.e., mechanism incentivizes lying

How do I give away 

such that

 Maximize social welfare

 Incentivize truthful bidding

Mechanism Design

a.k.a. inverse game theory

- Maximize social welfare
- Incentivize truthful bidding

Suppose we can charge **$$**

- Everyone bids
- Highest bidder wins
- Winner pays its bid

"1st price auction"

- Everyone bids
- Highest bidder wins
- Winner pays its bid

(boldface)

$\boldsymbol{v}$

- v_i true value of i-th bidder
- b_i : i-th bid $\qquad \mathbf{b} = (b_1 \ldots b_n)$
- **Allocation rule:** $\mathbf{x}(\mathbf{b}) := (x_1 \ldots x_n) \in \{0,1\}^n$
- **Payment rule:** $\mathbf{p}(\mathbf{b}) := (p_1 \ldots p_n)$

assume $p_i = 0$ if i is not winner

$p_i \geq 0 \quad \forall i$

- v_i : true value of i-th bidder
- b_i : i-th bid
- p_i : payment of bidder i
- **Utility** of bidder i:

$$u_i = \begin{cases} v_i - p_i & \text{if i wins} \\ 0 & \text{o.w.} \end{cases}$$

Example:

1st price auction: anyone who bids honestly has utility 0

Maximize social welfare:
give to whoever has the highest true value

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utility of i*

utility of auctioneer

Social welfare = v_{i*} - p_{i*} + p_{i*} = v_{i*}

where i* = winner

Does 1st price auction **incentivize truthful bidding?**

- Everyone bids
- Highest bidder wins
- Winner pays its bid

Does 1st price auction **incentivize truthful bidding?**

Incentivized to **underbid**

Example:

- My true value = 10
- I know that the 2nd highest bidder bids 9

What should I do?

9.1

2nd price auction a.k.a. Vickrey auction

- Highest bidder wins
- Winner pays **2nd highest bid**

example: bids: 10 7 6 5 1

winner: ↑ (10)

payment: 7

Claim: 2nd price auction is "**dominant-strategy incentive-compatible**", i.e.,

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for any valuations $v_1, v_2, \ldots, v_n$, for any player i, for any bid vector $\mathbf{b}_{-i}$ of all other players, and any v_i',

i's util when bidding truthfully — i's util when bidding an arbitrary bid

$$u_i(\text{Vickrey}(v_i, \mathbf{b}_{-i})) \geq u_i(\text{Vickrey}(v_i', \mathbf{b}_{-i}))$$

where $(v_i', \mathbf{b}_{-i})$ means letting everyone else's bids be $\mathbf{b}_{-i}$ and let the i-th bid be v_i'

Even after seeing others' bids, I still want to bid truthfully

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also called an ex-post Nash equilibrium

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Claim: 2nd price auction is **dominant-strategy incentive-compatible**

Proof: consider an arbitrary use i. let b^* be the highest bid among b_{-i}

Case 1: $v_i > b^*$:
- if i bids truthfully, i's util $= v_i - b^*$
- if i bids $> b^*$ — i's util $= v_i - b^*$
- if i bids $= b^*$ — i's util $\leq v_i - b^*$
- if i bids $< b^*$ — i's util $= 0$

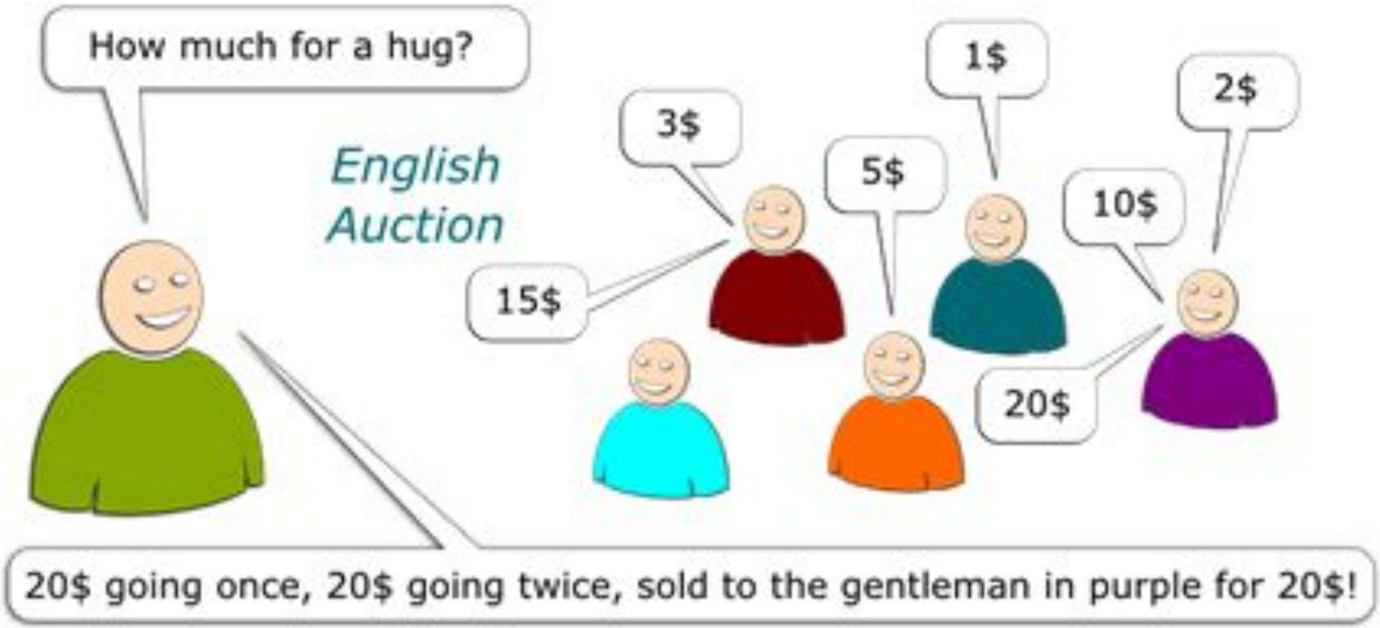
Case 2: $v_i = b^*$
- truthful: util $= 0$
- bid $> b^*$: util $= 0$
- bid $< b^*$: util $= 0$

Case 3: $v_i < b^*$

2nd price auction in the real world

ebay

→ 10
5+ε
5
1+ε
→ 1



Multiple identical items, say 2 items

Multiple identical items, say 2 items

7.5

10 8 7 5 2 1

Top 2 bidders are winners.

Top bidder pays 2nd price.

2nd top bidder pays 3rd price.

Is this dominant-strategy incentive compatible?

Second try

Top 2 bidders are winners.
Both winners pay 3rd price.

Is this dominant-strategy incentive compatible?

Vickrey-Clarke-Groves (VCG) Auction

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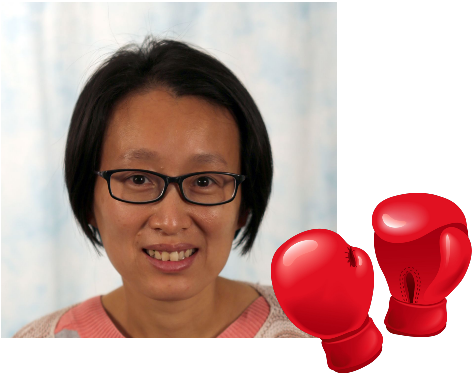
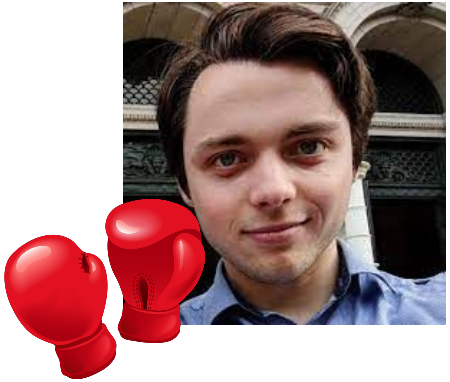
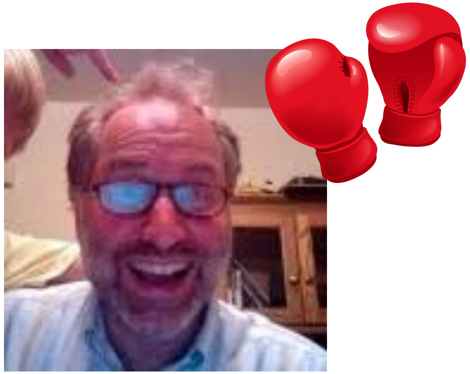




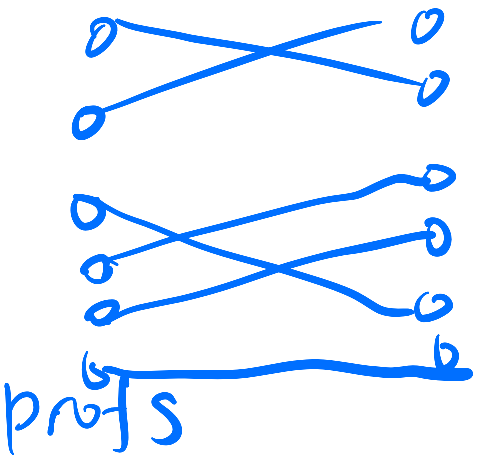



Dance Dance Revolution SuperNOVA
KONAMI

$\underline{A}$: set of alternatives (or "allocations")

example

$a \in A$ is a perfect matching btw. profs and offices



$$A\text{:} \quad \text{set of alternatives (or ”allocations”)}$$

$$v_i : A \to \mathbb{R}_{\geq 0}\text{:} \quad \text{maps allocations to valuations}$$

A:	set of alternatives (or "allocations")
$v_i : A \to \mathbb{R}_{\geq 0}$:	maps allocations to valuations
$u_i(a, p)$:	utility of user i, $u_i(a, p) = v_i(a) - p$
$SW(a)$:	social welfare $SW(a) = \sum_i v_i(a)$

Example

Direct revelation mechanism

- Everyone reveals their true values
  $\mathbf{v} = (v_{1}, v_{2}, \ldots v_{n})$
- **Allocation rule:** $f(\mathbf{v}) = a$
- **Payment rule:** $\mathbf{p}(\mathbf{v}) = (p_1, \ldots p_n)$

A direct revelation (f, p) mechanism is incentive compatible iff

for every $\mathbf{v} = (v_1, \ldots, v_n)$, every i, every v_i', we have

$$v_i(f(\mathbf{v})) - p_i(\mathbf{v}) \geq v_i(f(v_i', \mathbf{v}_{-i})) - p_i(v_i', \mathbf{v}_{-i})$$

i's util when bidding truthfully

i's util when it bid an arbitrary v_i'

A direct revelation (f, p) mechanism is incentive compatible iff

for every $\mathbf{v} = (v_1, \ldots, v_n)$, every i, every v'_i, we have

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Misreporting never helps

Claim: **There is a mechanism that**

- Is dominant-strategy incentive compatible
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"The VCG auction"

1st try

- f(**v**) = the allocation that maximizes social welfare w.r.t **v**
- p_i(**v**) = sum of everyone else's reported valuations

i.e., $$p_i(\mathbf{v}) = -\sum_{j \neq i} v_j(f(\mathbf{v}))$$

play i's goal: maximize $v_i(f(v)) + \sum_{j \neq i} v_j(f(v))$

$\sum_j v_j(f(v))$

Analysis

If player i reports truthfully, then

$$u_i = v_i(f(\mathbf{v})) + \sum_{j \neq i} v_j(f(\mathbf{v})) = \sum_j v_j(f(\mathbf{v})) = \max_a \sum_j v_j(a)$$

Analysis

If player i reports truthfully, then

$$u_i = v_i(f(\mathbf{v})) + \sum_{j \neq i} v_j(f(\mathbf{v})) = \sum_j v_j(f(\mathbf{v})) = \max_a \sum_j v_j(a)$$

What if player i misreports v'_i ?

Analysis

If player i reports truthfully, then

$$u_i = v_i(f(\mathbf{v})) + \sum_{j \neq i} v_j(f(\mathbf{v})) = \sum_j v_j(f(\mathbf{v})) = \max_a \sum_j v_j(a)$$

If player i misreports v'_i, then

$$u_i = v_i(f(\mathbf{v}')) + \sum_{j \neq i} v_j(f(\mathbf{v}')) = \sum_j v_j(f(\mathbf{v}')) \leq \max_a \sum_j v_j(a)$$

Misreporting does not help

If player i reports truthfully, then

$$u_i = v_i(f(\mathbf{v})) + \sum_{j \neq i} v_j(f(\mathbf{v})) = \sum_j v_j(f(\mathbf{v})) = \max_a \sum_j v_j(a)$$

If player i misreports v'$_i$, then

$$u_i = v_i(f(\mathbf{v}')) + \sum_{j \neq i} v_j(f(\mathbf{v}')) = \sum_j v_j(f(\mathbf{v}')) \leq \max_a \sum_j v_j(a)$$

Problem: auctioneer to give money to bidders

Add to each $p_i(v)$ something that depends only on $\mathbf{v}_{-i}$

VCG -- general version

- f(**v**) = the allocation that maximizes social welfare w.r.t **v**
- $p_i(\mathbf{v}) = h_i(\mathbf{v}_{-i}) - \sum_{j \neq i} v_j(f(\mathbf{v}))$

what should hi be??

How to choose $h_i(\mathbf{v}_{-i})$

- Auctioneer does not pay ✓
- Every bidder has non-negative utility ✓ (individual rationality)

max SW if the world is w/o i

$$h_i(\mathbf{v}_{-i}) = \max_a \sum_{j \neq i} v_j(a)$$

VCG -- standard version

- $f(\mathbf{v})$ = the allocation that maximizes social welfare w.r.t $\mathbf{v}$
- $p_i(\mathbf{v}) = \max_a \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(f(\mathbf{v}))$

"the cost I bring to the rest of the world"

"externality"

VCG -- standard version

- f(**v**) = the allocation that maximizes social welfare w.r.t **v**

- $p_i(\mathbf{v}) = \max_a \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(f(\mathbf{v}))$

$$= v_i - \left(\sum_j v_j(f(\mathbf{v})) - \max_a \sum_{j \neq i} v_j(a) \right)$$

Example

2nd price auction is a special case of VCG auction