15451 Spring 2023

Multiplicative Weights Algorithm

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Stock Market

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You discover it at the end of the day

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There are n experts to help you, each of them makes a prediction at the beginning of day

Goal

Design an algorithm whose performance is close to the **best expert in hindsight**

"minimize regret"

Discussion: why not compare with the best **algorithm** in hindsight

Goal

Design an algorithm whose performance is close to the **best expert in hindsight**

Warmup: you're promised that there exists an expert who is always right n : # experts

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Claim: there is an algorithm that makes at most $\log_2 n$ mistakes

Algorithm: take majority vote among experts
kick out whoever is wrong

If we make a mistake, then at least half of the remaining experts will be kicked out

Warmup: promised that the best expert makes **at most M** mistakes

Claim: there is an algorithm that makes at most **$(M+1)(\log_2 n + 1)$** mistakes

Algorithm: run the previous algorithm
once you have kicked out everyone, restart

phase: start $\longrightarrow$ kicked out everyone.

claim: in each phase, alg makes at most
$\log_2 n + 1$ mistakes

total mistakes

$$\leq M \cdot (\log_2 n + 1) + \log_2 n \leq (M+1)(\log_2 n + 1)$$

Warmup: promised that the best expert makes **at most M** mistakes

Claim: there is an algorithm that makes at most $\mathbf{2.41\,(M + \log_2 n)}$ mistakes

Deterministic Weighted Majority

- Initially, every expert has weight 1.
- When an expert makes a mistake, half its weight.
- Output the prediction of the weighted majority.

Analysis

sum of weights

Idea: relate Φ to perf of our alg / perf of the best expert

Let $\Phi := \sum_{i=1}^{n} w_i$

Analysis

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Claim: If our alg makes a mistake, then

$$\Phi_{new} \leq \frac{3}{4}\Phi_{old}.$$

Intuition: if alg makes many mistakes, then final sum of weights is small

Analysis

Let $\Phi := \sum_{i=1}^{n} w_i$

Claim: If our alg makes a mistake, then

$$\Phi_{new} \leq \frac{3}{4}\Phi_{old}.$$

Note: Φ can never increase

Claim: If our alg makes a mistake, then

$$\Phi_{new} \leq \frac{3}{4}\Phi_{old}.$$

Proof:

Claim: If our alg makes a mistake, then

$$\Phi_{new} \le \frac{3}{4}\Phi_{old}.$$

Proof: if we make a mistake, at least half of the weight will get halved

"**Intuition:** if alg makes many mistakes, then final sum of weights is small"

Formally, if we make **m** mistakes, then

$$\Phi_{final} \leq (3/4)^m \cdot \Phi_{init} = (3/4)^m \cdot n.$$

OTOH: if best expert makes few mistakes, final sum of weights must be high

Formally, if i* makes M mistakes, then

$$\Phi_{final} \geq w_{i^*} = (1/2)^M.$$

$$\Phi_{final} \leq (3/4)^m \cdot \Phi_{init} = (3/4)^m \cdot n.$$

$$\Phi_{final} \geq w_{i^*} = (1/2)^M.$$

$$m \leq \frac{1}{\log_2(4/3)} \cdot (\log_2 n + M)$$

$$\leq 2.41(\log_2 n + M)$$

$$(1/2)^M \leq (3/4)^m \cdot n \quad \Rightarrow \quad (4/3)^m \leq n2^M.$$

$$m \log_2(4/3) \leq \log_2 n + M$$

Claim: can improve the constant 2.41 to

$$2(1+\epsilon)M + O\left(\frac{\log n}{\epsilon}\right).$$

Idea: instead of halving, multiply by 1-ε

Assume $\varepsilon \le \frac{1}{2}$

Claim: can improve the constant 2.41 to

$$2(1+\epsilon)M + O(\frac{\log n}{\epsilon}).$$

Claim: 2 is optimal for any **deterministic** algorithm!

T: total # time steps

Why?

Construct an adversarial input that makes the det. alg. wrong every time.

Consider two experts, one always UP, one always Down

at least one of them makes $\leq \frac{1}{2}T$ mistakes

How can we overcome this 2 barrier?

How can we overcome this 2 barrier?

Use randomization!

(recall online paging in the last lecture)

Randomized weighted majority

(Multiplicative Weights Algorithm)

- Initially, every expert has weight 1.
- When an expert makes a mistake, multiply its weight by 1-ε
- Predict "up" with probability $\frac{\sum_{j \text{ says Up}} w_j}{\sum_j w_j}$;

Assume: ε ≤ 1/2

Randomized weighted majority

- Initially, every expert has weight 1.
- When an expert makes a mistake, multiply its weight by 1-ε
- Go with each expert i with prob $\frac{w_i}{\sum_j w_j}$

Theorem: suppose ε ≤ ½ and best expert makes M mistakes. Then the expected number of mistakes we make is at most

$$(1+\epsilon)M + \frac{\ln n}{\epsilon}.$$

$\le \sqrt{\frac{\ln n}{T}} \cdot T + \frac{\ln n}{\sqrt{\frac{\ln n}{T}}}$

$= 2\sqrt{T \cdot \ln n}$

T: total # days

$E[\text{our \# mistakes}] - OPT \le \varepsilon M + \frac{\ln n}{\varepsilon} \le \varepsilon \cdot T + \frac{\ln n}{\varepsilon}$

OPT ↑ best expert in hindsight

$\varepsilon T = \frac{\ln n}{\varepsilon}$

$\varepsilon = \sqrt{\frac{\ln n}{T}}$

Theorem: suppose ε ≤ ½ and best expert makes M mistakes. Then the expected number of mistakes we make is at most

$$(1+\epsilon)M + \frac{\ln n}{\epsilon}.$$

ε : learning rate

ε large: punish wrong expert more

Corollary:

Our expected # errors ≤ OPT + $2\sqrt{\frac{\ln n}{T}}$

Proof:

Proof: $\Phi := \sum_{i=1}^{n} w_i$

F_t: fraction of the total weight on the t th day on experts who make a mistake

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Q: What is the expected # total mistakes we make?

Proof: $\Phi := \sum_{i=1}^{n} w_i$

F_t: fraction of the total weight on the t th day on experts who make a mistake

Q: What is the expected # total mistakes we make?

A: $\sum_t F_t$.

E[# mistakes on day t] = Pr[mistake on day t] = F_t

linearity of expectation

Proof: $\Phi := \sum_{i=1}^{n} w_i$

day t: wrong $F_t \cdot \Phi_{old} \Rightarrow F_t \cdot \Phi_{old} \cdot (1-\epsilon)$; right $(1-F_t) \cdot \Phi_{old} \Rightarrow (1-F_t) \cdot \Phi_{old}$

F_t: fraction of the total weight on the t th day on experts who make a mistake

Claim: on the t-th day, $\Phi_{new} = \Phi_{old} \cdot (1 - \epsilon F_t)$.

$\Phi_{new} = F_t \cdot \Phi_{old} (1-\epsilon) + (1-F_t) \cdot \Phi_{old} = \Phi_{old} (F_t - \epsilon F_t + 1 - F_t)$

Intuition: if alg makes many mistakes in expectation, then final sum of weights is small

Proof: $\Phi := \sum_{i=1}^{n} w_i$

F_t: fraction of the total weight on the t th day on experts who make a mistake

Claim: on the t-th day, $\Phi_{new} = \Phi_{old} \cdot (1 - \epsilon F_t).$

Proof:

Intuition: if alg makes many mistakes in expectation, then final sum of weights is small

Formally,

initial sum weights

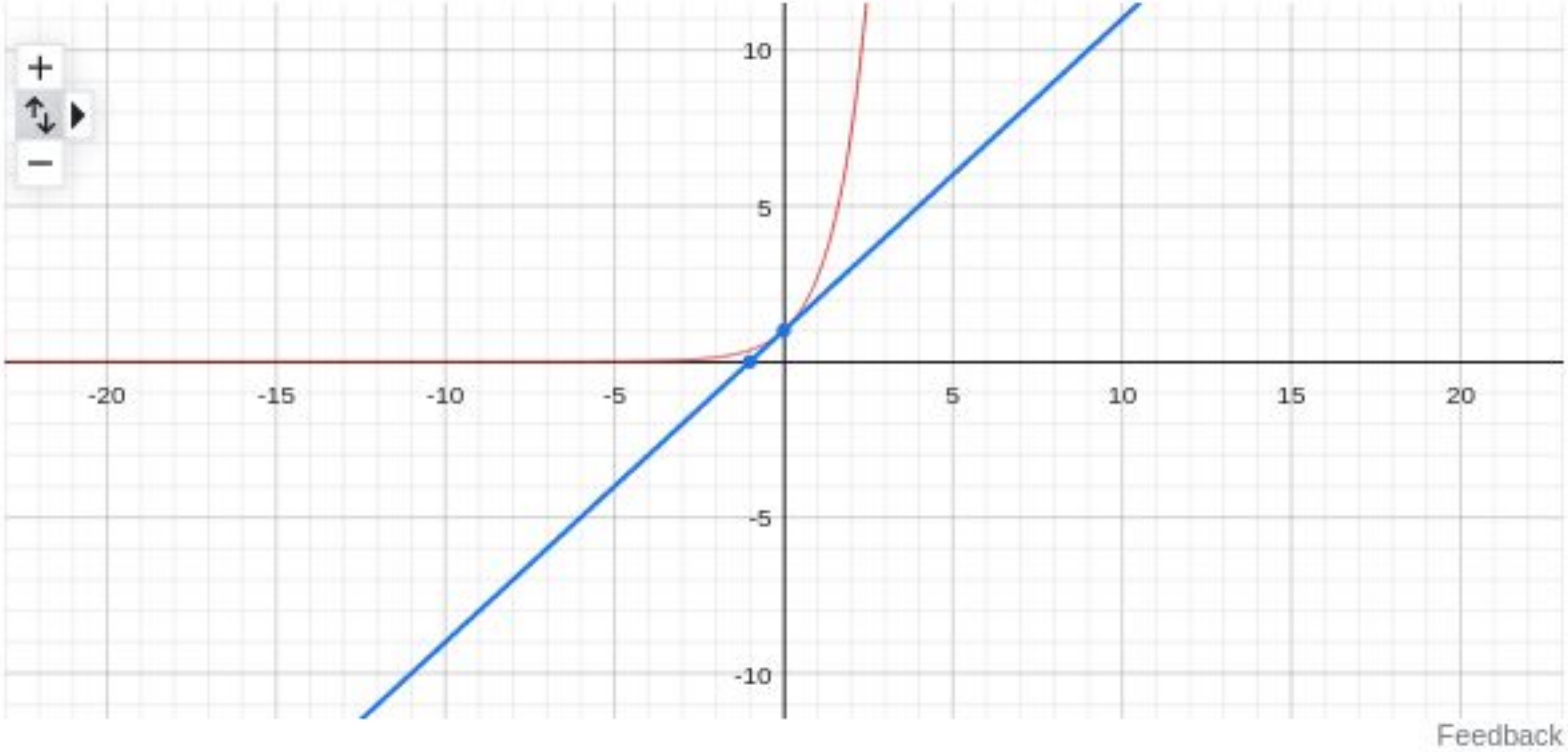
$$\Phi_{final} = n \cdot \prod_t (1 - \epsilon F_t) \leq n \cdot e^{-\epsilon \sum_t F_t}$$

$\leq n \cdot \prod_t e^{-\varepsilon F_t}$

$x = -\varepsilon F_t$

Recall: $1 + x \leq e^x$

Graph for **1+x**, e^x



OTOH: if best expert makes few mistakes, final sum of weights must be high

$$\Phi_{final} \geq (1-\epsilon)^M$$

$$\Phi_{final} \leq n \cdot e^{-\epsilon \sum_t F_t}$$

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$$\Phi_{final} \leq n \cdot e^{-\epsilon \sum_t F_t}$$

$$\Phi_{final} \geq (1 - \epsilon)^M$$

$$m \ln(1-\varepsilon) \leq \ln n - \varepsilon \cdot \textstyle\sum_t F_t$$

$$(1 - \epsilon)^M \leq n \cdot e^{-\epsilon \sum_t F_t} \quad \Rightarrow \quad \epsilon \sum_t F_t \leq M \ln \frac{1}{(1-\epsilon)} + \ln n.$$

$$\Phi_{final} \leq n \cdot e^{-\epsilon \sum_t F_t}$$

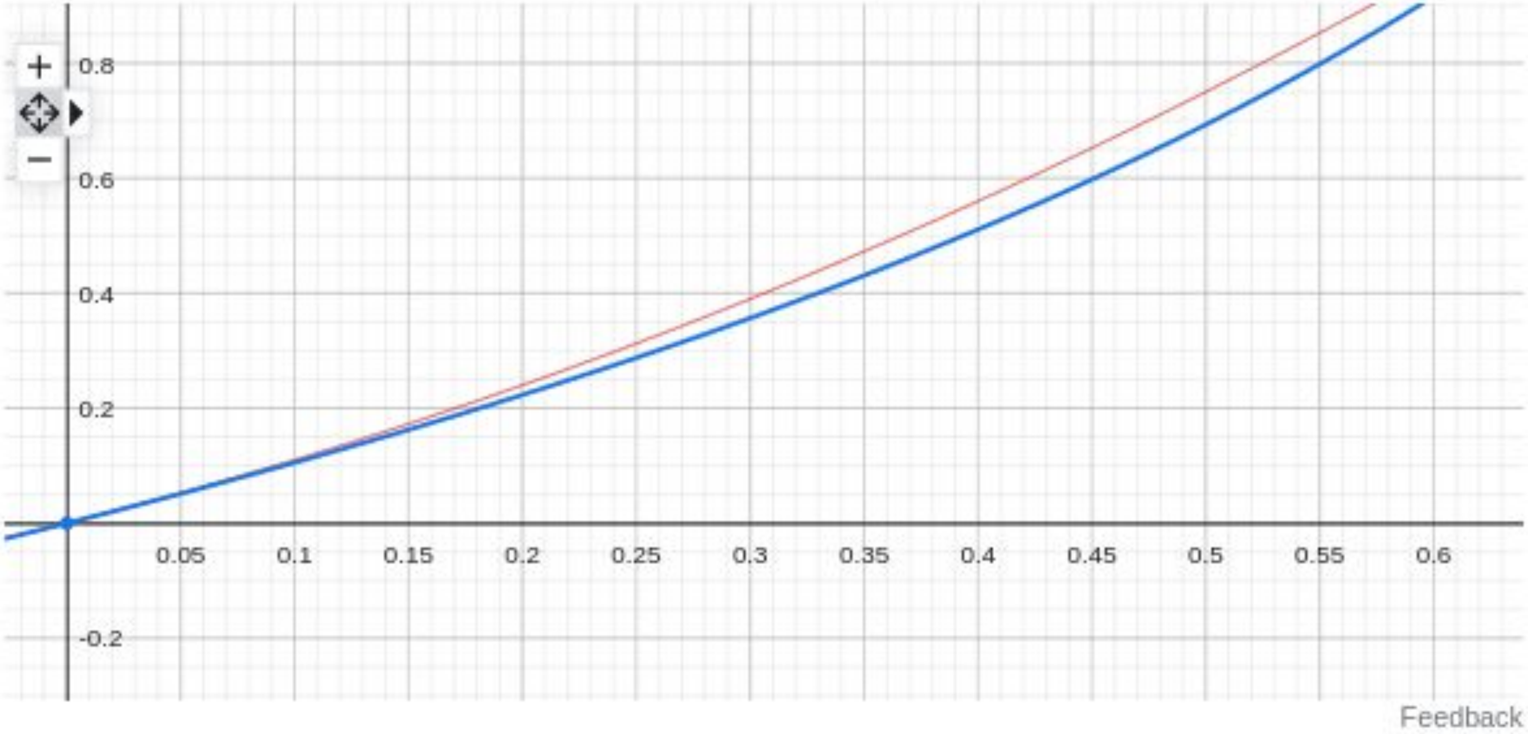
$$\Phi_{final} \geq (1-\epsilon)^M$$

$$(1-\epsilon)^M \leq n \cdot e^{-\epsilon \sum_t F_t} \quad \Rightarrow \quad \epsilon \sum_t F_t \leq M \ln \frac{1}{(1-\epsilon)} + \ln n.$$

Fact: $\ln \frac{1}{(1-\epsilon)} = -\ln(1-\epsilon) \leq \epsilon + \epsilon^2$

for $\epsilon \in [0, \frac{1}{2}]$

Graph for ln(1/(1-x)), x+x*x



$$\Phi_{final} \leq n \cdot e^{-\epsilon \sum_t F_t}$$

$\varepsilon \cdot \sum_t F_t \leq M(\varepsilon + \varepsilon^2) + \ln n$

$$\Phi_{final} \geq (1-\epsilon)^M$$

$\varepsilon + \varepsilon^2$

$$(1-\epsilon)^M \leq n \cdot e^{-\epsilon \sum_t F_t} \quad \Rightarrow \quad \epsilon \sum_t F_t \leq M \ln \frac{1}{(1-\epsilon)} + \ln n.$$

Fact: $\ln \frac{1}{(1-\epsilon)} = -\ln(1-\epsilon) \leq \epsilon + \epsilon^2$

for $\varepsilon \in [0, \frac{1}{2}]$

$$\sum_t F_t \leq M(1+\epsilon) + \frac{\ln n}{\epsilon}.$$

Extension: fractional rewards

reward $\in \{0, 1\}$
$\{-1, 1\}$

In each time step, each expert predicts some action, and at the end of the day, a reward ∈ [-1, 1] is revealed for each action

Performance of expert: sum of rewards over time

Applications of multiplicative weights

- Machine learning: AdaBoost, Winnow, Hedge
- Optimization (solving LP)
- Game theory
  - **see another proof of mini-max theorem in lecture notes**
- Operations research and online statistical decision-making
- Computational geometry
- Complexity theory
- Approximation algorithms
- Differential privacy