15451 Spring 2023

Approximation Algorithms

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What can we do for NPC problems?

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Design poly-time approximation algorithms

Consider the solution version (rather than the decision version)

m machines, n jobs, j-th job takes time p_j

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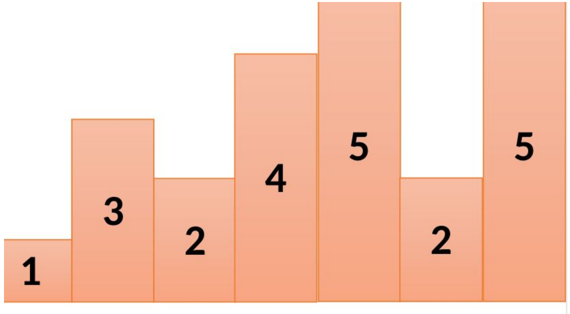
partition jobs to the machines to minimize the *makespan*, defined as

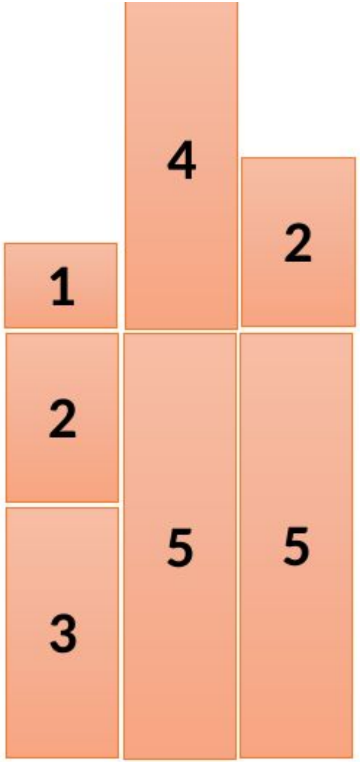
$$\max_{i \in [m]} \left(\sum_{j \in S_i} p_j \right)$$

where S_i is the set of jobs assigned to machine i

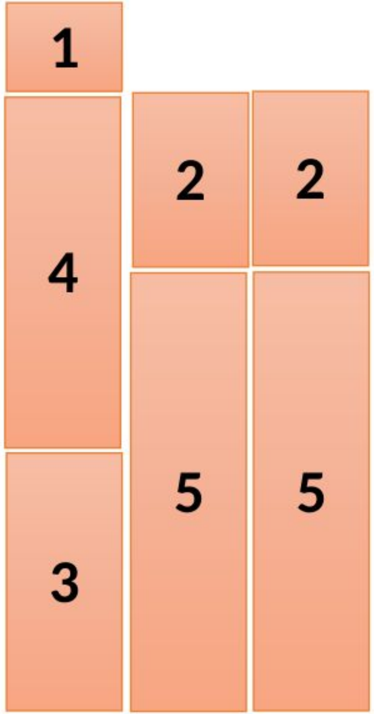
Example

m = 3





Makespan = 9



Makespan = 8

Claim:

The job assignment problem is NPC

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Proof: reduce subset sum to job assignment

Subset sum:

given $a_1, a_2, \dots a_n$, is there a subset that sum up to $\frac{1}{2} * (a_1 + a_2 + \dots + a_n)$?

Greedy:

Take any unassigned job, give it to the machine with current minimum load

Claim:

Greedy achieves **2**-approximation, i.e., achieves a makespan at most 2 times the optimal

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Observe that $OPT \geq p_{j^*}$, $OPT \geq L$

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What is a bad case?

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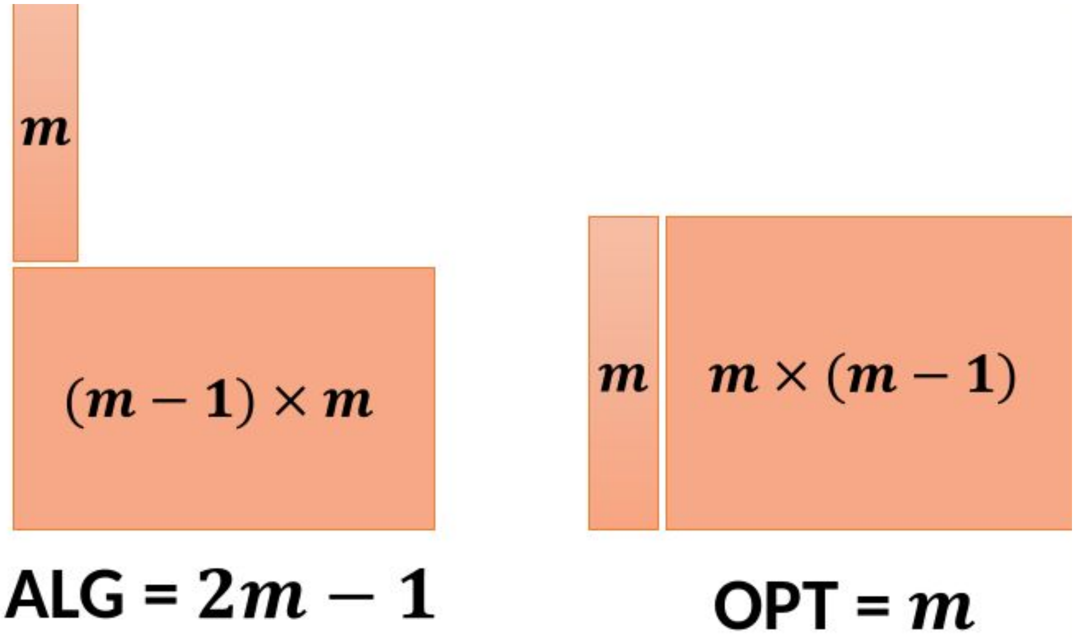
m(m-1) jobs of size 1,
1 job of size m



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1 job of size m

Sorted Greedy:

Run greedy largest unassigned job first

Let i* be the most loaded machine, let j* be the last job assigned to it

Want to show:

$$p_{j*} + L \leq 1.5 \cdot OPT$$

Suffices to show: $p_{j*} \leq OPT/2$

We may assume $L > 0$ (why?)

Every machine has at least one job assigned before, and its size is at least p_{j^*}

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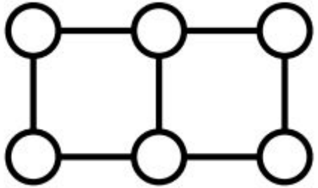
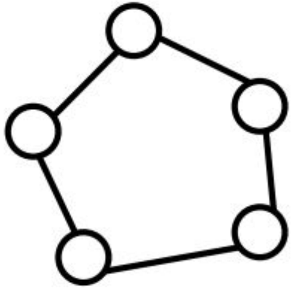
Every machine has at least one job assigned before, and its size is at least p_{j^*}

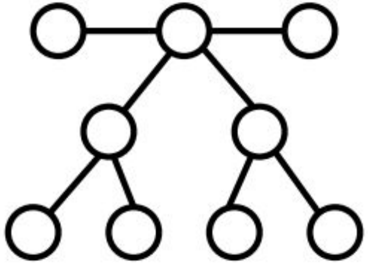
$$2p_{j^*} \leq OPT$$

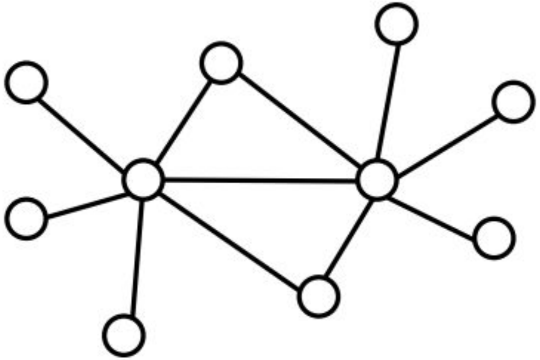
Vertex-Cover:
Given a graph G, find the smallest set of vertices such that every edge is incident to at least one of them.

Decision problem: Given G and k, does G contain a vertex cover of size ≤ k?









Pick an arbitrary vertex with at least one uncovered edge incident to it, put it into the cover, and repeat.

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Does this give a good approximation?

Pick the vertex that covers the most uncovered edges

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Achieves only O(log n) approximation

A 2-approximation algorithm

Pick an arbitrary edge. Add both endpoints.
Throw out all edges covered and repeat.
Continue until no uncovered edges left.

Proof:

Alg finds a matching M, with |M| edges

Observe that

$$|M| \leq OPT$$
$$Alg = 2\,|M|$$

Another 2-approximation algorithm

- **Find an LP solution (possibly fractional)**

  $0 <= x_i <= 1$, for each edge (i, j), $x_i + x_j >= 1$

  Minimize $\Sigma_i\, x_i$

- **Round the LP solution:**

  If $x_i >= ½$, set to 1, else set to 0

Example

Proof:

LP ≤ OPT

Rounded solution ≤ 2 LP

Known Results

Best known approximation:

$$2 - \Theta(1/\sqrt{\log n})$$

7/6 approximation is NP hard [Hastad]

Improved to 1.361 [Dinur and Safra]

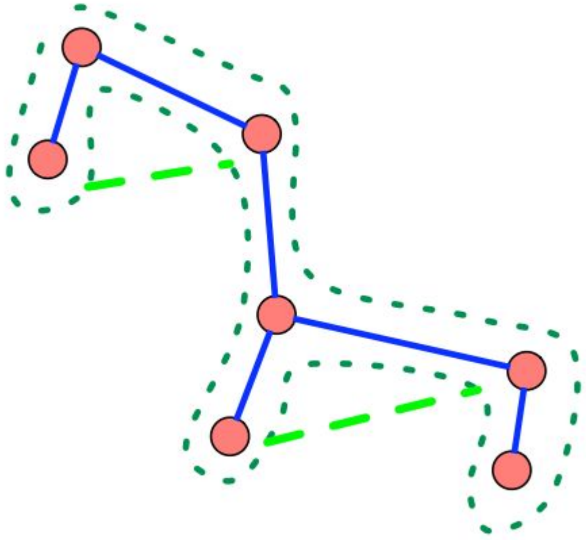
Metric Traveling Salesman Problem

Find the shortest path to visit n cities, each exactly once, returning to where you started.

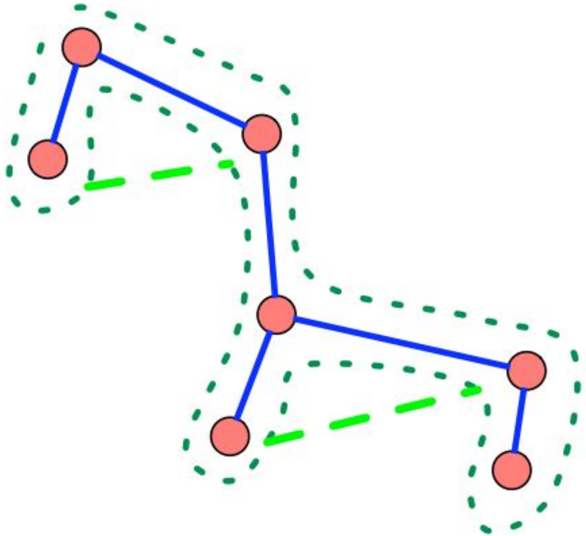
Metric: distances are symmetric and obey triangle inequality

Compute MST

Output a pre-order traversal of the MST



This algorithm achieves 2 approximation.

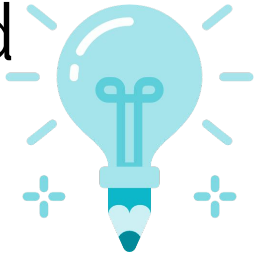


Christofide's algorithm

Compute the MST T.

Compute a minimum weight perfect matching M between the vertices of odd degree in T.

G = T ∪ M. Return the TSP tour constructed from shortcutting an Eulerian tour on G.



Karlin, Klein, and Oveis Gharan improves it to 1.49999999999999999999999999999999999 approximation.

(Not an April fools joke.
See https://arxiv.org/abs/2007.01409.)