15-451/651 Algorithm Design & Analysis
Spring 2023, Recitation #9

Objectives

- To review and practice the VCG mechanism
- To practice taking duals and interpret the results

Recitation Problems

1. **(VCG Examples)** Suppose there are 3 players and 3 allocations. Their values are as follows:

Player	Allocation		
	a_1	a_2	a_3
1	6	2	1
2	1	5	3
3	4	2	5

(a) Which allocation maximizes social welfare?

(b) Assume we are using VCG version 1.

VCG version 1: Given a vector of reported valuation functions $\mathbf{v}$:

- Let $f(\mathbf{v})$ be the allocation that maximizes social welfare with respect to $\mathbf{v}$.
  I.e., $f(\mathbf{v}) = \arg\max_{a \in A} \sum_j v_j(a)$.
- Pay each player i an amount equal to the sum of everyone else's reported valuations.
  I.e., $p_i(\mathbf{v}) = -\sum_{j \neq i} v_j(f(\mathbf{v}))$.

What are the values, payments, and utilities for each player? All players report their true values.

	Value	Payment	Utility
Player 1			
Player 2			
Player 3			

(c) Now suppose we are using the standard VCG.

> **VCG - standard version:** Given a vector of reported valuation functions $\mathbf{v}$,
>
> - Let $f(\mathbf{v})$ be the allocation that maximizes social welfare with respect to $\mathbf{v}$. I.e., $f(\mathbf{v}) = \arg\max_{a \in A} \sum_j v_j(a)$.
> - Let $p_i(\mathbf{v}) = \max_a (\sum_{j \neq i} v_j(a)) - \sum_{j \neq i} v_j(f(\mathbf{v}))$.

What are the values, payments, and utilities for each player? All players report their true values.

	Value	Payment	Utility
Player 1			
Player 2			
Player 3			

2. **(Duality Recap)**

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

its dual is:

$$\begin{aligned} \text{minimize} \quad & b^T y \\ \text{s.t.} \quad & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

For example, given

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 6x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 && \leq 5 \\ & 6x_1 + 3x_2 + 3x_3 && \leq 45 \\ & 2x_1 + x_2 + x_3 && \leq 3 \\ & x_1, x_2, x_3 && \geq 0 \end{aligned}$$

What is the dual?

3. **(Duality in graph problems)** Recall that a *vertex cover* of a graph $G = (V, E)$ is a subset of the vertices such that every edge in E is adjacent to at least one of the vertices in the subset. The minimum vertex cover is a vertex cover with the fewest possible vertices.

   (a) Write down a linear program for the vertex cover problem. You might need to make an assumption. What is not quite exact about this LP?

   (b) Write down the dual of this LP. What does it mean?

   (c) Based on your answer to Part (a), which of the following are true, and why?

   - ☐ For any graph G, size of minimum vertex cover = size of maximum matching
   - ☐ For any graph G, size of minimum vertex cover $\leq$ size of maximum matching
   - ☐ For any graph G, size of minimum vertex cover $\geq$ size of maximum matching

Further Review

1. **(Incentive-Compatibility)**

   (a) Suppose there is an election with 2 candidates. Each voter (player) has a preferred candidate: the value is 1 if the preferred candidate is elected and 0 otherwise. The candidate with the majority vote wins.
   Is this voting system incentive-compatible?

   (b) Now suppose there are 3 candidates. For each player, one candidate has value 1 and others have values 0. Is this incentive-compatible?

   (c) Now suppose there are preferences among the 3 candidates. For each player, one candidate has value 2, another has value 1, and the last has value 0. Voters will submit an order of preference for the three candidates. The candidate that receives the most votes wins (each voter can be thought of as giving two votes to a candidate and one vote to another). Is this incentive-compatible?

2. **(Duals of Non-Standard Forms)** For each of the following LPs that are not necessarily in standard form, find the dual of the LP. You may find it helpful to first convert to standard form.

   (a)

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

   (b)

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

   (c)

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

3. **(Maximum-flow, minimum-what?)** Let $G = (V, E)$ be a directed graph with edge capacities $c(u, v)$ for $(u, v) \in E$. Recall from lecture that the max-flow problem can be

written as an LP. Defining a flow variable f_{uv} representing $f(u,v)$ for every $(u,v) \in E$, we have the LP

$$\begin{aligned}
\text{maximize} \quad & \sum_{(s,u)\in E} f_{su} - \sum_{(u,s)\in E} f_{us}. && \text{(the net } s\text{-}t \text{ flow)} \\
\text{s.t.} \quad & \sum_{\substack{v \text{ s.t.} \\ (v,u)\in E}} f_{vu} - \sum_{\substack{v \text{ s.t.} \\ (u,v)\in E}} f_{uv} = 0. && \text{for all } u \notin \{s,t\} \text{ (flow conservation)} \\
& 0 \le f_{uv} \le c(u,v) && \text{for all } (u,v) \in E \text{ (capacity constraints)}
\end{aligned}$$

(a) Translate this LP to standard form.

(b) What is the dual of this problem?

(c) Simplify the dual LP as much as possible, which will involve making it *not* standard form. **Hints:**

- Remember that when we learned how to convert arbitrary LPs into standard form, we made substitutions like switching unbounded variables x with two non-negative variables $x^+ - x^-$. You might be able to simplify the resulting LP by doing the opposite.
- s and t are special cases (they don't have conservation constraints) which means they will also end up as special cases in the dual. Creating some extra variables for them will help to eliminate redundant constraints.

(d) Intuit that this corresponds to min-cut. If you assume you get an integer solution to this LP, describe what the variables and their values represent, and how each constraint forces the solution to be a minimum cut. It is possible to prove that you actually always will get an integer solution, but its a hard proof, so we won't do it.

4. **(Nash Equilibria & Duality)**

In the last further review, we defined the concept of a *Nash equilibrium* as a pair of strategies $(\mathbf{p}^*, \mathbf{q}^*)$ for the row and column players where $\mathbf{p}^*$ is an optimal response to $\mathbf{q}^*$, and $\mathbf{q}^*$ is an optimal response to $\mathbf{p}^*$.

To formalize this notion, we recall that the expected payoff to the row and column players when playing strategies $\mathbf{p}$ and $\mathbf{q}$ in a game with payoff matrices (R, C) are

$$V_R(\mathbf{p}, \mathbf{q}) = \mathbf{p}^\intercal R\mathbf{q} \qquad V_C(\mathbf{p}, \mathbf{q}) = \mathbf{p}^\intercal C\mathbf{q}$$

Therefore we can say that $(\mathbf{p}^*, \mathbf{q}^*)$ is a Nash equilibrium if

$$(\forall \mathbf{p})\,(\mathbf{p}^{*\intercal} R\mathbf{q}^* \ge \mathbf{p}^\intercal R\mathbf{q}^*) \quad \wedge \quad (\forall \mathbf{q})\,(\mathbf{p}^{*\intercal} C\mathbf{q}^* \ge \mathbf{p}^{*\intercal} R\mathbf{q})$$

Just from this definition, it's not immediately clear a Nash equilibrium necessarily even exists. It is possible, for example, that the best response to $\mathbf{p}_1$ is $\mathbf{q}_1$, but the best response to $\mathbf{q}_1$ is $\mathbf{p}_2$, whose best response is $\mathbf{q}_2$, whose best response is $\mathbf{p}_1$, forming a sort of loop without any stable point. In this problem, we use LP duality to prove that Nash equilibria exist in a variety of games.

First, suppose that Efe is playing the zero-sum game with payoff matrices (R^1, C^1) as the row player against Aditya as the column player.

(a) Using LP duality, prove that there exists a Nash equilibrium in this game, and explain how to find it in polynomial time.

Unfortunately, just one game isn't enough to satiate Efe's massive intellect. Therefore he starts playing a second zero-sum game with payoff matrices (R^2, C^2) as the column player against Abby as the row player. He plays these two games simultaneously, with the restriction that he must play the same strategy in both games. Therefore, you can consider this as a limited form of 3-player game.

(b) How would you formalize the definition of a Nash equilibrium to this setting?

(c) Now prove that there exists a Nash equilibrium in this game, and explain how to find it in polynomial time.

Then, to take it up even one notch further, Aditya begins playing a third game with payoff matrices (R^3, C^3) as the row player against Abby as the column player, where they also must play the same strategies in both of their games. In addition, we relax the requirement that the first two games be zero-sum. Instead, we require only that all games together be zero-sum: that $R^1_{ij} + R^2_{ij} + R^3_{ij} = 0$.

(d) How would you formalize the definition of a Nash equilibrium to this setting?

(e) Now prove that there exists a Nash equilibrium in this game, and explain how to find it in polynomial time.